Operator Algebras and Noncommutative Geometric Aspects in Conformal Field Theory

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Recent work based on joint papers
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Things to discuss

- Getting inspired by black hole entropy
- Symmetry and supersymmetry
- Local conformal nets
- Modularity and asymptotic formulae
- Fermi and superconformal nets
- Neveu-Schwarz and Ramond representations
- Fredholm index and Jones index
- Noncommutative geometrization
- Model analysis (in progress)

Prelude. Black hole entropy

Bekenstein: The entropy S of a black hole is proportional to the area A of its horizon

$$\boxed{S = A/4}$$

- S is geometric
- S is proportional to the *area*, not to the volume as a naive microscopic interpretation of entropy would suggest (logarithmic counting of possible states).
- This dimensional reduction has led to the *holographic principle* by t'Hooft, Susskind, ...
- The horizon is not a physical boundary, but a submanifold where coordinates pick critical values $\rightarrow$ *conformal symmetries*
- The proportionality factor $1/4$ is fixed by Hawking temperature (*quantum* effect).

Black hole entropy

Discretization of the horizon (Bekenstein): horizon is made of cells or area ℓ^2 and k degrees of freedom ($\ell =$ Planck length):

$$A = n\ell^2,$$
$$\text{Degrees of freedom} = k^n,$$
$$S = Cn \log k = C\frac{A}{\ell^2}\log k,$$
$$\mathrm{d}S = C \log k$$

Conclusion.

Black hole entropy

$\downarrow$

Two-dimensional conformal quantum field theory

with a "fuzzy" point of view

Legenda: Fuzzy = *noncommutative geometrical*

Symmetries in Physics

Spacetime symmetries
Lorentz, Poincaré,...

Internal symmetries
Gauge, ...

SUSY
Bose-Fermi

SUSY: $H = Q^2$, Q odd operator, $[\cdot, Q]$ graded super-derivation interchanging Boson and Fermions

Among consequences: Cancellation of some Higgs boson divergence

Conformal and superconformal

- Low dimension, conformal $\rightarrow$ infinite dim. symmetry
- Low dimension, conformal $+$ SUSY $\rightarrow$ Superconformal symmetry (very stringent)

About three approaches to CFT

$$\begin{tikzcd}
\substack{\text{Vertex Algebras} \\ \textit{(algebraic)}} \arrow[rr, leftrightarrow, "\textit{Kac}"] \arrow[dr, leftrightarrow, "\textit{Carpi}-\textit{Weiner}"'] & & \substack{\text{Wightman fields} \\ \textit{(analytic)}} \arrow[dl, leftrightarrow, "\textit{Fredenhagen}-\textit{Jorss}"] \\
& \substack{\text{Operator Algebras} \\ \textit{(algebraic \& analytic)}} &
\end{tikzcd}$$

partial relations known

von Neumann algebras

$\mathcal{H}$ Hilbert space, $B(\mathcal{H})$ *algebra of all bounded linear operators on $\mathcal{H}$.

Def. A *von Neumann algebra* M is a weakly closed non-degenerate *-subalgebra of $B(\mathcal{H})$.

• von Neumann density thm. $\mathfrak{A} \subset B(\mathcal{H})$ non-degenerate *-subalgebra

$$\mathfrak{A}^- = \mathfrak{A}''$$

where $'$ denotes the commutant

$$\mathfrak{A}' = \{T \in B(\mathcal{H}) : TA = AT \ \forall A \in \mathfrak{A}\}$$

Double aspect, analytical and algebraic

M is a factor if its center $M \cap M' = \mathbb{C}$.

The tensor category End(M)

M an infinite factor $\rightarrow$ End(M) is a *tensor C^*-category*:

- *Objects*: End(M)
- *Arrows*: $\mathrm{Hom}(\rho, \rho') \equiv \{t \in M : t\rho(x) = \rho'(x)t \;\forall x \in M\}$
- *Tensor product of objects*: $\rho \otimes \rho' = \rho\rho'$
- *Tensor product of arrows*: $\sigma, \sigma' \in \mathrm{End}(M)$, $t \in \mathrm{Hom}(\rho, \rho')$, $s \in \mathrm{Hom}(\sigma, \sigma')$,

$$t \otimes s \equiv t\rho(s) = \rho'(s)t \in \mathrm{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma')$$

- *Conjugation*: $\exists$ isometries $v \in \mathrm{Hom}(\iota, \rho\bar{\rho})$ and $\bar{v} \in \mathrm{Hom}(\iota, \bar{\rho}\rho)$ such that

$$(\bar{v}^* \otimes 1_{\bar{\rho}}) \cdot (1_{\bar{\rho}} \otimes v) \equiv \bar{v}^* \bar{\rho}(v) = \frac{1}{d}$$
$$(v^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \bar{v}) \equiv v^* \rho(\bar{v}) = \frac{1}{d}$$

for some $d > 0$.

