product of real spectral triples

G. Dossena
(joint work with L. Dąbrowski)

SISSA, Trieste

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introduction

Riemannian spin geometries on compact manifolds are encoded by *commutative* real spectral triples [Connes, arxiv 2008].

Riemannian spin geometry		real spectral triples
Cartesian products	$\Longleftrightarrow$	?

additional (more physical) motivation: products of real spectral triples are used in the construction of the noncommutative standard model of particle physics [Connes-Lott, Chamseddine-Connes, etc.].

definition of a spectral triple $(A, \mathcal{H}, D)$

1. A is a real or complex associative unital $*$-algebra
2. $\rho\colon A \to \mathcal{B}(\mathcal{H})$ is a faithful unital $*$-rep on the C^*-algebra of bounded linear ops $\mathcal{B}(\mathcal{H})$ on some separable $\mathbb{C}$-Hilbert space $\mathcal{H}$;
3. D (Dirac op) is a densely defined self-adjoint op:

$$D\colon \mathrm{dom} D \subset \mathcal{H} \to \mathcal{H} \tag{1}$$

   with compact resolvent (i.e. for $\lambda \notin \mathrm{sp}(D)$ the operator $(D - \lambda I)^{-1}\colon \mathcal{H} \to \mathrm{dom} D$ is compact) and such that
   - $\rho(a)\, \mathrm{dom} D \subset \mathrm{dom} D$ for each $a \in A$ and
   - $[D, \rho(a)]$ is a (a priori densely defined) bounded op for each $a \in A$;

4. J (charge conjugation) is an antiunitary on $\mathcal{H}$ [i.e. J is antilinear bijective and $(Jv|Jw) = (w|v)$] such that:

$$J^2 = \pm I \text{ and } J_{\pm} D = \pm D J_{\pm} \quad ; \tag{2}$$

moreover we request:

$$[\rho(a), J\rho(b^*)J^{-1}] = 0 \tag{3}$$

for each $a, b \in A$;

remarks

- note that the map $b \mapsto J\rho(b^*)J^{-1}$ is a unital $*$-rep of the opposite algebra A° on $\mathcal{B}(\mathcal{H})$;
- a spectral triple with J is called real.

5. [optional] $\chi \in \mathcal{B}(\mathcal{H})$ (chirality) is a self-adjoint unitary such that:

$$\begin{aligned} \chi D &= -D\chi \\ [\chi, \rho(a)] &= 0 \quad \forall a \in A \quad ; \end{aligned} \tag{4}$$

remarks

- χ implements the $\mathbb{Z}/2$-grading of $\mathcal{H}$;
- a spectral triple with χ is called even.

$$J^2 = \epsilon 1, \quad JD = \epsilon' DJ, \quad J\chi = \epsilon'' \chi J \tag{5}$$

Table: Case J_+

$n \mod 8$	0	1	2	3	4	5	6	7
ϵ	$+$		$-$	$-$	$-$		$+$	$+$
ϵ'	$+$		$+$	$+$	$+$		$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

Table: Case J_-

$n \mod 8$	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$+$		$-$	$-$	$-$	
ϵ'	$-$	$-$	$-$		$-$	$-$	$-$	
ϵ''	$+$		$-$		$+$		$-$	

remarks

- tables are constructed in accordance with the commutative case (mod 8 $\leftrightarrow$ Clifford periodicity)
- integers mod 8 define the so-called KO-dimension of the triple, a priori different from the metric dimension; recall a triple has metric dimension n if the compact op

$$|D|^{-n} \colon \mathcal{H}/\ker D \to \mathcal{H} \tag{6}$$

is a first order infinitesimal, i.e. the eigenvalues

$$\lambda_0 \leq \lambda_1 \leq \dots \tag{7}$$

behave asymptotically as $\lambda_m = O(1/m)$ and $\sum_{m<N} \lambda_m = O(logN))$; for simplicity we will consider these two dimensions as equal.

products

given two real spectral triples $(A_i, D_i, \mathcal{H}_i, J_i, (\chi_i))_{i=1,2}$, the product triple has algebra:

$$A := A_1 \otimes A_2 \tag{8}$$

where

$$\otimes = \begin{cases} \otimes_{\mathbb{R}} & \text{at least one of the algebras is real} \\ \otimes_{\mathbb{C}} & \text{both algebras are complex} \end{cases} \tag{9}$$

products and adjoints are component-wise:

$$\begin{gathered} (a_1 \otimes a_2)(b_1 \otimes b_2) := (a_1 b_1) \otimes (a_2 b_2) \quad , \\ (a_1 \otimes a_2)^* := (a_1^*) \otimes (a_2^*) \quad . \end{gathered} \tag{10}$$

all other ingredients depend on the parity of the dimensions involved, as follows.

even-even case

$$\begin{aligned} \mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \rho &:= \rho_1 \otimes \rho_2 \colon A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ D &:= D_1 \otimes I + \chi_1 \otimes D_2 \\ \widetilde{D} &:= D_1 \otimes \chi_2 + I \otimes D_2 \text{ (two choices)} \\ J &:= J_1 \otimes J_2 \\ \chi &:= \chi_1 \otimes \chi_2 \end{aligned} \tag{11}$$

remarks

1. $D^2 = D_1^2 \otimes I + I \otimes D_2^2 \implies$ metric dimensions add;
2. all properties of a real spectral triple are preserved; some caution for the domain of self-adjointness of D (or $\widetilde{D}$): this is given by a suitable closure of the dense domain $\mathrm{dom} D_1 \otimes \mathrm{dom} D_2$.

