Inequivalent bundle representations for the Noncommutative Torus

Chern numbers: from "abstract" to "concrete"

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Outline

- The seminal paper [Thouless, Kohmoto, Nightingale & Niji '82] paved the way for the explanation of the Quantum Hall Effect (QHE) in terms of geometric quantities. The model is a 2-dimensional magnetic-Bloch-electron (2DMBE), i.e. an electron in a lattice potential plus a uniform magnetic field. In the limit $B \to 0$, assuming a rational flux M/N and via the Kubo formula the quantization of the Hall conductance is related with the (first) Chern numbers of certain vector bundles related to the energy spectrum of the model. A duality between the limit cases $B \to 0$ and $B \to \infty$ is proposed:

$$N\ C_{B\to\infty} + M\ C_{B\to 0} = 1 \quad \text{(TKNN-formula)}.$$

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- In the papers [Bellissard '87], [Helffer & Sjőstrand '89], [D. & Panati t.b.p.] is rigorously proved that the effective models for the 2DMBE in the limits $B \to 0, \infty$ are elements of two different representations of the (rational) Noncommutative Torus (NCT).

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- Rigorous proof and generalization of the TKNN-formula.

The NCT with deformation parameter $\theta \in \mathbb{R}$ is the "abstract" C^*-algebra $\mathfrak{A}_\theta$ generated by:

$$\mathfrak{u}^* = \mathfrak{u}^{-1}, \qquad \mathfrak{v}^* = \mathfrak{v}^{-1}, \qquad \mathfrak{u}\mathfrak{v} = e^{i2\pi\theta}\mathfrak{v}\mathfrak{u}$$

and closed in the *(universal) norm*:

$$\|\mathfrak{a}\| := \sup\{\|\pi(\mathfrak{a})\|_{\mathscr{H}} \mid \pi \text{ representation of } \mathfrak{A}_\theta \text{ on } \mathscr{H}\}.$$

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The canonical trace $⨍ : \mathfrak{A}_\theta \to \mathbb{C}$ (*unique* if $\theta \notin \mathbb{Q}$) is defined by:

$$⨍ (\mathfrak{u}^n\mathfrak{v}^m) = \delta_{n,0}\delta_{m,0}.$$

It is a *state* (linear, positive, normalized), *faithful* $⨍ (\mathfrak{a}^*\mathfrak{a}) = 0 \Leftrightarrow \mathfrak{a} = 0$ with the *tracial property* $⨍ (\mathfrak{a}\mathfrak{b}) = ⨍ (\mathfrak{b}\mathfrak{a})$.

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The canonical derivations $\partial_j : \mathfrak{A}_\theta \to \mathfrak{A}_\theta$, $j = 1,2$ are defined by

$$\partial_1(\mathfrak{u}^n\mathfrak{v}^m) = in(\mathfrak{u}^n\mathfrak{v}^m), \qquad \partial_2(\mathfrak{u}^n\mathfrak{v}^m) = im(\mathfrak{u}^n\mathfrak{v}^m).$$

Symmetric $\partial_j(\mathfrak{a}^*) = \partial_j(\mathfrak{a})^*$, *commuting* $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$ and $⨍ \circ \partial_j = 0$.

A $\mathfrak{p} \in \mathfrak{A}_\theta$ is a projection if $\mathfrak{p} = \mathfrak{p}^* = \mathfrak{p}^2$. Let $\mathrm{Proj}(\mathfrak{A}_\theta)$ the collection of the projections of $\mathfrak{A}_\theta$.
If $\theta = \frac{M}{N}$ with $M \in \mathbb{Z}$, $N \in \mathbb{N}^*$ and $\mathrm{g.c.d}(M, N) = 1$, then $⨍\colon \mathrm{Proj}(\mathfrak{A}_{M/N}) \to \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$.