Dimension

The *minimal* d is the *dimension* $d(\rho)$

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definition of the Jones index)

$$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$$
$$d(\rho_1 \rho_2) = d(\rho_1) d(\rho_2)$$
$$d(\bar{\rho}) = d(\rho)$$

End(M) is a "universal" tensor category (cf. Popa, Yamagami)

(generalising the Doplicher-Haag-Roberts theory)

Local conformal nets

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \; = \; \mathcal{A}(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

First consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I)=B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I),\Omega)$, are

$$U(\Lambda_I(2\pi t))=\Delta_I^{it},\ t\in\mathbb{R}, \qquad \text{dilations}$$
$$U(r_I)=J_I \qquad \text{reflection}$$

  (Frölich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)'=\mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I\subset\cup_i I_i \implies \mathcal{A}(I)\subset\vee_i\mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Local conformal nets

$\mathrm{Diff}(S^1) \equiv$ group of orientation-preserving smooth diffeomorphisms of S^1

$\mathrm{Diff}_I(S^1) \equiv \{g \in \mathrm{Diff}(S^1) : g(t) = t \ \forall t \in I'\}$.

A local conformal net $\mathcal{A}$ is a Möbius covariant net s.t.

F. *Conformal covariance*. $\exists$ a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb s.t.

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}_{I'}(S^1), \end{aligned}$$

$\longrightarrow$ unitary representation of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$[L_n, c] = 0,\ L_n^* = L_{-n}$.

Representations

A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$I \in \mathcal{I} \mapsto \pi_I, \text{ normal rep. of } \mathcal{A}(I) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}$$

π is automatically diffeomorphism *covariant*: $\exists$ a projective, pos. energy, unitary rep. U_π of $\mathrm{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$, $g \in \mathrm{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

DHR argument: given I, there is an endomorphism of $\mathcal{A}$ localized in I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$.

- $\mathrm{Rep}(\mathcal{A})$ is a *braided tensor category* (DHR, FRS, L.)

Index-statistics theorem

$$\text{DHR dimension } d(\rho) = \sqrt{\text{Jones index } \mathrm{Ind}(\rho)}$$

$$\text{tensor category } \mathrm{Rep}_I(\mathcal{A}) \xrightarrow[\text{restriction}]{\text{full functor}} \text{tensor category } \mathrm{End}(\mathcal{A}(I))$$

Black hole incremental free energy

Define the *incremental free energy* $F(\varphi_\sigma|\varphi_\rho)$ between the thermal states φ_σ and φ_ρ in reps ρ, σ localized in I (β^{-1} = Hawking temperature)

$$F(\varphi_\sigma|\varphi_\rho) = \varphi_\rho(H_\rho) - \beta^{-1}S(\varphi_\sigma|\varphi_\rho)$$

$S(\varphi_\sigma|\varphi_\rho) = -(\log \Delta_{\xi_\sigma,\xi_\rho}\xi_\rho, \xi_\rho)$ is Araki relative entropy

Then

$$\begin{aligned} F(\varphi_\sigma|\varphi_\rho) =& \frac{1}{2}\beta^{-1}\big(d(\sigma) - d(\rho)\big) \\ =& \frac{1}{2}\beta^{-1}(\log m - \log n) \end{aligned}$$

If the charges ρ, σ come from a spacetime of dimension $d \geq 2+1$ then n, m integers by DHR restriction on the values $d(\rho)$, $d(\sigma)$.

Complete rationality

I_1, I_2 intervals $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, $E \equiv I_1 \cup I_2$.

$$\mu\textit{-index}: \quad \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')' : \mathcal{A}(E)]$$

(Jones index). $\mathcal{A}$ conformal:

$$\mathcal{A} \textit{ completely rational} \stackrel{\text{def}}{=} \mathcal{A} \text{ split } \& \ \mu_{\mathcal{A}} < \infty$$

Thm. (Y. Kawahigashi, M. Müger, R.L.) $\mathcal{A}$ completely rational: then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum over all irreducible sectors. (F. Xu in $SU(N)$ models);

• $\mathcal{A}(E) \subset \mathcal{A}(E')' \sim$ LR inclusion (quantum double);

• Representations form a modular tensor category (i.e. non-degenerate braiding).

