Temperature for double-cones in 2D boundary CFT

P. Martinetti

Göttingen Universität

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Outline:

1. Physical interpretation of the modular group as a flow of time
2. Wedges in Minkowski space-time
3. Double-cones in Minkowski space-time
4. Double-cones in 2d boundary conformal field theory

1. Time flow from the modular group

Time, state and temperature

$\mathcal{A}$: algebra of observables of a system,

α_t : time evolution (e.g. $\alpha_t a = e^{-iHt} a e^{iHt}$).

An equilibrium state ω at temperature β^{-1} is a state that satisfies the KMS condition:

$$\omega((\alpha_t a) b) = \omega(b(\alpha_{t+i\beta} a)) \qquad \forall a, b \in \mathcal{A}.$$

"Von Neumann algebras naturally evolve with time" (*Connes*)

$$\left.\begin{array}{l} \text{- a von Neumann algebra } \mathcal{A} \text{ acting on } \mathcal{H} \\ \\ \text{- a vector } \Omega \text{ in } \mathcal{H} \text{ cyclic and separating} \end{array}\right\} \Rightarrow \begin{array}{l} \text{Tomita's operator:} \\ \qquad S\, a\Omega \to a^*\Omega \\ \text{yields a 1-parameter group} \\ \sigma \text{ of automorphisms of } \mathcal{A} \\ \text{(modular group)} \end{array}$$

The state $\omega : a \mapsto \langle \Omega, a\Omega \rangle$ is KMS with respect to σ_s,

$$\omega((\sigma_s a)b) = \omega(b(\sigma_{s-i} a)) \quad \forall a, b \in \mathcal{A},\ s \in \mathbb{R}.$$

Hence ω is thermal at temperature -1 with respect to the evolution σ_s.

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Tomita's operator:

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Writing $\alpha_{-\beta s} \doteq \sigma_s$,

$$\omega((\alpha_{-\beta s} a)b) = \omega(b(\alpha_{-\beta s + i\beta} a))$$

An equilibrium state at temperature β^{-1} is a faithful state over the algebra of observables whose modular group σ_s is the physical time translation, up to rescaling
$$t = -\beta s.$$

$$\left\{\begin{array}{l}\text{time flow } \alpha_t \\ \text{temperature } \beta^{-1}\end{array}\right. \xRightarrow[\text{KMS}]{} \text{equilibrium state } \omega$$

$$\left\{\begin{array}{l}\text{state } \omega \\ \text{temperature } \beta^{-1}\end{array}\right. \xRightarrow[\text{modular theory}]{} \text{time flow } \alpha_{-\beta s}$$

<u>The thermal time hypothesis</u> (Connes, Rovelli 1993): assuming the system is in a thermal state at temperature β^{-1}, then the physical time t is the modular flow up to rescaling $t = -\beta s$.

If another notion of time is available (e.g. geometrical time τ), one should check that $\tau = t$, i.e. $\beta = -\frac{\tau}{s}$.

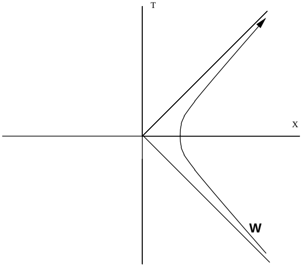
$$\left\{\begin{array}{l}\text{state} \\ \text{time}\end{array}\right. \Longrightarrow \text{temperature}$$

2. Temperature for the wedge

Bisognano, Wichman, Sewell

$$W \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{A}(W) \\ \text{vacuum modular group } \sigma_s^W \rightarrow \text{ boosts} \rightarrow \text{geometrical action} \end{cases}$$

$$\begin{array}{ccc} \text{uniformly accelerated observer's trajectory} & & \text{orbit of the modular group} \\ \tau \in]-\infty, +\infty[& = & s \in]-\infty, +\infty[\end{array}$$



$$\beta = |\frac{\tau}{s}| = \frac{2\pi}{a} = T_{\text{Unruh}}^{-1}.$$

The temperature is constant along a given trajectory.

Same analysis for other open regions $\mathcal{O}$ of Minkowski space-time ?

- The vacuum modular group $\sigma_s^{\mathcal{O}}$ must have a geometrical action.
- The orbits must coincide with the trajectories of some observers with proper time τ.
- The ratio $\frac{\tau}{s}$ should be constant along each orbit.

