Gap-labelling of the pinwheel tiling

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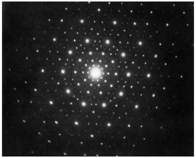
Vietri Sul Mare, August 31 2009



Plan

- Pinwheel tiling, tiling spaces and the gap labelling conjecture

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  Douglas, Hurder and Kaminker, 1991

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  The gap-labelling is given by $\dfrac{1}{264}\mathbb{Z}\left[\frac{1}{5}\right]$

Pinwheel tiling, tiling spaces and the gap-labelling conjecture

Definitions

Definition :

- **Tiling** of the plane : countable family $P = \{t_1, t_2, \ldots\}$ of non empty polygons t_i, called *tiles* s.t. :
  - $t_1, t_2, \ldots$ cover the Euclidean plane.
  - Two tiles only meet on their border.

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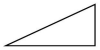
  - Two tiles only meet on their border.
- **Patch** : finite union of tiles of the tiling.

Pinwheel tiling



(a)

Fig. 1: Construction of a (1,2)-pinwheel tiling

Pinwheel tiling

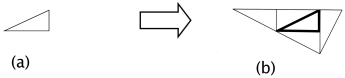


FIG. 1: Construction of a (1,2)-pinwheel tiling

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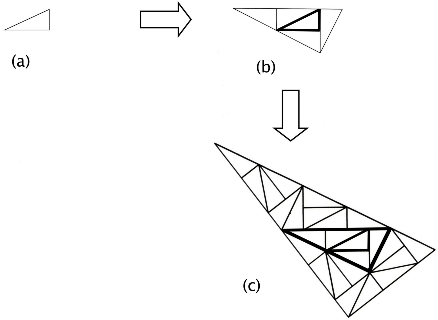

FIG. 1: Construction of a (1,2)-pinwheel tiling

Pinwheel tiling

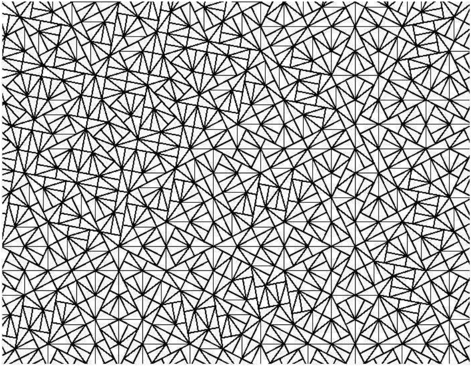

FIG. 2: (1,2)-pinwheel tiling

Repetitivity

$G = \mathbb{R}^2 \rtimes S^1$ group of rigid motions.

- **Aperiodic** tiling P : no translation of $\mathbb{R}^2$ fixes P.
- **Finite** G**-type** tiling : $\forall R > 0$, there exists a finite number of patches with diameter smaller than R modulo the action of G.
- G-**Repetitive** tiling P : for any patch $\mathcal{A}$ of P, $\exists R(\mathcal{A}) > 0$ s.t. any ball of radius $R(\mathcal{A})$ intersects P on a patch containing a G-copy of $\mathcal{A}$.

Tiling space Ω

P a pinwheel tiling.

- $\Omega =$ completion of $P \cdot (\mathbb{R}^2 \rtimes S^1)$.
- Ω is a compact metric space.
- $\left(\Omega, \mathbb{R}^2 \rtimes S^1\right)$ is a minimal dynamical system.
- $C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1 =$ completion of $C_c(\mathbb{R}^2 \rtimes S^1 \times \Omega)$.

The canonical transversal Ξ

- $\Xi := \{P' \in \Omega \mid 0 \in Punct(P') \;\&\; P' \text{ is well oriented}\}$.
- Ξ is a Cantor set
- Ω is a foliated space and Ξ is a transversal of Ω.

Gap-labelling conjecture

- Ω is endowed with a G-invariant ergodic probability measure μ.
- μ induces an invariant transverse measure μ^t on Ξ defined locally by the quotient of μ by the Lebesgue measure .
- $\tau^\mu(f) := \int_\Omega f(0,0,\omega)d\mu(\omega)$ for $f \in C_c(\mathbb{R}^2 \rtimes S^1 \times \Omega)$ defines a trace on $C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1$.

