A classification of all finite-index subfactors for a class of II_1 factors.

Steven Deprez

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Outline

Basics about subfactors

Consider a II_1 subfactor $P \subset M$.

Definition (Jones, 1983)

The index of a subfactor $P \subset M$ is

$$[M : P] = \dim_P(L^2(M))$$

Examples

subfactor	index
$\mathcal{L}(H) \subset \mathcal{L}(G)$	$[G : H]$
$L^\infty(X,\mu) \rtimes H \subset L^\infty(X,\mu) \rtimes G$	$[G : H]$
$L^\infty(X,\mu) \rtimes G \xhookrightarrow{u_g \mapsto \pi(g) \otimes u_g} M_n(\mathbb{C}) \otimes L^\infty(X,\mu) \rtimes G$	n^2
$L^\infty(X,\mu) \rtimes G \xhookrightarrow{f \mapsto 1 \otimes f} L^\infty(I \times X, c \times \mu) \rtimes G$	$\lvert I \rvert$

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The invariants $\mathcal{I}$ and $\mathcal{C}$

Definition (Jones, 1983)

For a II_1 factor M, we have the following invariants.

$$\mathcal{I}(M) = \{[M:P] \mid P \subset M \text{ a subfactor}, [M:P] < \infty\}$$
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Theorem (Jones, 1983)

$$\mathcal{C}(M) \subset \mathcal{I}(M) \subset \mathcal{I} = \{4\cos^2(\pi/n) \mid n = 3, 4, \ldots\} \sqcup [4, \infty[$$

$$\mathcal{I}(R) = \mathcal{I} \text{ (Jones, 1983)}, \mathcal{C}(R): \text{ open question (Connes)}$$
$$\mathcal{C}(\mathcal{L}(\mathbb{F}_\infty)) = \mathcal{I} \text{ (Radulescu, 1994, Shlyakhtenko-Ueda, 2002)}$$
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Good actions of good groups

Definition (Vaes, 2008)

An action of a countable group Γ on a countable set I is a good action of a good group if

- T Γ admits an infinite (almost) normal subgroup with relative property (T)
- C1 $\text{Stab}(i) \cdot j$ is infinite if $i \neq j$.
- C2 there is no sequence $i_1, i_2, \ldots \in I$ with $\text{Stab}\{i_1, \ldots, i_n\}$ stictly decreasing with n.
- C3 For all $g \in G$, $\text{Fix}(g) \subset I$ has inifinite index.

Examples

(C2) there is no sequence $i_1, i_2, \ldots \in I$ with $\mathrm{Stab}\{i_1, \ldots, i_n\}$ stictly decreasing with n.

Examples (of actions with C2)

- $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n \curvearrowright \mathbb{Z}^n$, $\mathrm{SL}_n(\mathbb{Q}) \ltimes \mathbb{Q}^n \curvearrowright \mathbb{Q}^n$, $n \geq 2$
- $\mathrm{PSL}_{n+1}(\mathbb{Z}) \curvearrowright \mathbb{P}_n(\mathbb{Q})$, $\mathrm{PSL}_{n+1}(\mathbb{Q}) \curvearrowright \mathbb{P}_n(\mathbb{Q})$, $n \geq 2$
- Left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$ with minimal condition on centralizers (linear groups, word-hyperbolic groups, $C'(1/6)$-small cancellation groups).
- $G \times H \curvearrowright I \times J$ if $G \curvearrowright I$ and $H \curvearrowright J$ satisfy C2.
- $\Gamma_0 \curvearrowright I_0$ if $\Gamma_0 \subset \Gamma$, $I_0 \subset I$ and $\Gamma \curvearrowright I$ satisfies C2.

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Generalized Bernoulli actions

Definition (Generalized Bernoulli action)

Given $\Gamma \curvearrowright I$ and (X_0, μ_0), define $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$ by $(g \cdot x)(i) = x(g^{-1} \cdot i)$.

Notation

For a good action of a good group $\Gamma \curvearrowright I$, and a base space (X_0, μ_0), set

$$M(\Gamma \curvearrowright I) = \mathrm{L}^\infty\left((X_0, \mu_0)^I\right) \rtimes \Gamma$$

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The theorem

Theorem (D.-Vaes, 2009)

Let $\Gamma \curvearrowright I$ be a good action of a good group. Then every irreducible finite-index subfactor $P \subset M(\Gamma \curvearrowright I)$ is of the form

$$P^{nm} \cong \mathrm{L}^\infty(X,\mu) \rtimes_\Omega G \qquad \left((X,\mu) = (X_0,\mu_0)^I\right)$$

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$$\cong (\mathrm{L}^\infty(X,\mu) \rtimes_\Omega G_0)^m$$

$$\underset{\pi}{\hookrightarrow} (M_n(\mathbb{C}) \otimes \mathrm{L}^\infty(X,\mu) \rtimes G_0)^m \qquad (\pi(g)\pi(h) = \Omega(g,h)\pi(gh))$$

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P^{nm} &\cong \mathrm{L}^\infty(X,\mu) \rtimes_\Omega G && \Big((X,\mu) = (X_0,\mu_0)^I\Big) \\
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In particular: $[M(\Gamma \curvearrowright I) : P] = [G : G_0] \dim(\pi)^2 [\Gamma : G_0] \in \mathbb{N}$.

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We use a purely atomic base space (X_0, μ_0) with unequal weights.

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Theorem (Vaes, 2008)

Let $\Gamma \curvearrowright I$ and $G \curvearrowright J$ be good actions of good groups and set $M = M(\Gamma \curvearrowright I)$, $N = M(G \curvearrowright J)$.

Let ${}_M\mathrm{H}_N$ be an irreducible bimodule, then H is an elementary bimodule: a connes tensor product of

- *a restriction of M to $\mathrm{L}^\infty(X, \mu) \rtimes \Gamma_0$.*
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From bimodules to subfactors

$P \subset M = M(\Gamma \curvearrowright I)$ a subfactor with index t.

Consider ${}_M\mathrm{L}^2(\mathrm{M}_1)_M$.

$\Rightarrow$ a direct sum of elementary bimodules

$\Rightarrow$ describe $M \underset{\psi}{\hookrightarrow} M_2 \cong M^t$ in terms of subgroups, representations and conjugations.

But $\psi(M) \subset M_1 \cong P^t$.

$\Rightarrow$ there is $p \in P^t \cap \psi(\mathrm{L}^\infty(X,\mu))'$ such that $\psi(\mathrm{L}^\infty(X,\mu))p$ is maximal abelian in pP^tp.

Set

$$G = \{\Delta \in \mathrm{comm}_{\mathrm{Aut}(X,\mu)}(\Gamma) \,|\, \exists\, u \in pP^tp : u\psi(f)u^* = \psi(f \circ \Delta)\}$$

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