Table: D

1 \ 2	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+	2_+	4_+	6_+				
2_+					2_+	4_+	6_+	0_+
4_+	4_+	6_+	0_+	2_+				
6_+					6_+	0_+	2_+	4_+
0_-					0_-	2_-	4_-	6_-
2_-	2_-	4_-	6_-	0_-				
4_-					4_-	6_-	0_-	2_-
6_-	6_-	0_-	2_-	4_-				

remark

the two top blocks correspond to the even-even cases considered in [Vanhecke, 1999].

Table: $\widetilde{D}$

2 \ 1	0_+	2_+	4_+	6_+	0_-	2_-	4_-	6_-
0_+	0_+		4_+			2_-		6_-
2_+	2_+		6_+			4_-		0_-
4_+	4_+		0_+			6_-		2_-
6_+	6_+		2_+			0_-		4_-
0_-		2_+		6_+	0_-		4_-	
2_-		4_+		0_+	2_-		6_-	
4_-		6_+		2_+	4_-		0_-	
6_-		0_+		4_+	6_-		2_-	

even-odd case

$$
\begin{aligned}
\mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2 \\
\rho &:= \rho_1 \otimes \rho_2 \colon A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\
D &:= D_1 \otimes I + \chi_1 \otimes D_2 \quad \text{or} \\
\widetilde{D} &:= D_1 \otimes \chi_2 + I \otimes D_2 \\
J &:= J_1 \otimes J_2
\end{aligned}
\tag{12}
$$

we get analogous tables as before; half of the cases were considered in [Vanhecke, 1999].

odd-odd case

$$
\begin{aligned}
A &:= A_1 \otimes A_2 \\
\mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2 \\
\rho &:= \rho_1 \otimes \rho_2 \otimes I \\
D &:= D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2 \\
J_\pm &:= J_1 \otimes J_2 \otimes M_\pm K \\
\chi &:= 1 \otimes 1 \otimes \sigma_3
\end{aligned}
\tag{13}
$$

where

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}
$$

$M_\pm$ are two complex matrices specified by the next table and K is the complex conjugation operator defined for the canonical basis of $\mathbb{C}^2$.

Table: odd-odd case

1 \ 2	1_-	3_+	5_-	7_+
1_-	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
3_+	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2
5_-	σ_2, σ_1	σ_3, σ_0	σ_2, σ_1	σ_3, σ_0
7_+	σ_0, σ_3	σ_1, σ_2	σ_0, σ_3	σ_1, σ_2

remarks

- every entry contains the pair M_+, M_-; σ_0 is the identity matrix;
- this odd-odd construction still works under any permutation of the Pauli matrices (e.g., one can take $D := D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_3$ and $\chi := 1 \otimes 1 \otimes \sigma_2$).

additional properties & their preservation under products

in Connes' reconstruction theorem additional properties are used to recover the spin manifold; what about their preservation under products?

first order

$$[[D, \rho(a)], J\rho(b^*)J^{-1}] = 0 \tag{15}$$

preserved (proof by computation, using commutation between the reps of A_i and A_i° and, for even-odd or even-even, commutation between χ_i and the rep of A_i).

orientation

$$\exists \text{ Hochschild cycle } c \in Z_n(A, A \otimes A^\circ) \text{ s.t. } \pi_D(c) = \chi \tag{16}$$

where

$$\begin{aligned} \pi_D(a_0 \otimes a_1 \otimes \cdots a_n) &:= R_J(a_0)[D, \rho(a_1)] \cdots [D, \rho(a_n)] \\ R_J(a_0) &:= \rho(a_0') J \rho(a_0''^*) J^{-1}, \quad a_0 = a_0' \otimes a_0'' \end{aligned} \tag{17}$$

preserved; Hochschild cycle on the product is given by

$$\begin{aligned} c &:= \tfrac{1}{r} c_1 \times c_2 \\ r &:= \tfrac{1}{2}(n_1 + n_2 - 1)(n_1 + n_2) \cdot \begin{cases} 1 & \text{when } n_1 n_2 \text{ is even} \\ i & \text{when } n_1 n_2 \text{ is odd} \end{cases} \quad . \end{aligned} \tag{18}$$

where $\times$ is the shuffle product:

$$(a_0^1, a_1^1, \ldots, a_p^1) \times (a_0^2, a_1^2, \ldots, a_q^2) := \sum_\tau (-1)^\tau \tau \cdot (a_0^1 \otimes a_0^2, a_1^1 \otimes 1, \ldots, a_p^1 \otimes 1, 1 \otimes a_1^2, \ldots, 1 \otimes a_q^2) \tag{19}$$

$$\tau \cdot (a_0, a_1 \ldots, a_n) := (a_0, a_{\tau^{-1}(1)}, \ldots, a_{\tau^{-1}(n)}) \tag{20}$$

the sum is over all (p, q)-shuffles, i.e. permutations of $\{1, \ldots, p+q\}$ preserving the order of $\{1, \ldots, p\}$ and $\{p+1, \ldots, p+q\}$ separately.

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examples/outlook

- Chamseddine-Connes model for particle physics
- noncommutative tori
- θ-deformations