Gap-labelling conjecture

Gap-Labelling conjecture : (Bellissard, 1989)

$$\tau_*^\mu\Big(K_0\big(C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1\big)\Big) = \mu^t\big(C(\Xi, \mathbb{Z})\big)$$

Index theorem to solve the gap-labelling conjecture

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Theorem : (M., 2009)
$\forall\, b \in K_0\big(C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1\big)$, $\exists [u] \in K_1\big(C(\Omega)\big)$ s.t.

$$\tau_*^{\mu}\big(b\big) = \tau_*^{\mu}\big([u] \otimes_{C(\Omega)} [D_3]\big)$$

Index theorem to solve the gap-labelling conjecture

$$K_1\big(C(\Omega)\big) \xrightarrow{\otimes[D_3]} K_0\big(C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1\big)$$

Index theorem to solve the gap-labelling conjecture

$$
\begin{array}{ccc}
K_1\big(C(\Omega)\big) & & \\
\downarrow \otimes[d_1] & \searrow \otimes[D_3] & \\
K_0\big(C(\Omega) \rtimes S^1\big) & \xrightarrow[\otimes[D_2]]{} & K_0\big(C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1\big)
\end{array}
$$

Index theorem to solve the gap-labelling conjecture

$$
\begin{array}{ccccc}
\check{H}^3(\Omega;\mathbb{Z}) \oplus \check{H}^1(\Omega;\mathbb{Z}) & \cong & K_1\big(C(\Omega)\big) & & \\
 & & \Big\downarrow \otimes[d_1] & \searrow \otimes[D_3] & \\
\check{H}^2(\Omega/S^1;\mathbb{Z}) \oplus \mathbb{Z} & \cong & K_0\big(C(\Omega) \rtimes S^1\big) & \xrightarrow[\otimes[D_2]]{} & K_0\big(C(\Omega) \rtimes \mathbb{R}^2 \rtimes S^1\big)
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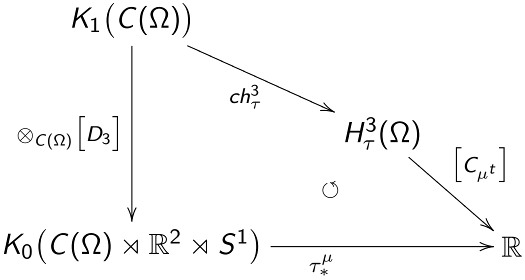
$$\tau_*^{\mu}(b) = \tau_*^{\mu}([u] \otimes_{C(\Omega)} [D_3]) = \Big[Ch_\ell^3([u]) \mid [C_{\mu^t}] \Big]$$

where $[C_{\mu^t}] \in H_3^\ell(\Omega)$ is the Ruelle-Sullivan current associated to μ^t and $Ch_\ell^3 : K_1(C(\Omega)) \to H_\ell^3(\Omega)$ is the degree 3 component of the longitudinal Chern character.

Interger group of coinvariants



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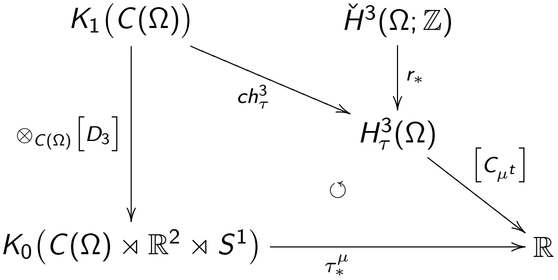

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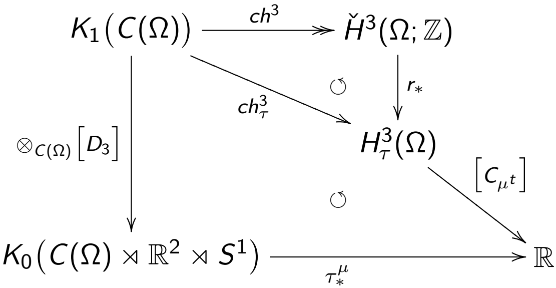

Interger group of coinvariants

Theorem :

$$\check{H}^3(\Omega;\mathbb{Z}) \simeq H^2(\Omega/S^1;\mathbb{Z})$$

and

$$\Omega/S^1 = \varprojlim B_n$$

with B_n simplicial complexes of dimension 2.

