K-theory for certain expanders

Joint work with G. Yu (Vanderbilt University)

H. Oyono-Oyono

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Coarse structures

Definition

(X, d) and (Y, d) metric spaces,

- *$f : X \to Y$ is a coarse map if*
  1. *$\forall r > 0, \exists s > 0$ s.t $d(x, y) < r \Rightarrow d(f(x), f(y)) < s$;*
  2. *$Z \subset Y$ bounded $\Rightarrow f^{-1}(Z)$ bounded.*
- *X and Y are coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ and $M > 0$ such that $d(f \circ g(y), y) < M$ and $d(g \circ f(x), x) < M \quad \forall x \in X$ and $\forall y \in Y$.*

Example

1. $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent;
2. If a finitely generated discrete group Γ equipped with the word metric acts isometrically, properly and cocompactly on a proper metric space X, then Γ and X are coarsely equivalent.

Quick review on Roe algebras

Definition

A discrete metric space Σ has bounded geometry if $\forall R > 0$, $\exists N \in \mathbb{N}$ s.t $\#B(x, R) \leq N$ for all x in Σ;

- Σ: discrete metric space with bounded geometry.
- $C[\Sigma]$: $*$-algebra of loc. cpct operators on $\ell^2(\Sigma)\otimes H$ with finite propagation (H separable Hilbert space) , i.e $T = (T_{x,y})_{(x,y)\in\Sigma^2}$ bounded with
  - $T_{x,y}$ cpct operator H;
  - there exists $r > 0$ s.t that $T_{x,y} = 0$ if $d(x, y) > r$.
- $C^*(\Sigma) = \overline{C[\Sigma]} \subset B(\ell^2(\Sigma)\otimes H)$;
- $C^*(\Sigma)$ encodes the large scale geometry of Σ.
- $K_*(C^*(\Sigma))$ is the receptacle for higher order indices of elliptic operators on manifolds coarsely equivalent to Σ;

The general case

For X proper metric space and μ borelian measure on X with $Supp\,\mu = X$, the preceding construction can be generalised to $L^2(\mu) \otimes H$ (H sep. Hilbert) $\Rightarrow C^*(X, \mu)$

Definition

Let (X, d) be a proper metric space. A discrete set $\Sigma \subset X$ is a net in X if $\exists\ \varepsilon > 0$ and $R > 0$ s.t

- *$d(y, y') > \varepsilon$ for all y and y' in Σ;*
- *For all x in X, there exists y in Σ s.t $d(x, y) < R$;*

Example : $\mathbb{Z}$ is a net in $\mathbb{R}$.
If μ is any borelian measure on X with $Supp\,\mu = X$, Σ a net in X, then $L^2(\mu) \otimes H \cong \ell^2(\Sigma) \otimes H$ by an isometry that preserves finite propagation, hence $C^*(X, \mu) \cong C^*(\Sigma)$ (non canonical);
The isomorphism is canonical in K-theory (Higson-Roe)

How can we construct elements in $K_*(C^*(\Sigma))$ (Higson-Roe)?

- Let X be a proper metric space that contains a net Σ with bounded geometry;
- Let $x \in K_*(X) = KK_*(C_0(X), \mathbb{C})$ given by a K-cycle $(\mathcal{H}, \rho, F)$ where
  - $\mathcal{H}$ is a Hilbert space;
  - $\rho : C_0(X) \to B(\mathcal{H})$ is a representation;
  - $F \in B(\mathcal{H})$ satisfies the K-cycle condition i.e $\rho(f)(F^2 - Id_{\mathcal{H}})$, $\rho(f)(F - F^*)$ and $[\rho(f), F]$ are compact operators for all f in $C_0(X)$.

We can assume without loss of generality that

- $\mathcal{H} = L^2(\mu) \otimes H$ with μ borelian measure on X with $Supp\,\mu = X$ and H separable Hilbert space;
- ρ is induced by the representation $C_0(X) \hookrightarrow L^2(\mu)$;
- F has finite propagation (replace F by $\sum_i f_i^{1/2} F f_i^{1/2}$ for some suitable partition of the unit (f_i));

In the even case set $W = \begin{pmatrix} I_d & F \\ 0 & I_d \end{pmatrix} \begin{pmatrix} I_d & 0 \\ -F^* & I_d \end{pmatrix} \begin{pmatrix} I_d & F \\ 0 & I_d \end{pmatrix} \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$

Then

$$W \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} W^{-}1 - \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} \in C^*(X, \mu) \otimes M_2(\mathbb{C})$$

and hence $\left[W \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} W^{-}1\right] - \left[\begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}\right]$ defines an element in $K_0(C^*(X, \mu))$

We get an assembly map

$$K_*(X) \to K_*(C^*(X, \mu)) \cong K_*(C^*(\Sigma))$$

The coarse Baum-Connes assembly map

Definition (The rips complex)

Σ discrete with bounded geometry. $P_r(\Sigma)$ is the set of probabilities on Σ with support of diameter less than r

Σ is a net in $P_r(\Sigma)$ (viewed as Dirac measures).