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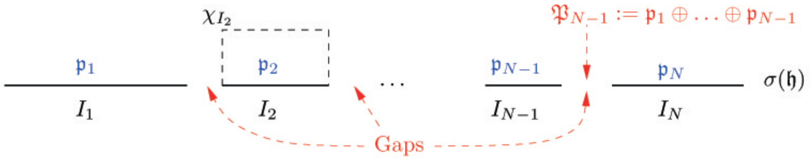
The "abstract" (first) Chern number of $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta)$ is defined by:

$$\mathfrak{Ch}(\mathfrak{p}) := \frac{i}{2\pi}\, ⨍ \left(\mathfrak{p}[\partial_1(\mathfrak{p}); \partial_2(\mathfrak{p})]\right).$$

A selfadjoint $\mathfrak{h} \in \mathfrak{A}_\theta$ has a band spectrum if it is a locally finite union of closed intervals in $\mathbb{R}$, i.e. $\sigma(\mathfrak{h}) = \bigcup_{j\in\mathbb{Z}} I_j$. The open interval which separates two adjacent bands is called *gap*.

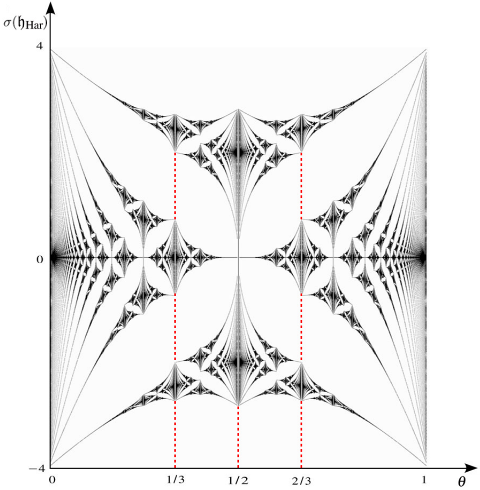


Let χ_{I_j} the characteristic functions for the *spectral band* I_j, then $\chi_{I_j} \in C(\sigma(\mathfrak{h})) \simeq C^*(\mathfrak{h}) \subset \mathfrak{A}_\theta$. One to one correspondence between I_j and *band projection* $\mathfrak{p}_j \in \mathrm{Proj}(\mathfrak{A}_\theta)$. Gap projection $\mathfrak{P}_j := \bigoplus_{k=1}^{j} \mathfrak{p}_j$. If $\theta = M/N$ then $1 \leqslant j \leqslant N$.

An important example of selfadjoint element in $\mathfrak{A}_\theta$ is the Harper Hamiltonian

$$\mathfrak{h}_{\text{Har}} := \mathfrak{u} + \mathfrak{u}^{-1} + \mathfrak{v} + \mathfrak{v}^{-1}.$$

Spectrum of $\mathfrak{h}_{\text{Har}}$ for $\theta \in \mathbb{Q}$ [Hofstadter '76]:



$\Pi_{0,1} : \mathfrak{A}_\theta \to \mathscr{B}(\mathscr{H}_{0,1})$ with $\mathscr{H}_{0,1} := L^2(\mathbb{T}^2)$ defined by:

$$\begin{cases} \Pi_{0,1}(\mathfrak{u}) =: U_{0,1} : \psi_{n,m} \mapsto \mathrm{e}^{i\pi\theta n}\psi_{n,m+1} \\ \Pi_{0,1}(\mathfrak{v}) =: V_{0,1} : \psi_{n,m} \mapsto \mathrm{e}^{-i\pi\theta m}\psi_{n+1,m} \end{cases}$$

$\psi_{n,m}(k_1, k_2) := (2\pi)^{-1}\mathrm{e}^{i(nk_1+mk_2)}$ *Fourier basis* of $\mathscr{H}_{0,1}$.

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It is the GNS representation related to $⨍$, indeed
$\psi_{n,m} \leftrightarrow \mathrm{e}^{i\pi\theta nm}\mathfrak{v}^n\mathfrak{u}^m$, cyclic vector $\psi_{0,0} = (2\pi)^{-1}$,

$$⨍\,(\mathfrak{u}^a\mathfrak{v}^b) = \delta_{a,0}\delta_{b,0} = (\psi_{0,0}; \Pi_{0,1}(\mathfrak{u}^a\mathfrak{v}^b)\psi_{0,0}) = \mathrm{e}^{i\pi\theta b}\int_{\mathbb{T}^2}\psi_{b,a}(k)\,dk$$

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$\Pi_{0,1}$ is injective since $⨍$ is faithful.