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This last assumption may be relaxed:
to identify ∂_s to ∂_τ, one only needs ∂_s to be normalised,

$$\partial_t \doteq \frac{\partial_s}{\beta} \text{ with } \beta \doteq \|\partial_s\|\,.$$

Putting $\partial_t = \partial_\tau$ then yields

$$\partial_\tau = \frac{\partial_s}{\beta} \Rightarrow \beta = |\frac{d\tau}{ds}|.$$

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- For wedges, $\frac{d\tau}{ds} =$ constant along a given orbit $= \frac{\tau}{s}$.
- β still makes sense when it is no longer constant $\Rightarrow$ *local* equilibrium temperature.

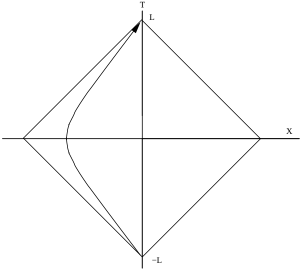
3. Temperature for the double-cone

P. M., Rovelli

$$D \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{A}(D) \\ \text{vacuum modular group } \sigma_s^D \end{cases}$$

$D = \varphi(W)$ for a conformal map φ. So for conformal qft (Hislop, Longo):

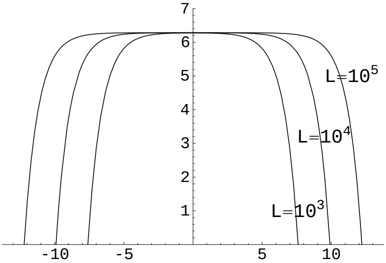
$$\begin{array}{c} \text{uniformly accelerated observer's trajectory} \\ \tau \in]-\tau_0, +\tau_0[\end{array} = \begin{array}{c} \text{orbit of the modular group} \\ s \in]-\infty, +\infty[\end{array}$$



Ratio $\frac{\tau}{s}$ no longer constant,

$$\beta(\tau) = |\frac{d\tau}{ds}| = \frac{2\pi}{La^2}\,(\sqrt{1 + a^2L^2} - \mathrm{ch}\, a\tau).$$

Along a given orbit, the inverse temperature $\beta(\tau)$, $-\tau_0 < \tau < \tau_0$ varies:



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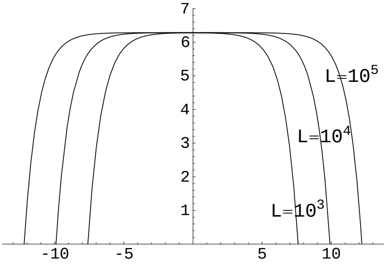


The conformal map $\varphi : W \to D$ induces on W a metric $\tilde{g}$,

$$\tilde{g}(U, V) = g(\varphi_* U, \varphi_* V) = C^2 g(U, V).$$

The double-cone temperature is proportional to the inverse of C,

$$\beta(x) = \frac{2\pi}{a} C(\varphi^{-1}(x))$$

where a is the acceleration characterizing the modular orbit of $\varphi^{-1}(x)$.

3. Double-cone in 2d boundary CFT

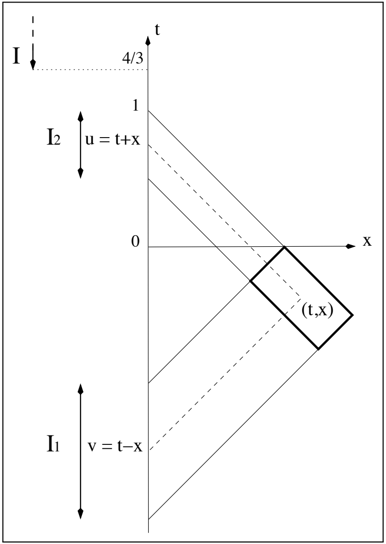
work in progress with R. Longo and K. H. Rehren

A CFT on the half plane $(t, x > 0)$ yields a chiral net

$$\mathcal{I} \mapsto \mathcal{A}(\mathcal{I}), \quad \mathcal{I} =]A, B[\in \mathbb{R},$$

and generates a net of double-cone algebras

$$\mathcal{O} = I_1 \times I_2 \mapsto \mathcal{A}(\mathcal{O}).$$



One can build on $\mathcal{A}(\mathcal{O})$ a state whose associated modular group has a geometrical action.