Definition (Coarse Baum-Connes assembly map)

$$\mu_{\Sigma,*} : \lim_{r>0} K_*(P_r(\Sigma)) \longrightarrow K_*(C^*(\Sigma)).$$

Theorem (Higson, Roe, Skandalis-Tu-Yu)

For Γ finitely generated group, if $\mu_{|\Gamma|,*}$ is an isomorphism, then the Novikov conj. holds for Γ. ($|\Gamma|$ is the underlying metric space equipped with the word metric)

Definition

A discrete metrioc space Σ is uniformly embeddable into a Hilbert space H if there exists a map $f : \Sigma \to H$ such that for every $R > 0$, there exists $S > 0$ such that

- $d(x, y) < R \Rightarrow \|f(x), f(y)\| < S$;
- $\|f(x), f(y)\| < R \Rightarrow d(x, y) < S$.

Theorem (Yu)

If Σ is uniformly embeddable into a Hilbert space, then Σ satisfies the coarse Baum-connes conj. (i.e μ_Σ is an iso).

In particular if Γ is a finitely generated discrete group uniformly embeddable into a Hilbert space, then Γ satisfies Novikov conjecture.

Example

1. Linear groups (Guentner)
2. Exact groups (Yu).

Counterexamples for the coarse Baum-Connes conjecture (CBC)

Infinite families of expander graphs are not uniformly embeddable into a Hilbert space (Gromov).

Theorem (Higson-Lafforgue-Skandalis)

Infinite families of expander graphs are counterexamples for CBC

Conjecture (Baum-Connes)

If A is a Γ-C^*-algebra, $\mu_{\Gamma,A,*} : K_*^{top}(\Gamma, A) \longrightarrow K_*(A \rtimes_r \Gamma)$ is an isomorphism, with $K_*^{top}(\Gamma, A) = \lim_{r>0} KK^\Gamma(C_0(P_r(\Gamma)), A)$.

Theorem (Higson-Lafforgue-Skandalis)

If the Cayley graph of Γ quasi-contains an infinite families of expander graphs, then $\mu_{\Gamma,A,*}$ fails to be an iso for some Γ-C^*-algebra A.

How can we construct families of expander graphs?

Recall that a family of k-regular graphs $(X_i)_{i\in I}$ is a family of expanders if there exists $\varepsilon > 0$ s.t $\lambda_1(\Delta_{X_i}) > \varepsilon$ for all i.

Definition

Γ has the property (τ) with respect to a family $(\Gamma_i)_{i\in\mathbb{N}}$ of finite index normal subgroups if the trivial rep. is isolated from irr. representation that factorises through some Γ/Γ_i.

Example

For $n \geq 2$, $\Gamma = SL_n(\mathbb{Z})$ with $\Gamma_i = \ker SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/\mathbb{Z}_i)$.

Theorem

The family of Caley Graphs of Γ/Γ_i is a family of expanders iff Γ has property (τ) holds with respect to $(\Gamma_i)_{i\in\mathbb{N}}$.

Assembly maps and expanders

- For Γ f.g and residually finite with respect to a family $(\Gamma_i)_{i\in\mathbb{N}}$ of finite index normal subgroups (i.e $\cap_{i\in\mathbb{N}}\Gamma_i = \{e\}$), we set

$$X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$$

  equipped with a left invariant metric d
    - Induced by the word metric of Γ on Γ/Γ_i;
    - $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) \geq i + j$.
- The behaviours of $X(\Gamma)$ with respect to CBC and of Γ with respect to BC substancially differ :
- for $\Gamma = SL_2(\mathbb{Z})$
    - $X(\Gamma)$ is a family of expander graphs so is a counterexample for CBC
    - Γ has the Haagerup property and thus satisfies BC with coefficients (Higson-Kasparov).
    - If we consider maximal C^*-algebras, the behaviours become quite similar!