$\Pi_{0,1} : \mathfrak{A}_\theta \to \mathscr{B}(\mathscr{H}_{0,1})$ with $\mathscr{H}_{0,1} := L^2(\mathbb{T}^2)$ defined by:

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It is the GNS representation related to $\fint$, indeed
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$\mathfrak{A}_\theta^{0,1} := \Pi_{0,1}(\mathfrak{A}_\theta)$ describes the effective models for the 2DMBE in the limits $B \to 0$ ($\theta \propto f_B :=$ flux trough the unit cell).

The commutant $\mathfrak{A}_\theta^{0,1\prime}$ is generated by:

$$\begin{cases} F_1 : \psi_{n,m} \mapsto e^{i\pi\theta n}\psi_{n,m-1} \\ \widetilde{F}_2 : \psi_{n,m} \mapsto e^{i\pi\theta m}\psi_{n+1,m}. \end{cases}$$

The commutant $\mathfrak{A}_\theta^{0,1\,\prime}$ is generated by:

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If $\theta = M/N$ then

$$\mathfrak{S}_{0,1} := C^*(F_1, F_2 := \widetilde{F}_2^N)$$

is a maximal commutative C^*-subalgebra of $\mathfrak{A}_\theta^{0,1\,\prime}$.

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$$\mathfrak{S}_{0,1} := C^*(F_1, F_2 := \widetilde{F}_2^N)$$

is a maximal commutative C^*-subalgebra of $\mathfrak{A}_\theta^{0,1\,\prime}$.
$\mathfrak{S}_{0,1}$ is a unitary representation of $\mathbb{Z}^2$ on $\mathscr{H}_{0,1}$, moreover $\{\phi_j\}_{j=0}^{N-1} \subset \mathscr{H}_{0,1}$ with $\phi_j := \psi_{j,0}$ is wandering for $\mathfrak{S}_{0,1}$, i.e.

$$\mathfrak{S}_{0,1}\left[\{\phi_j\}_{j=0}^{N-1}\right] = \mathscr{H}_{0,1}, \quad (\phi_i; F_1^a F_2^b \phi_j) = \delta_{i,j}\delta_{a,0}\delta_{b,0}.$$

$\Pi_{q,r} : \mathfrak{A}_\theta \to \mathscr{B}(\mathscr{H}_q)$ with $\mathscr{H}_q := L^2(\mathbb{R}) \otimes \mathbb{C}^q \simeq L^2(\mathbb{R}; \mathbb{Z}_q)$ defined by:

$$\begin{cases} \Pi_{q,r}(\mathfrak{u}) =: U_{q,r} = T_1 \otimes \mathbb{U} \\ \Pi_{q,r}(\mathfrak{v}) =: V_{q,r} = T_2{}^{\theta - \frac{r}{q}} \otimes \mathbb{V}^r \end{cases} \qquad r < q \in \mathbb{N} \ \ \text{g.c.d.}(r,q) = 1$$

where T_1 and T_2 are the *Weyl operators* on $L^2(\mathbb{R})$, i.e.

$$T_1 := \mathrm{e}^{i2\pi Q}, \quad T_2 := \mathrm{e}^{-i2\pi P}, \quad Q := \text{multiplication by } x, \quad P := \frac{-i}{2\pi}\frac{\partial}{\partial x}$$

while $\mathbb{U}$ and $\mathbb{V}$ act on $\mathbb{C}^q$ as:

$$\mathbb{U} := \begin{pmatrix} 1 & & & \\ & \varpi_q & & \\ & & \ddots & \\ & & & \varpi_q^{q-1} \end{pmatrix}, \qquad \mathbb{V} := \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

where $\varpi_q := \mathrm{e}^{i2\pi\frac{1}{q}}$, $\mathbb{U}\mathbb{V} = \varpi_q \mathbb{V}\mathbb{U}$.

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where $\varpi_q := \mathrm{e}^{i2\pi\frac{1}{q}}$, $\mathbb{U}\mathbb{V} = \varpi_q \mathbb{V}\mathbb{U}$.