Weyl's theorem

M compact oriented Riemann manifold, Δ Laplace operator on $L^2(M)$.

Theorem (Weyl)

Heat kernel expansion as $t \to 0^+$:

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots)}$$

The *spectral invariants* n and $a_0, a_1, \ldots$ encode geometric information and in particular

$$a_0 = \mathrm{vol}(M), \qquad a_1 = \frac{1}{6}\int_M \kappa(m) d\mathrm{vol}(m),$$

κ scalar curvature. $n = 2$: a_1 is proportional to the Euler characteristic $= \frac{1}{2\pi}\int_M \kappa(m) d\mathrm{vol}(m)$ by Gauss-Bonnet theorem.

Modularity

With ρ rep. of $\mathcal{A}$, set $L_{0,\rho}$ conf. Hamiltonian of ρ,

$$\chi_\rho(\tau) = \mathrm{Tr}\left(e^{2\pi i\tau(L_{0,\rho}-c/24)}\right) \quad \mathrm{Im}\,\tau > 0.$$

specialized character, c the central charge.
$\mathcal{A}$ is *modular* if $\mu_{\mathcal{A}} < \infty$ and

$$\chi_\rho(-1/\tau) = \sum_\nu S_{\rho,\nu}\chi_\nu(\tau),$$
$$\chi_\rho(\tau+1) = \sum_\nu T_{\rho,\nu}\chi_\nu(\tau).$$

with S, T the (algebraically defined) Kac-Peterson, Verlinde Rehren matrices generating a representation of $SL(2,\mathbb{Z})$. One has:

- Modularity $\implies$ complete rationality
- Modularity holds in all computed rational case, e.g. $SU(N)_k$-models
- $\mathcal{A}$ modular, $\mathcal{B} \supset \mathcal{A}$ irreducible extension $\implies$ $\mathcal{B}$ modular.
- All conformal nets with central charge $c < 1$ are modular.

Asymptotics

$\mathcal{A}$ modular. The following asymptotic formula holds as $t \to 0^+$:

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{\pi c}{12}\frac{1}{t} - \frac{1}{2}\log \mu_{\mathcal{A}} - \frac{\pi c}{12} t$$

In any representation ρ, as $t \to 0^+$:

$$\boxed{\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12}\frac{1}{t} + \frac{1}{2}\log \frac{d(\rho)^2}{\mu_{\mathcal{A}}} - \frac{\pi c}{12} t}$$

Modular nets as NC manifolds (∞ degrees of freedom)

2-dim. cpt manifold M	conformal net $\mathcal{A}$
$\mathrm{supp}(f) \subset I$	$x \in \mathcal{A}(I)$
Laplacian Δ	conf. Hamiltonian L_0
Δ elliptic	L_0 log-elliptic
area $\mathrm{vol}(M)$	NC area $a_0(2\pi L_0)$
Euler charact. $\chi(M)$	NC Euler char. $12a_1$

Entropy. Physics and geometric viewpoints:

Inv.	*Value*	*Geometry*	*Physics*
a_0	$\pi c/12$	NC area	Entropy
a_1	$-\frac{1}{2}\log \mu_{\mathcal{A}}$	NC Euler charact.	1^{st} order entr.
a_2	$-\pi c/12$	2^{nd} spectral invariant	2^{nd} order entr.

Rem. Physical literature: proposals for $2\pi c/12 = A/4$.

Question: What can we say for SUSY? (Dirac operator case)

Quantum calculus with infinitely many degrees of freedom

CLASSICAL	Classical variables Differential forms Chern classes	Variational calculus Infinite dimensional manifolds Functions spaces Wiener measure
QUANTUM	Quantum geometry Fredholm operators Index Cyclic cohomology	Subfactors Correspondences, Endomorphisms Multiplicative index Supersymmetric QFT, $(\mathfrak{A}, \mathcal{H}, Q)$

McKean-Singer formula

Γ be a selfadjoint unitary on a Hilbert space $\mathcal{H}$, thus $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is graded.