Maximal Roe algebras

Lemma (Gong-Wang-Yu)

For any T in $C[\Sigma]$, there exists a real c_T such that for any $*$-representation ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ, then $\|\phi(T)\| \leq c_T$.

Definition (Gong-Wang-Yu)

$C^*_{max}(\Sigma)$ is the completion of $C[\Sigma]$ with respect to the $*$-norm

$$\|T\| = \sup_{(\phi, H_\phi)} \|\phi(T)\|,$$

when (ϕ, H_ϕ) runs through representations ϕ of $C[\Sigma]$ on a Hilbert space H_ϕ.

We can define a maximal coarse assembly map

$$\mu_{\Sigma,\max} : \lim_{r>0} K_*(P_r(\Sigma)) \longrightarrow K_*(C^*_{\max}(\Sigma))$$

Main result

Proposition (O-Yu)

For $A_\Gamma = \ell^\infty(X(\Gamma), \mathcal{K}(H))/C_0(X(\Gamma), \mathcal{K}(H))$*, we have an exact sequence*

$$0 \longrightarrow \mathcal{K}(\ell^2(X(\Gamma)) \otimes H) \longrightarrow C^*_{max}(X(\Gamma)) \longrightarrow A_\Gamma \rtimes_{max} \Gamma \longrightarrow 0$$

which gives rise to a $\mathbb{Z}/2\mathbb{Z}$*-graded exact sequence*

$$0 \longrightarrow \mathbb{Z} \oplus \{0\} \longrightarrow K_*(C^*_{max}(X(\Gamma))) \longrightarrow K_*(A_\Gamma \rtimes_{max} \Gamma) \longrightarrow 0.$$

Not exact for $C^*(X(\Gamma))$ and $A_\Gamma \rtimes_{red} \Gamma$ (Higson-Lafforgues-Skandalis).

Theorem (O-Yu)

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} \oplus \{0\} & \longrightarrow & \lim_r K_*(P_r(X(\Gamma))) & \longrightarrow & K_*^{top}(\Gamma, A_\Gamma) & \longrightarrow & 0 \\
 & & = \big\downarrow & & \mu_{X(\Gamma),max} \big\downarrow & & \mu_{\Gamma, A_\Gamma, max} \big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} \oplus \{0\} & \longrightarrow & K_*(C^*_{max}(X(\Gamma))) & \longrightarrow & K_*(A_\Gamma \rtimes_{max} \Gamma) & \longrightarrow & 0
\end{array}$$

Applications

Theorem

$$\mu_{X(\Gamma),max,*} : \lim_r K_*(P_r(X(\Gamma))) \to K_*(C^*_{max}(X(\Gamma)))$$

is an isomorphism iff

$$\mu_{\Gamma,A_\Gamma,max,*} : K_*^{top}(\Gamma, A_\Gamma) \to K_*(A_\Gamma \rtimes_{max} \Gamma)$$

is an isomorphism.

Examples: $\Gamma = SL_2(\mathbb{Z})$ and more generally if Γ has the Haagerup property, Γ fundamental group of a compact oriented 3-manifold.

Theorem

If $\mu_{\Gamma,A_\Gamma,*} : K_*^{top}(\Gamma, A_\Gamma) \to K_*(A_\Gamma \rtimes_{red} \Gamma)$ *is one-to-one, then* $\mu_{X(\Gamma),*} : \lim_r K_*(P_r(X(\Gamma))) \to K_*(C^*(X(\Gamma)))$ *is also one-to-one.*

Example : Γ uniformly embeddable in a Hilbert space.

Further connection with Baum-Connes conjecture: quantitative assembly maps

- In what follows, Γ is f.g and residually finite with respect to a family $(\Gamma_i)_{i\in\mathbb{N}}$ of finite index normal subgroups;
- We have a family of assembly maps

$$\mu_{\Gamma_i,\mathbb{C},\max,*}: K_*^{\mathrm{top}}(\Gamma_i,\mathbb{C})\longrightarrow K_*(C^*_{\max}(\Gamma_i))$$

with $K_*^{\mathrm{top}}(\Gamma_i,\mathbb{C}) = \lim_{r>0} KK_*^{\Gamma_i}(C_0(P_r(\Gamma)),\mathbb{C})$.

