Projective representations of compact quantum groups

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Twists

Definition

Let (A, Δ_A) and (D, Δ_D) be two Hopf algebras. We say that (D, Δ_D) is a *twisting* of (A, Δ_A) if there exists

$$\Delta_E : E := \begin{pmatrix} D & B \\ C & A \end{pmatrix} \rightarrow \begin{pmatrix} D \odot D & B \odot B \\ C \odot C & A \odot A \end{pmatrix} \subseteq E \odot E$$

with $BC = A$ and $CB = D$. We then call (B, Δ_B), as a right A-module coalgebra, an associated *twist*, and (E, Δ_E) an associated linking structure.

Categorical interpretation

Theorem (B. Pareigis)

There is a one-to-one correspondence between isomorphism classes of

1. *linking structures between* (A, Δ_A) *and* (D, Δ_D)*, and*
2. *comonoidal equivalences between A-Mod and D-Mod.*

Remark: The equivalence F is the functor $B \underset{A}{\odot} -$, while the monoidal structure is given by

$$u : B \underset{A}{\odot} (V \odot W) \rightarrow (B \underset{A}{\odot} V) \odot (B \underset{A}{\odot} W) :$$

$$b \underset{A}{\otimes} (v \otimes w) \rightarrow (b_{(1)} \underset{A}{\otimes} v) \otimes (b_{(2)} \underset{A}{\otimes} w).$$

2-cocycles

Definition

A *2-cocycle* for a Hopf algebra (A, Δ_A) is an invertible element $\Omega \in A \odot A$ such that

$$(\Omega \otimes 1)(\Delta_A \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta_A)(\Omega).$$

Two 2-cocycles Ω and $\tilde{\Omega}$ are called *cohomologous* if there exists an invertible element $u \in A$ s.t.

$$\tilde{\Omega} = (u \otimes u)\Omega\Delta_A(u^{-1}).$$

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Definition

A *2-coboundary* is a 2-cocycle Ω which is cohomologous to $1_A \otimes 1_A$:

$$\Omega = (u \otimes u)\Delta_A(u^{-1})$$

for $u \in A$ invertible.

Cleft twists

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Proposition

1. *Any 2-cocycle Ω for (A, Δ_A) determines a cleft twist $(B_A, \Delta_B) := (A_A, \Omega\Delta_A(\cdot))$.*
2. *Conversely, any cleft twist (B, Δ_B) determines a unique cohomology class of 2-cocycles.*

Remark: If (B, Δ_B) is cleft by some 2-cocycle Ω, then the twisted Hopf algebra is

$$(D, \Delta_D) \cong (A, \Delta_\Omega := \Omega\Delta_A(\cdot)\Omega^{-1}).$$

From 2-coboundaries to 2-cocycles

Sometimes, one can get non-trivial 2-cocycles from 2-coboundaries, for example when

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2. the 2-cocycle is only ‘formally’ a 2-coboundary.

Compact quantum groups and their manifestations

A compact quantum group $\mathbb{G}$ has several avatars:

1. a Hopf-*-algebra $(\mathrm{Pol}(\mathbb{G}), \Delta)$ with an invariant state φ,
2. a unital C*-bi-algebra $(C(\mathbb{G}), \Delta)$ satisfying the cancelation property,
3. a von Neumann bi-algebra $(\mathscr{L}^\infty(\mathbb{G}), \Delta)$ with a normal invariant state φ (i.e. $(\iota \otimes \varphi)(\Delta(x)) = \varphi(x)1 = (\varphi \otimes \iota)(\Delta(x))$).

In the C*-algebraic setting, there are two *canonical* extremal ones:

1. a reduced C*-algebraic one, $(C_r(\mathbb{G}), \Delta)$, and
2. a universal C*-algebraic one, $(C_u(\mathbb{G}), \Delta)$.

If these coincide, the compact quantum group is called *coamenable*.

Tensor products of CQG

Proposition (Wang)

If $\mathbb{G}_n$ is a sequence of compact quantum groups, then one can form the infinite direct product $\prod_{n=1}^{\infty} \mathbb{G}_n$, with

$$Pol(\prod_{n=1}^{\infty} \mathbb{G}_n) = \bigodot_{n=1}^{\infty} Pol(\mathbb{G}_n),$$

$$(\mathscr{L}^{\infty}(\prod_{n=1}^{\infty} \mathbb{G}_n), \varphi) = \bigotimes_{n=1}^{\infty} (\mathscr{L}^{\infty}(\mathbb{G}_n), \varphi_n),$$

and the comultiplication given as

$$\bigodot_{n=1}^{\infty} Pol(\mathbb{G}_n) \overset{\otimes \Delta_n}{\rightarrow} \bigodot_{n=1}^{\infty} (Pol(\mathbb{G}_n) \odot Pol(\mathbb{G}_n)) \cong (\bigodot_{n=1}^{\infty} Pol(\mathbb{G}_n)) \odot (\bigodot_{n=1}^{\infty} Pol(\mathbb{G}_n)).$$