Q selfadjoint *odd* operator: $\Gamma Q \Gamma^{-1} = -Q$ or

$$Q = \begin{bmatrix} 0 & Q_- \\ Q_+ & 0 \end{bmatrix}$$

$\mathrm{Tr}_\mathsf{s} = \mathrm{Tr}(\Gamma\,\cdot)$ the *supertrace*.

If e^{-tQ^2} is trace class then $\mathrm{Tr}_\mathsf{s}(e^{-tQ^2})$ is an integer independent of t:

$$\mathrm{Tr}_\mathsf{s}(e^{-tQ^2}) = \mathrm{ind}(Q_+) \quad \forall t > 0$$

$\mathrm{ind}(Q_+) \equiv \mathrm{Dim}\ker(Q_+) - \mathrm{Dim}\ker(Q_+^*)$ is the Fredholm index of Q_+.

Fermi conformal nets

$\mathcal{A}$ is a Fermi net if locality is replaced by twisted locality:
$\exists$ self-adjoint unitary Γ, $\Gamma\Omega = \Omega$, $\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$; if $I_1 \cap I_2 = \varnothing$

$$[x, y] = 0, \quad x \in \mathcal{A}(I_1),\ y \in \mathcal{A}(I_2)\ .$$

$[x, y]$ is the graded commutator w.r.t. $\gamma = \mathrm{Ad}\Gamma$. Then

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*$$

indeed $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$ *twisted duality* $\left(\text{where } Z \equiv \frac{1-i\Gamma}{1-i}\right)$
The *Bose subnet* $\mathcal{A}_b \equiv \mathcal{A}^\gamma$ of is local.
Spin-statistics:

$$U(2\pi) = \Gamma\ .$$

Therefore, in the Fermi case, U is representation of $\mathrm{Diff}^{(2)}(S^1)$.

Nets on a cover of S^1

A conformal net $\mathcal{A}$ on $S^{1(n)}$ is a isotone map

$$I \in \mathcal{I}^{(n)} \mapsto \mathcal{A}(I) \subset B(\mathcal{H})$$

with a projective unitary, positive energy representation U of $\mathrm{Diff}^{(\infty)}(S^1)$ on $\mathcal{H}$ with

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(\dot{g}I), \quad I \in \mathcal{I}^{(n)},\ g \in \mathrm{Diff}^{(\infty)}(S^1)$$

conformal net $\mathcal{A}$ on $S^1 \xrightarrow{\text{promotion}}$ conformal net $\mathcal{A}^{(n)}$ on $S^{1(n)}$

Representations of a Fermi net

Let $\mathcal{A}$ be a Fermi net on S^1. A general representation λ of $\mathcal{A}$ is a representation the cover net of $\mathcal{A}^{(\infty)}$ such that $\lambda_b \equiv \lambda|_{\mathcal{A}_b}$ is a DHR representation $\mathcal{A}_b$.

λ is indeed a representation of $\mathcal{A}^{(2)}$. The following alternative holds:

(*a*) λ is a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is not a scalar.

(*b*) λ is the restriction of a representation of $\mathcal{A}^{(2)}$ and λ is not a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is a scalar.

Case (*a*): Neveu-Schwarz representation
Case (*b*): Ramond representation

Super-Virasoro algebra

The super-Virasoro algebra governs the superconformal invariance:

local conformal $\leftrightarrow$ Virasoro

superconformal $\leftrightarrow$ super-Virasoro

Two super-Virasoro algebras: They are the super-Lie algebras generated by L_n, $n \in \mathbb{Z}$ (even), G_r (odd), and c (central):

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$$

$$[G_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

Neveu-Schwarz case: $r \in \mathbb{Z} + 1/2$, Ramond case: $r \in \mathbb{Z}$.

Note: $G_0^2 = 2L_0 - c/12$ in Ramond sectors

FQS: admissible values for central charge c and lowest weight h

Either $c \geq 3/2$, $h \geq 0$ ($h \geq c/24$ in the Ramond case) or

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right), \ m = 2, 3, \ldots$$

and

$$h = h_{p,q}(c) \equiv \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{\varepsilon}{8}$$

where $p = 1, 2, \ldots, m-1$, $q = 1, 2, \ldots, m+1$ and $p - q$ is even or odd corresponding to the Neveu-Schwarz case ($\varepsilon = 0$) or Ramond case ($\varepsilon = 1/2$).

Neveu-Schwarz algebra has a vacuum representation, the Ramond algebra has *no vacuum representation*.