Cayley transform

$$z = \frac{1+ix}{1-ix} \in S^1 \iff x = \frac{(z-1)/i}{z+1} \in \mathbb{R} \cup \{\infty\}.$$

Square and square root:

$$\begin{aligned} z \mapsto z^2 \quad &\iff \quad x \mapsto \sigma(x) \doteq \frac{2x}{1-x^2}, \\ z \mapsto \pm\sqrt{z} \quad &\iff \quad x \mapsto \rho_\pm(x) = \frac{\pm\sqrt{1+x^2}-1}{x}. \end{aligned}$$

Modular group

For a pair of symmetric intervals I_1, I_2 , i.e.

$$\sigma(I_1) = \sigma(I_2) = I,$$

the modular group has a geometrical action

$$(u, v) \in \mathcal{O} \;\mapsto\; (u_s, v_s) \in \mathcal{O} \qquad s \in \mathbb{R},$$

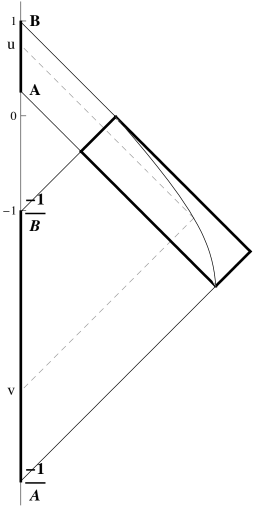
with orbits

$$\begin{aligned} u_s &= \rho_+ \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(u), \\ v_s &= \rho_- \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(v), \end{aligned}$$

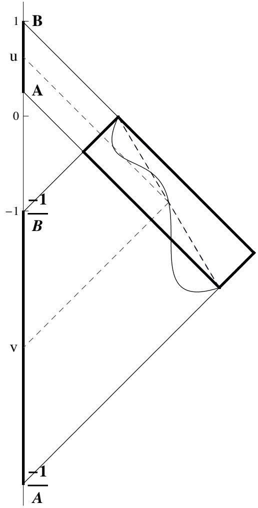
where $\lambda_s(x) = e^s x$ is the dilation of $\mathbb{R}$, and m is a Möbius transformation which maps $\mathbb{R}_+$ to I.

Implicit equation of the orbits

$$\frac{(u_s - A)(Au_s + 1)}{(u_s - B)(Bu_s + 1)} \cdot \frac{(v_s - B)(Bv_s + 1)}{(v_s - A)(Av_s + 1)} = \text{const},$$



- This equation only depends on the end points of $I_2 =]A, B[, \ I_1 =] - \frac{1}{A}, -\frac{1}{B}[$.
- All orbits are time-like, hence $\beta = |\frac{d\tau}{ds}|$ makes sense as a temperature.
- One and only one orbit is a boost ($\text{const} = 1$) and thus is the trajectory of a uniformly accelerated observer.



Other orbits have more complicated dynamics (e. g. the sign of the acceleration may change).

Temperature on the boost trajectory

Constant acceleration: $d\tau^2 = du\,dv$ hence

$$\beta = \frac{d\tau}{ds} = \sqrt{u'v'}$$

with $' = \frac{d}{ds}$. On the boost orbit, $v_s = -\frac{1}{u_s}$ hence

$$\beta = \frac{u'}{u} = \frac{d}{ds}\ln u_s \Longrightarrow \tau(s) = \ln u_s - \ln u_0 \Longrightarrow u_s = u_o e^{\tau(s)}.$$

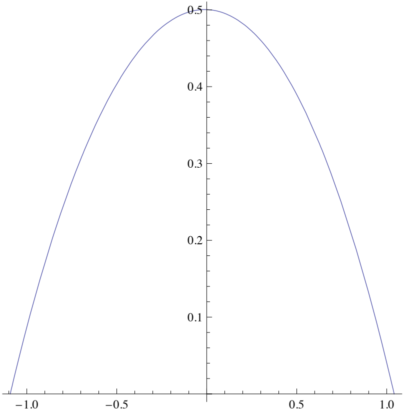
Knowing

$$u_s' = f_{AB}(u_s) \doteq \frac{(u_s - A)(Au_s + 1)(B - u_s)(Bu_s + 1)}{(B - A)(1 + AB)\cdot(1 + u_s^2)}.$$

one finally gets

$$\beta(\tau) = \frac{f_{AB}(u_o e^{\tau})}{u_o e^{\tau}}.$$

Contrary to double-cones in Minkowski space-time, the temperature along the boost-orbit does not present any plateau region.



- What happens far from the boundary (require double-cone defined by a non-symmetric pair of intervals) ?