$\Pi_{q,r}$ is injective and $\mathfrak{A}_\theta^{q,r} := \Pi_{q,r}(\mathfrak{A}_\theta)$ describes the 2DMBE in the limits $B \to \infty$ $(q = 1,\; r = 0,\; \theta \propto f_B{}^{-1})$.

The commutant $\mathfrak{A}_\theta^{q,r\prime}$ is generated by [Takesaki '69]:

$$\begin{cases} G_1 := T_1{}^{\frac{1}{q\theta - r}} \otimes \mathbb{U}^a \\ \widetilde{G}_2 := T_2{}^{\frac{1}{q}} \otimes \mathbb{V}^{-1}. \end{cases} \qquad bq - ar = 1.$$

The commutant $\mathfrak{A}_\theta^{q,r'}$ is generated by [Takesaki '69]:

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If $\theta = M/N$ (with g.c.d.(N,q)=1) then

$$\mathfrak{S}_{q,r} := C^*(G_1, G_2 := \widetilde{G}_2^{N_0}), \quad N_0 := qM - rN, \quad G_2 := T_2{}^{\frac{N_0}{q}} \otimes \mathbb{V}^{Nr}$$

is a maximal commutative C^*-subalgebra of $\mathfrak{A}_\theta^{q,r'}$.

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is a maximal commutative C^*-subalgebra of $\mathfrak{A}_\theta^{q,r\,\prime}$.
$\mathfrak{S}_{q,r}$ is a unitary representation of $\mathbb{Z}^2$ on $\mathscr{H}_q \simeq L^2(\mathbb{R};\mathbb{Z}_q)$,
moreover $\{\phi_j\}_{j=0}^{N-1} \subset \mathscr{H}_q$ with

$$\phi_j := \begin{pmatrix} \chi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \chi_j(x) := \begin{cases} 1 & \text{if} \quad j\dfrac{N_0}{N} \leqslant x \leqslant (j+1)\dfrac{N_0}{N} \\ 0 & \text{otherwise.} \end{cases}$$

is wandering for $\mathfrak{S}_{q,r}$, i.e.

$$\mathfrak{S}_{q,r}\left[\{\phi_j\}_{j=0}^{N-1}\right] = \mathscr{H}_q, \quad (\phi_i; G_1^a G_2^b \phi_j) = \delta_{i,j}\delta_{a,0}\delta_{b,0}.$$

Ingredients:

1. separable Hilbert space $\mathscr{H}$ *(space of physical states)*;
2. a C^*-algebra $\mathfrak{A}$ of bounded operators on $\mathscr{H}$ *(physical observables)*;
3. a (maximal) commutative C^*-subalgebra $\mathfrak{S}$ of the commutant $\mathfrak{A}'$ *(simultaneous implementable symmetries)*.

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Technical assumptions:

1. $\mathfrak{S}$ is generated by a unitary representation of $\mathbb{Z}^d$, i.e. $\mathfrak{S} := C^*(U_1, \ldots, U_d)$ with $U_j^* = U_j^{-1}$;
2. $\mathfrak{S}$ has the wandering property, i.e. it exists a (countable) subset $\{\phi_j\} \subset \mathscr{H}$ of orthonormal vectors such that

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Consequences:

1. $\mathfrak{S}$ is algebraically compatible, i.e. $\sum a_{n_1,\ldots,n_d} U_1^{n_1} \ldots U_d^{n_d} = 0$ iff $a_{n_1,\ldots,n_d} = 0$;
2. the Gel'fand spectrum of $\mathfrak{S}$ is $\mathbb{T}^d$;
3. the Haar measure $dz := \frac{dt^d}{(2\pi)^d}$ on $\mathbb{T}^d$ is basic for $\mathfrak{S}$ (abs. cont. with respect the spectral measures).