Super-Virasoro nets

c an admissible value, $h = 0$. Bose and Fermi stress-energy tensors:

$$T_B(z) = \sum_n z^{-n-2} L_n, \qquad T_F(z) = \frac{1}{2} \sum_r z^{-r-3/2} G_r$$

in any NS/Ramond rep. same commutation relations ($w \equiv z_2/z_1$):

$$[T_F(z_1), T_F(z_2)] = \frac{1}{2} z_1^{-1} T_F(z_1)\delta(w) + z_1^{-3} w^{-\frac{3}{2}} \frac{c}{12} \big(w^2 \delta''(w) + \frac{3}{4}\delta(w)\big)$$

In the NS vacuum define the *Super-Virasoro net* of vN algebras:

$$SVir(I) \equiv \{e^{iT_B(f_1)}, e^{iT_F(f_2)} : f_1, f_2 \in C^\infty(S^1) \text{ real}, \text{supp} f_1, \text{supp} f_2 \subset I\}''$$

Neveu-Schwarz rep. of *SVir* net $\longleftrightarrow$ rep. of Neveu-Schwarz algebra
Ramond rep. of *SVir* net $\longleftrightarrow$ rep. of Ramond algebra

- $SVir_b$ is modular (F. Xu)
- $SVir_b = \big(SU(2)_{N+2}\big)' \cap \big(SU(2)_2 \otimes SU(2)_N\big)$ (GKO)

Supersymmetric representations

A general representation λ of the Fermi conformal net $\mathcal{A}$ is *supersymmetric* if λ is graded

$$\lambda(\gamma(x)) = \Gamma_\lambda \lambda(x) \Gamma_\lambda^*$$

and the conformal Hamiltonian H_λ satisfies

$$\tilde{H}_\lambda \equiv H_\lambda - c/24 = Q_\lambda^2$$

where Q_λ is a selfadjoint odd w.r.t. Γ_λ.

Then

$$H_\lambda \geq c/24$$

McKean-Singer lemma:

$$\mathrm{Tr}_s(e^{-t(H_\lambda - c/24)}) = \dim\ker(H_\lambda - c/24)\ ,$$

the multiplicity of the lowest eigenvalue $c/24$ of H_λ.

Super-Virasoro net:

λ *supersymmetric* $\Rightarrow \lambda$ *Ramond* (irr. iff h= c/24 i.e. minimal)

SUSY, Fredholm and Jones index

Assume $\mathcal{A}_b$ modular $\lambda|_{\mathcal{A}_b} = \rho \oplus \rho'$.

$$\mathrm{Tr_s}(e^{-2\pi t\tilde{H}_\lambda}) = 2 \sum_{\nu\,\mathrm{Ramond}} S_{\rho,\nu}\,\mathrm{Tr}(e^{-2\pi\tilde{L}_{0,\nu}/t}) \ .$$

If λ is supersymmetric then

$$\mathrm{ind}(Q_{\lambda+}) = 2 \sum_{\nu\,\mathrm{Ramond}} S_{\rho,\nu}\mathrm{null}(\nu, c/24)$$

Therefore, writing Rehren definition of the S matrix, we have

$$\mathrm{ind}(Q_{\lambda+}) = \frac{d(\rho)}{\sqrt{\mu_{\mathcal{A}}}} \sum_{\nu\,\mathrm{Ramond}} K(\rho,\nu)d(\nu)\mathrm{null}(\nu, c/24)$$

The *Fredholm index* of the supercharge operator $Q_{\lambda+}$ and the *Jones index* both appear

Some consequences

- An identity for the S matrix:

$$\sum_{\nu \text{ Ramond}} S_{\rho,\nu} d(\nu) = 0$$

- If $\mathrm{ind}(Q_{\lambda+}) \neq 0$ there exists a Ramond sector ν such that $c/24$ is an eigenvalue of $L_{0,\nu}$.
- Suppose that ρ is the only Ramond sector with lowest eigenvalue $c/24$ modulo integers. Then

$$S_{\rho,\rho} = \frac{d(\rho)^2}{\sqrt{\mu_{\mathcal{A}_b}}} K(\rho,\rho) = \frac{1}{2} \ .$$

Classification (S. Carpi, Y. Kawahigashi, R. L.)