- Aim: Introduce some propagation control on $(\mu_{\Gamma_i,\mathbb{C},\max})_{i\in\mathbb{N}}$ to relate BC max for the Γ_i with CBC max for $X(\Gamma) = \coprod_{i\in\mathbb{N}} \Gamma/\Gamma_i$;
- The relevant propagation comes indeed from $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$ under the Morita equivalence between $C^*_{\max}(\Gamma_i)$ and $C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma$;
- Up to this Morita equivalence, we have a family of assembly maps

$$\mu_{\Gamma_i,\mathbb{C},\max,*}: K_*^{\mathrm{top}}(\Gamma_i,\mathbb{C})\longrightarrow K_*(C(\Gamma/\Gamma_i)\rtimes_{\max}\Gamma)$$

Almost projection with finite propagation

Definition

For a unital Γ-C^*-algebra A and $r > 0$, then $f \in A \rtimes_{max} \Gamma$ has propagation less than r if $f \in C_c(\Gamma, A)$ and $\mathrm{Supp}\, f \subset B(e, r)$ (where Γ is equipped with the word metric).

Idea : control the propagation in $K_*(C(\Gamma/\Gamma_i) \rtimes_{\max} \Gamma$ by the family $(KK^{\Gamma_i}(C_0(P_r(\Gamma), \mathbb{C}))_{r>0}$.

Definition

For a unital Γ-C^*-algebra A, $\varepsilon \in (0, 1/4)$ and $r > 0$, $p \in A \rtimes_{max} \Gamma$ is an ε-r-projection if

- $p = p^*$, $\|p^2 - p\| < \varepsilon$;
- *p has propagation less than r;*

p is an ε-r proj $\Rightarrow$ p has a spectral gap and if $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is cont. s.t $\phi_\varepsilon(t) = 0$ for $t < \frac{1-\sqrt{1-4\varepsilon}}{2}$ and $\phi_\varepsilon(t) = 1$ for $t > \frac{1+\sqrt{1-4\varepsilon}}{2}$, then $\phi_\varepsilon(p)$ is a proj.

Almost projection and K-theory

Definition

Let p_0 in some $M_{k_0}(A \rtimes_{max} \Gamma)$ and p_1 in $M_{k_1}(A \rtimes_{max} \Gamma)$ be ε-r-projections and let n_0 and n_1 be integers, then

$$(p_0, n_0) \sim_{\varepsilon,r} (p_1, n_1)$$

if for some n, $\begin{pmatrix} p_0 & & \\ & I_{k_0+n} & \\ & & 0 \end{pmatrix}$ and $\begin{pmatrix} p_1 & & \\ & I_{k_1+n} & \\ & & 0 \end{pmatrix}$ are homotopic as ε-r-projections in some $M_N(A \rtimes_{max} \Gamma)$ with $N \geq k_0 + n_0 + n,\ k_1 + n_1 + n$.

Remark

If $(p_0, n_0) \sim_{\varepsilon,r} (p_1, n_1)$ then $[\phi_\varepsilon(p_0)] - [I_{n_0}] = [\phi_\varepsilon(p_1)] - [I_{n_1}]$ in $K_0(A \rtimes_{\max} \Gamma)$.

A Quantitative assembly map

- Fix for every r a Γ-inv. measure ν_r on $P_r(\Gamma)$ with $Supp\,\nu_r = P_r(\Gamma)$;
- Every element in $KK_*^{\Gamma_i}(C_0(P_r(\Gamma)), \mathbb{C})$ can be represented by a K-cycle $(L^2(\nu_r) \otimes H, \rho_r, F_i)$ where
  - ρ_r is induced by the representation $C_0(P_r(\Gamma)) \hookrightarrow L^2(\nu_r)$;
  - F_i has finite propagation and is Γ_i-equivariant;

$\Psi^{\Gamma_i}((P_r(\Gamma))$: set of op. on $L^2(\nu_r) \otimes H$ satisfying the (even) K-cycle condition for $KK_0^{\Gamma_i}(C_0(P_r(\Gamma)), \mathbb{C})$, of finite prop., Γ_i-equiv. and $\|F_i\| \leq 1$.

Proposition (O-Yu)

Let F_i in $\Psi^{\Gamma_i}((P_r(\Gamma))$. Then for every $\varepsilon \in (0, 1/72)$, there exists

- *r (depending only on ε, d and the propagation of F_i);*
- *an integer $n_{F_i,\varepsilon}$*
- *a ε-r-projection $p_{F_i,\varepsilon}$ of $M_{2n_{F_i,\varepsilon}}(C(\Gamma/\Gamma_i) \rtimes_{max} \Gamma)$;*

s.t if x_{F_i} in $K_0^{top}(\Gamma_i, \mathbb{C})$ comes from the K-cycle $(L^2(\nu_r) \otimes H, \rho_r, F_i)$ then $\mu_{\Gamma_i,\mathbb{C},max,*}(x_{F_i}) = [\phi_\varepsilon(p_{F_i,\varepsilon})] - [I_{n_{F_i,\varepsilon}}]$ in $K_0(C(\Gamma/\Gamma_i) \rtimes_{max} \Gamma)$.