Theorem [D. & Panati t.b.p.]

i) Let Φ the nuclear space obtained as the inductive limit of vector spaces spanned by finite collections of vectors in $\mathfrak{S}\left[\{\phi_j\}\right]$. The map

$$\Phi \ni \varphi \longmapsto (\mathscr{U}\varphi)(t) := \sum e^{-in_1 t_1} \ldots e^{-in_d t_d} U_1^{n_1} \ldots U_d^{n_d} \varphi \in \Phi'$$

is well defined and $(\mathscr{U}\varphi)(t)$ is a generalized eigenvector of U_j with eigenvalue e^{it_j}.

ii) Let $\mathscr{K}(t) \subset \Phi'$ the space spanned by $\left\{\xi_j(t) := (\mathscr{U}\phi_j)(t)\right\}$, then

$$\mathscr{H} \xrightarrow{\mathscr{U}} \int_{\mathbb{T}^d}^{\oplus} \mathscr{K}(t)\, dz(t) \qquad (L^2 - \text{sections})$$

is a unitary map between Hilbert spaces.

iii) Φ is a pre-C^*-module over $C(\mathbb{T}^d)$ and is mapped by $\mathscr{U}$ in a dense set of continuous sections of the vector bundle $\mathcal{E}_{\mathfrak{S}} \to \mathbb{T}^d$ with fiber $\mathscr{K}(t)$ and frame of sections $\{\xi_j\}$.

iv) $\mathfrak{A}$ is mapped by $\mathscr{U}$ in the continuous sections of $\mathrm{End}(\mathcal{E}_{\mathfrak{S}}) \to \mathbb{T}^d$.

Vector bundle representation of $\mathfrak{A}^{(0,1)}_{M/N}$ (GNS)

$$\mathscr{H}_{0,1} := L^2(\mathbb{T}^2) \xrightarrow{\mathscr{U}} \Gamma_{L^2}(\mathcal{E}_{0,1}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^N \, dz$$

$\mathcal{E}_{0,1} \to \mathbb{T}^2$ is a rank-N vector bundle with typical fiber $\mathscr{K}(t) \simeq \mathbb{C}^N$, N is the cardinality of the wandering system $\{\phi_j\}_{j=0}^{N-1}$. The vector bundle is trivial, indeed the frame of sections $\{\xi_j := \mathscr{U}\phi_j\}_{j=0}^{N-1}$ satisfies:

$$\xi_j(t_1,t_2) = \xi_j(t_1+2\pi,t_2) = \xi_j(t_1,t_2+2\pi) \quad \forall\, j=0,\ldots,N-1 \quad (t_1,t_2) \in \mathbb{T}^2,$$

then $C(\mathcal{E}_{0,1}) = 0$, where C is the (first) Chern number.

Vector bundle representation of $\mathfrak{A}^{(0,1)}_{M/N}$ (GNS)

$$\mathscr{H}_{0,1} := L^2(\mathbb{T}^2) \xrightarrow{\mathscr{U}} \Gamma_{L^2}(\mathcal{E}_{0,1}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^N \, dz$$

$\mathcal{E}_{0,1} \to \mathbb{T}^2$ is a rank-N vector bundle with typical fiber $\mathscr{K}(t) \simeq \mathbb{C}^N$, N is the cardinality of the wandering system $\{\phi_j\}_{j=0}^{N-1}$. The vector bundle is trivial, indeed the frame of sections $\{\xi_j := \mathscr{U}\phi_j\}_{j=0}^{N-1}$ satisfies:

$$\xi_j(t_1,t_2) = \xi_j(t_1+2\pi, t_2) = \xi_j(t_1, t_2+2\pi) \quad \forall\, j = 0,\ldots,N-1 \quad (t_1,t_2) \in \mathbb{T}^2,$$

then $C(\mathcal{E}_{0,1}) = 0$, where C is the (first) Chern number.

$\mathfrak{A}^{(0,1)}_{M/N} \xrightarrow{\mathscr{U}\ldots\mathscr{U}^{-1}} \mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathcal{E}_{0,1})) \simeq \Gamma(\mathrm{End}(\mathcal{E}_{0,1}))$ generated by:

$$U_{0,1}(t) := e^{-it_1} \begin{pmatrix} 1 & & & \\ & \rho & & \\ & & \ddots & \\ & & & \rho^{N-1} \end{pmatrix} \quad V_{0,1}(t) := \begin{pmatrix} 0 & & & e^{it_2} \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

with $\rho := e^{i2\pi \frac{M}{N}}$.