Complete list of superconformal nets, i.e. Fermi extensions of the super-Virasoro net, with $c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$

1. The super Virasoro net: (A_{m-1}, A_{m+1}).
2. Index 2 extensions of the above: $(A_{4m'-1}, D_{2m'+2})$, $m = 4m'$ and $(D_{2m'+2}, A_{4m'+3})$, $m = 4m' + 2$.
3. Six exceptionals: (A_9, E_6), (E_6, A_{13}), (A_{27}, E_8), (E_8, A_{31}), (D_6, E_6), (E_6, D_8).

Remark. Follows the classification of local conformal nets with $c < 1$ with the construction of new models (Kawahigashi, L., also F. Xu and K.H. Rehren)

Relation with Connes Noncommutative Geometry

Def. A (θ-summable) graded spectral triple $(\mathfrak{A}, \mathcal{H}, Q)$ consists of a graded Hilbert space $\mathcal{H}$, with selfadjoint grading unitary Γ, a unital *-algebra $\mathfrak{A} \subset B(\mathcal{H})$ graded by $\gamma \equiv \mathrm{Ad}(\Gamma)$, and an odd selfadjoint operator Q on $\mathcal{H}$ as follows:

- $\mathfrak{A}$ is contained in $D(\delta)$, the domain of the superderivation $\delta = [Q, \cdot]$;
- For every $\beta > 0$, $\mathrm{Tr}(e^{-\beta Q^2}) < \infty$ (θ-summability).

The operator Q is called the *supercharge* operator, its square the *Hamiltonian*. Q is also called *Dirac operator* and denoted by D.

A spectral triple is a fundamental object to define NCG.

Jaffe-Lesniewski-Osterwalder cocycle

Assume we have a quantum algebra (essentially a spectral triple)
Then the JLO cocycle (Chern character) on the Bose algebra

$$\tau_n(a_0, a_1, \ldots, a_n) \equiv$$

$$(-1)^{-\frac{n}{2}} \int_{0 \le t_1 \le \cdots \le t_n \le 1} \mathrm{Tr}_s\big(e^{-H} a_0 \alpha_{it_1}(\delta a_1) \alpha_{it_2}(\delta a_2) \ldots \alpha_{it_n}(\delta a_n)\big) \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

(n even) is an entire cyclic coclycle, so it gives an element in Connes entire cyclic cohomology that pairs with K-theory.

Spectral triples in CFT

A supersymmetric representation ρ of a Fermi net $\mathcal{A}$ gives rise to a θ-summable spectral triple if the superderivation δ

$$\delta(a) \equiv [a, Q_\rho]$$

has a *dense domain* in the representation ρ (θ-summability is essentially automatic)
Then the JLO cocycle (Chern character) on the Bose algebra

$$\tau_n^\rho(a_0, a_1, \ldots, a_n) \equiv$$
$$(-1)^{-\frac{n}{2}} \int_{0 \le t_1 \le \cdots \le t_n \le 1} \mathrm{Tr}_s\big(e^{-H_\rho} a_0 \alpha_{it_1}(\delta a_1) \alpha_{it_2}(\delta a_2) \ldots \alpha_{it_n}(\delta a_n)\big) \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

(n even) is entire cyclic coclycle

Noncommutative geometrization

We want to associate to each supersymmetric sector the above Chern character

$$\rho \to \tau^{\rho}$$

Thm. (Carpi, Hillier, Kawahigashi, R.L.)

The supersymmetric Ramond sectors of SVir give rise to θ-summable spectral triple (δ has a dense domain)

For the super-Virasoro net the index map

$$\rho \to \sum \tau^{\rho}_{n}(1, 1, \dots, 1) = \mathrm{Tr_s}(e^{-tH_\rho})$$

for Ramond sectors is given by

$$\mathrm{Index}(\rho_{h=c/24}) = 1, \qquad \mathrm{Index}(\rho_{h\neq c/24}) = 0$$

Further model analysis (in progress)

- Free supersymmetric CFT on the circle: all Buchholz-Mack-Todorov sectors give the same JLO cocycle (deformation argument). Even JLO cocycle is trivial, odd JLO cocycle probably non-trivial
- Extension of $U(1)_2 \otimes (\mathrm{Fermions})$ gives a non-trivial JLO cocycle (there is a unitary that does not distinguish sectors)
- Super-Virasoro net: Ramond lowest energy sector is non-trivial (see above), other Ramond sectors give trivial JLO cocycles (NS case is not interesting).
- Richer structure is expected in the $N = 2$ superconformal case