Asymptotic statements

For any integer i and any positive real r, r', s, s' and any ε in $(0, 1/72)$

$\mathbf{QI_0(i, r, r', s, \varepsilon)}$ for any (even) K-cycle F of $\Psi^{\Gamma_i}(P_r(\Gamma))$, then $(p_{F,\varepsilon}, n_{F,\varepsilon}) \sim_{18\varepsilon,s} (0,0) \Rightarrow$ the class corresponding to F lies in the kernel of

$$KK_0^{\Gamma}(P_r(\Gamma), C(\Gamma/\Gamma_i)) \longrightarrow KK_0^{\Gamma}(P_{r'}(\Gamma), C(\Gamma/\Gamma_i))$$

induced by the inclusion $P_r(\Gamma) \hookrightarrow P_{r'}(\Gamma)$.

$\mathbf{QS_0(i, r, s, s', \varepsilon)}$ For any ε-s-projection p in some $M_k(C(\Gamma/\Gamma_i) \rtimes_{\max} \Gamma)$ and any integer n, there exists a (even) K-cycle F of $\Psi^{\Gamma_i}(P_r(\Gamma))$ such that $(p_{F,\varepsilon}, n_{F,\varepsilon}) \sim_{18\varepsilon,s} (p, n)$

Remark

We have similar statement in the odd case in term of ε-r-unitaries (u with propagation less than r and $\|u^*u - 1\| < \varepsilon$ and $\|uu^* - 1\| < \varepsilon$

Application to injectivity

Theorem (O-Yu)

The following statements are equivalent:

1. *For any positive real r the following condition holds : there is an ε in $(0, 1/72)$ such that for any positive real s, there exists an integer j and a positive real r' for which $QI_*(i, r, r', s, \varepsilon)$ is true for all $i \geq j$.*
2. $\mu_{X(\Gamma),max,*}$ *is injective.*

Example $SL_n(Z)$ and more generally if Γ embeds uniformally in a Hilbert space

Application to surjectivity

Theorem (O-Yu)

Assume that $\exists\, \varepsilon \in (0, 1/72)$ *s.t the following holds:* $\forall s > 0$*,* $\exists r > 0$ *and* $s' > 0$ *and an integer* j *such that* $QS_*(i, r, s, s', \varepsilon)$ *is true for all integer* $i \geq j$*. Then* $\mu_{X(\Gamma),max,*}$ *is surjective.*

Up to reoganise the sequence $(\Gamma_i)_{i\in/N}$, we have the following converse

Theorem (O-Yu)

Set $X^\infty(\Gamma) = \coprod_{i\in\mathbb{N}} \coprod_{j\geq i} \Gamma/\Gamma_j$ *and assume that*

$$\mu_{X^\infty(\Gamma),max,*} : \lim_r K_*(P_r(X^\infty(\Gamma)), \mathbb{C}) \to K_*(C^*_{max}(X^\infty(\Gamma)))$$

is onto. Then $\exists \varepsilon \in (0, 1/72)$ *s.t the following holds:* $\forall s > 0$*,* $\exists r > 0$ *and* $s' > 0$ *and an integer* j *s.t* $QS_*(i, r, s, s', \varepsilon)$ *is true for all integer* $i \geq j$*.*

Examples : $SL_2(\mathbb{Z})$ and more generally Γ with the Haagerup property; Fundamental groups of compact oriented 3-manifolds.

A hint for surjectivity defect of the Baum-Connes assembly maps

Conditions *QI* and *QS* can obviously be defined with reduced cross products instead of maximal cross products.

Proposition (O-Yu)

Set $A_\Gamma^\infty = \ell^\infty(X^\infty(\Gamma), \mathcal{K}(H))/C_0(X^\infty(\Gamma), \mathcal{K}(H))$ *and assume that* Γ *is K-exact and that following condition is satisfied:*

for all ε *in* $(0, 1/72)$ *there exists* $s > 0$ *such that for all* $r >$ *and* $s' > 0$ *and* j *integer, there exists an integer* i *with* $i \geq j$ *for which* $QS_*^{red}(i, r, s, s', \varepsilon)$ *does not hold.*

Then $\mu_{\Gamma, A_\Gamma^\infty, *}$ *is not onto.*