Vector bundle representation of $\mathfrak{A}^{(q,r)}_{M/N}$ (Weyl)

$$\mathscr{H}_{q,r} := L^2(\mathbb{R}) \otimes \mathbb{C}^q \xrightarrow{\mathscr{U}} \Gamma_{L^2}(\mathcal{E}_{q,r}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^N \, dz$$

$\mathcal{E}_{q,r} \to \mathbb{T}^2$ is a rank-N vector bundle, N is the cardinality of the wandering system. The vector bundle is non-trivial, indeed

$$\begin{aligned} \xi_j(t_1, t_2) &= g(t_2)^{-1} \xi_j(t_1 + 2\pi, t_2) \\ \xi_j(t_1, t_2) &= \xi_j(t_1, t_2 + 2\pi) \end{aligned} \qquad g(t_2) := \begin{pmatrix} 0 & & & e^{iqt_2} \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

then $C(\mathcal{E}_{r,q}) = q$.

Vector bundle representation of $\mathfrak{A}_{M/N}^{(q,r)}$ (Weyl)

$$\mathscr{H}_{q,r} := L^2(\mathbb{R}) \otimes \mathbb{C}^q \xrightarrow{\mathscr{U}} \Gamma_{L^2}(\mathcal{E}_{q,r}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathbb{C}^N \, dz$$

$\mathcal{E}_{q,r} \to \mathbb{T}^2$ is a rank-N vector bundle, N is the cardinality of the wandering system. The vector bundle is non-trivial, indeed

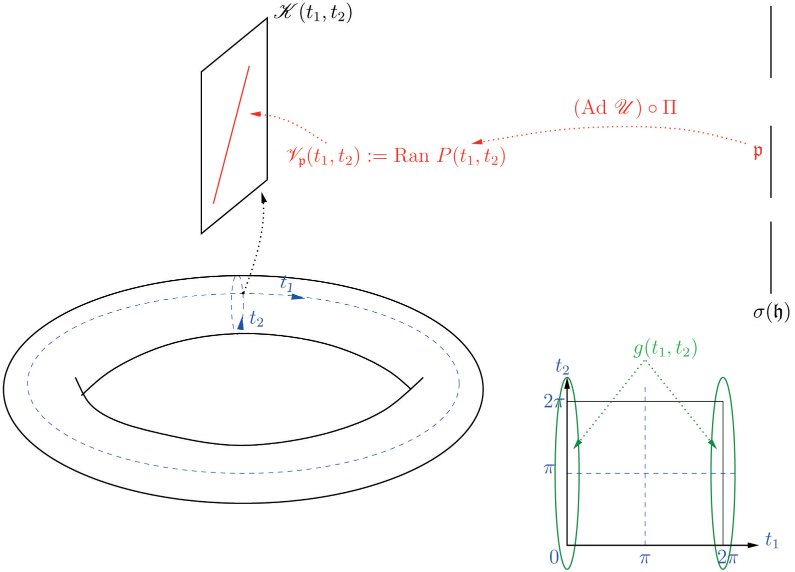
$$\begin{aligned} \xi_j(t_1,t_2) &= g(t_2)^{-1}\xi_j(t_1+2\pi,t_2) \\ \xi_j(t_1,t_2) &= \xi_j(t_1,t_2+2\pi) \end{aligned} \qquad g(t_2) := \begin{pmatrix} 0 & & & e^{iqt_2} \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

then $C(\mathcal{E}_{r,q}) = q$.

$\mathfrak{A}_{M/N}^{(q,r)} \xrightarrow{\mathscr{U}\ldots\mathscr{U}^{-1}} \mathrm{End}_{C(\mathbb{T}^2)}(\Gamma(\mathcal{E}_{q,r})) \simeq \Gamma(\mathrm{End}(\mathcal{E}_{q,r}))$ generated by:

$$U_{q,r}(t) = e^{i\frac{N_0}{N}t_1} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \rho^{q(N-1)} \end{pmatrix} \quad V_{q,r}(t) = e^{-ist_2} \left(\begin{array}{c|c} 0 & e^{iqt_2}\mathbb{1}_\ell \\ \hline \mathbb{1}_{N-\ell} & 0 \end{array} \right)$$

where: $\ell q - sN = 1$ (g.c.d.(N,q)=1).