Thus

$$\check{H}^3(\Omega;\mathbb{Z}) \simeq \varinjlim \check{H}^2(B_n;\mathbb{Z})$$

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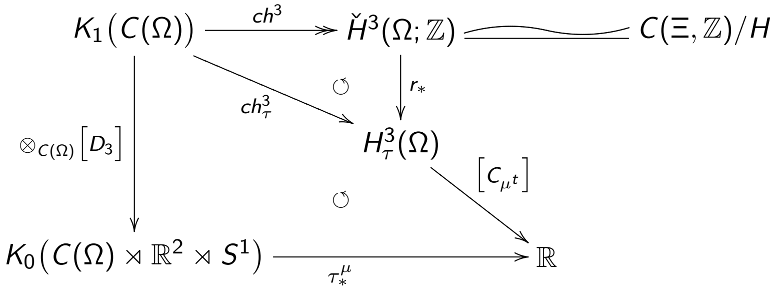
Interger group of coinvariants

Theorem : (M., 2009)

$$\varinjlim H^2(B_n;\mathbb{Z}) \simeq C(\Xi,\mathbb{Z})/H$$

with $\forall h \in H$, $\mu^t(h) = 0$.

Interger group of coinvariants



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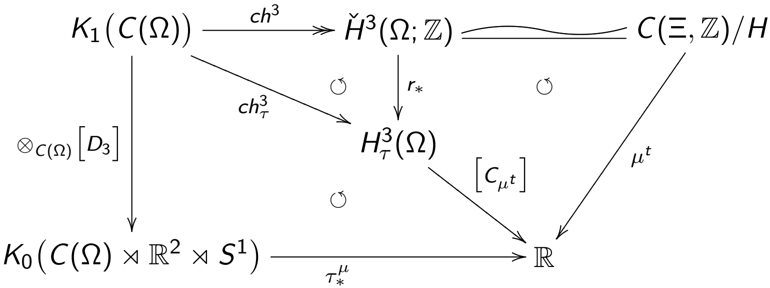

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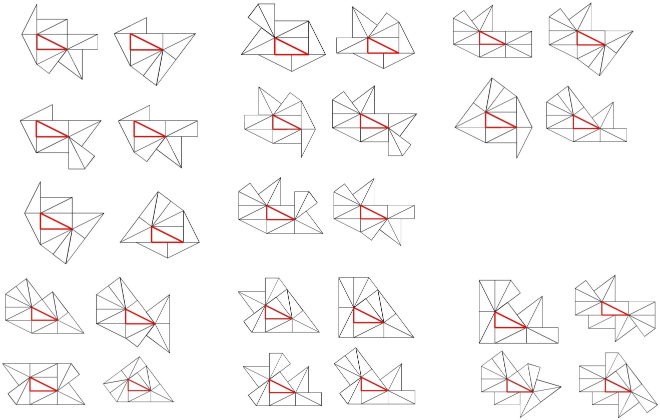
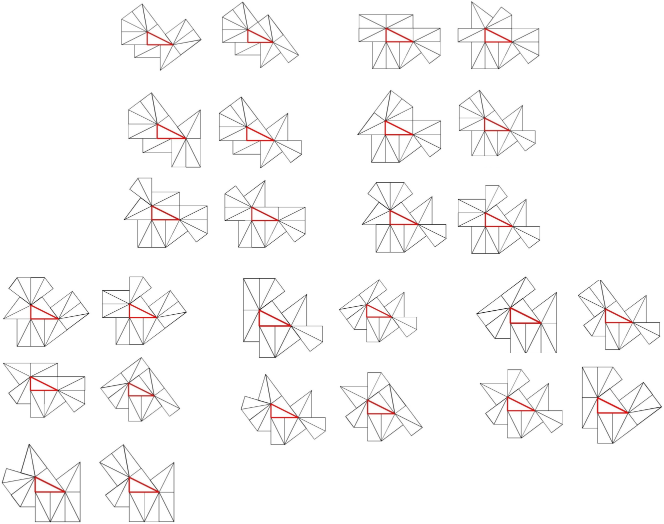
Computation

Definitions

P a pinwheel tiling.

- **First corona** of a tile : union of all the tiles intersecting it in P .
- **collared prototile** of P : equivalence class of tiles with the same first corona up to rigid motions.





Computation

A = matrix with $a_{i,j}$ = number of collared tiles of type i in the substitution of the collared prototile of type j.

Proposition : (M. ,2009)

$$\varinjlim(\mathbb{Z}^{108}, A') \simeq C(\Xi, \mathbb{Z})/H'$$

with $\forall h \in H'$, $\mu^t(h) = 0$.

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Conclusion