$\mathscr{K}(t_1, t_2)$
$(\mathrm{Ad}\,\mathscr{U}) \circ \Pi$
$\mathscr{V}_{\mathfrak{p}}(t_1, t_2) := \mathrm{Ran}\ P(t_1, t_2)$
$\mathfrak{p}$
$\sigma(\mathfrak{h})$
t_1
t_2
$g(t_1, t_2)$
2π
π
0

$$
\begin{tikzcd}
& P_{0,1} \arrow[r, "\mathrm{Ad}\mathscr{U}"] & P_{0,1}(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathfrak{L}_{0,1}[\mathfrak{p}] \arrow[r, "C"] & C_{0,1}(\mathfrak{p}) \\
\mathrm{Proj}(\mathfrak{A}_\theta) \ni \mathfrak{p} \arrow[ur, "\Pi_{0,1}"] \arrow[dr, "\Pi_{q,r}"'] & \mathscr{B}(\mathscr{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
& P_{q,r} \arrow[r, "\mathrm{Ad}\mathscr{U}"'] & P_{q,r}(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathfrak{L}_{q,r}[\mathfrak{p}] \arrow[r, "C"'] & C_{q,r}(\mathfrak{p})
\end{tikzcd}
$$

$C :=$ first Chern number.

$$
\begin{tikzcd}
 & P_{0,1} \arrow[r, "\mathrm{Ad}\mathscr{U}"] & P_{0,1}(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathfrak{L}_{0,1}[\mathfrak{p}] \arrow[r, "C"] & C_{0,1}(\mathfrak{p}) \\
\mathrm{Proj}(\mathfrak{A}_\theta) \ni \mathfrak{p} \arrow[ur, "\Pi_{0,1}"] \arrow[dr, "\Pi_{q,r}"'] & \mathscr{B}(\mathscr{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_{q,r} \arrow[r, "\mathrm{Ad}\mathscr{U}"'] & P_{q,r}(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathfrak{L}_{q,r}[\mathfrak{p}] \arrow[r, "C"'] & C_{q,r}(\mathfrak{p})
\end{tikzcd}
$$

What is the relation between $C_{0,1}(\mathfrak{p})$ and $C_{q,r}(\mathfrak{p})$ for a given $\mathfrak{p} \in$ Proj$(\mathfrak{A}_\theta)$?

$$
\begin{tikzcd}
 & P_{0,1} \arrow[r, "\mathrm{Ad}\,\mathscr{U}"] & P_{0,1}(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathfrak{L}_{0,1}[\mathfrak{p}] \arrow[r, "C"] & C_{0,1}(\mathfrak{p}) \\
\mathrm{Proj}(\mathfrak{A}_\theta) \ni \mathfrak{p} \arrow[ur, "\Pi_{0,1}"] \arrow[dr, "\Pi_{q,r}"'] & \mathscr{B}(\mathscr{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_{q,r} \arrow[r, "\mathrm{Ad}\,\mathscr{U}"'] & P_{q,r}(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathfrak{L}_{q,r}[\mathfrak{p}] \arrow[r, "C"'] & C_{q,r}(\mathfrak{p})
\end{tikzcd}
$$

Transforms of the torus:

$$
\mathbb{T}^2 \ni (t_1, t_2) \xmapsto{f} (qNt_1, t_2) \in \mathbb{T}^2
$$
$$
\mathbb{T}^2 \ni (t_1, t_2) \xmapsto{g} (-N_0 t_1, qt_2) \in \mathbb{T}^2.
$$

Pullback of vector bundles:

$$
f^*(\mathfrak{L}_{q,r}[\mathfrak{p}]) \simeq g^*(\mathfrak{L}_{0,1}[\mathfrak{p}]) \otimes l_{q^2}
$$

$l_{q^2} :=$ line bundle with Chern number q^2 (global twist of $\mathcal{E}_{q,r}$).

$$
\begin{tikzcd}
 & P_{0,1} \arrow[r, "\mathrm{Ad}\mathscr{U}"] & P_{0,1}(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathfrak{L}_{0,1}[\mathfrak{p}] \arrow[r, "C"] & C_{0,1}(\mathfrak{p}) \\
\mathrm{Proj}(\mathfrak{A}_\theta) \ni \mathfrak{p} \arrow[ur, "\Pi_{0,1}"] \arrow[dr, "\Pi_{q,r}"'] & \mathscr{B}(\mathscr{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_{q,r} \arrow[r, "\mathrm{Ad}\mathscr{U}"'] & P_{q,r}(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathfrak{L}_{q,r}[\mathfrak{p}] \arrow[r, "C"'] & C_{q,r}(\mathfrak{p})
\end{tikzcd}
$$

Functoriality of C:

$$
\boxed{C_{q,r}(\mathfrak{p}) = q\frac{R(\mathfrak{p})}{N} - \left(q\frac{M}{N} - r\right) C_{0,1}(\mathfrak{p})}
$$

$R(\mathfrak{p}) := \mathrm{Rk}(g^*(\mathfrak{L}_{0,1}[\mathfrak{p}]) = \mathrm{Rk}(\mathfrak{L}_{0,1}[\mathfrak{p}])$.

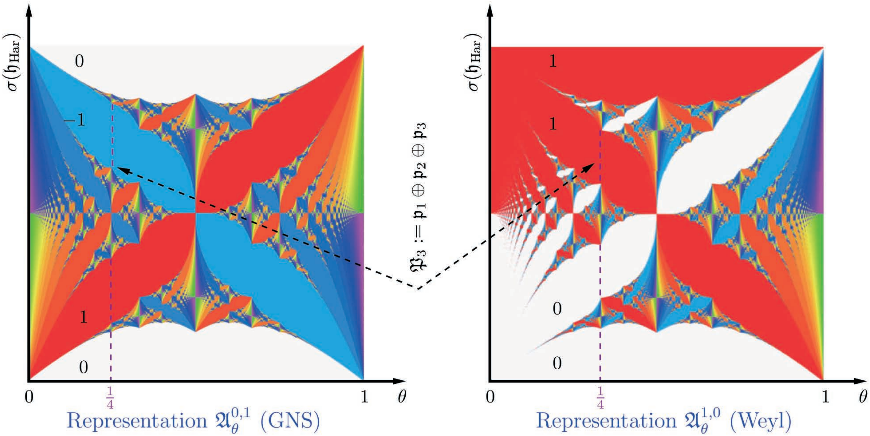
$$
\begin{tikzcd}
 & P_{0,1} \arrow[r, "\mathrm{Ad}\mathscr{U}"] & P_{0,1}(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathfrak{L}_{0,1}[\mathfrak{p}] \arrow[r, "C"] & C_{0,1}(\mathfrak{p}) \\
\mathrm{Proj}(\mathfrak{A}_\theta) \ni \mathfrak{p} \arrow[ur, "\Pi_{0,1}"] \arrow[dr, "\Pi_{q,r}"'] & \mathscr{B}(\mathscr{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_{q,r} \arrow[r, "\mathrm{Ad}\mathscr{U}"'] & P_{q,r}(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathfrak{L}_{q,r}[\mathfrak{p}] \arrow[r, "C"'] & C_{q,r}(\mathfrak{p})
\end{tikzcd}
$$

"Abstract" version:

$$
\frac{R(\mathfrak{p})}{N} = \fint(\mathfrak{p}), \qquad C_{0,1}(\mathfrak{p}) = \mathfrak{Ch}(\mathfrak{p})
$$

$$
\boxed{C_{q,r}(\mathfrak{p}) = q \fint(\mathfrak{p}) + (r - q\theta)\mathfrak{Ch}(\mathfrak{p})}
$$

Duality between gap projections of $\mathfrak{h}_{\text{Har}}$



Courtesy of J. Avron

Thank you for your attention