Construction of a unitary 2-cocycle

Fix a *square summable* sequence $0 < q_n < 1$, and put

$$SU_{\mathbf{q}}(2) := \prod_{n=1}^{\infty} SU_{q_n}(2).$$

Then one can find unitaries $w_n \in \mathscr{L}^\infty(SU_{q_n}(2))$ such that

1. the element $\bigotimes_{n=1}^{\infty} w_n$ is *ill-defined* in $\mathscr{L}^\infty(SU_{\mathbf{q}}(2))$, but
2. the unitary

$$\Omega = \bigotimes_{n=1}^{\infty} ((w_n \otimes w_n)\Delta_n(w_n^*))$$

is a *well-defined* 2-cocycle in $\mathscr{L}^\infty(SU_{\mathbf{q}}(2)) \otimes \mathscr{L}^\infty(SU_{\mathbf{q}}(2))$.

An invariant weight on the twisted QG

One can *explicitly* write down a formula for an invariant nsf weight φ_Ω on $(\mathscr{L}^\infty(SU_{\mathbf{q}}(2)), \Delta_\Omega)$.

Hence, it is a *von Neumann algebraic quantum group* (in the sense of Kustermans-Vaes).

Consequences

Theorem

The twisted von Neumann algebraic quantum group

$$(\mathscr{L}^\infty(SU_\Omega(2)), \Delta_\Omega) := (\mathscr{L}^\infty(SU_{\mathbf{q}}(2)), \Delta_\Omega)$$

is no longer compact (i.e. φ_Ω is not finite).

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Corollary

The C^*-algebra $C_0(SU_\Omega(2)), \Delta_\Omega)$ underlying $(\mathscr{L}^\infty(SU_\Omega(2)), \Delta_\Omega)$ is different from $C(SU_{\mathbf{q}}(2))$.

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Remark: There is a natural embedding

$$(C(SU_{\mathbf{q}}(2)), \Delta_{\mathbf{q}}) \hookrightarrow (\mathscr{L}^\infty(SU_\Omega(2)), \Delta_\Omega),$$

$$\otimes x_i \rightarrow \otimes (w_i x_i w_i^*).$$

A connecting structure

Inside $\mathscr{L}^\infty(SU_{\mathbf{q}}(2)) \otimes M_2(\mathbb{C})$, there is a natural (non-unital) C*-algebra

$$E = \begin{pmatrix} C_0(SU_\Omega(2)) & C_0(SU_{\Omega,\mathbf{q}}(2)) \\ C_0(SU_{\mathbf{q},\Omega}(2)) & C(SU_{\mathbf{q}}(2)) \end{pmatrix}$$

with (non-unital) comultiplication

$$\Delta_E : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \rightarrow \begin{pmatrix} \Omega\Delta_{\mathbf{q}}(x)\Omega^* & \Omega\Delta_{\mathbf{q}}(y) \\ \Delta_{\mathbf{q}}(z)\Omega^* & \Delta_{\mathbf{q}}(w) \end{pmatrix} \in M(E \otimes E).$$

Consequence for the W*-RepCat

The functor

$$\begin{array}{ccc} \mathrm{Rep}^*(C(SU_{\mathbf{q}}(2))) & \longrightarrow & \mathrm{Rep}^*(C_0(SU_{\Omega}(2))) \\ \uparrow \cong & & \downarrow \cong \\ \text{U-Rep}(\widehat{SU_{\mathbf{q}}(2)}) & \longrightarrow & \text{U-Rep}(\widehat{SU_{\Omega}(2)}) \end{array}$$

determined by the C*-bi-correspondence $C_0(SU_{\Omega,\mathbf{q}}(2))$, extends to a *comonoidal equivalence* by use of $\Delta_{C_0(SU_{\Omega,\mathbf{q}}(2))}$.

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Corollary

There exist a discrete and a non-discrete quantum group whose unitary representation categories coincide as abstract monoidal W-categories.*

Projective representations

Definition

Let $\mathscr{H}$ be a Hilbert space. A *projective representation* of a compact quantum group $\mathbb{G}$ on $\mathscr{H}$ is a coaction

$$\alpha : B(\mathscr{H}) \to \mathscr{L}^\infty(\mathbb{G}) \otimes B(\mathscr{H}).$$

It is called *irreducible* if the coaction is ergodic (i.e. the space of invariants is trivial).

Returning to the example

Fix a *square summable* sequence $0 < q_n < 1$.

On $\mathbb{C}^2$, consider the fundamental unitary representation of $SU_{q_n}(2)$:

$$U_n : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix} \otimes \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then the induced coactions on $M_2(\mathbb{C})$ can be tensored to give an ergodic coaction

$$\alpha : \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \omega_n) \rightarrow \mathscr{L}^{\infty}(SU_{\mathbf{q}}(2)) \otimes (\bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \omega_n)).$$

By the convergence rate of the q_n, we have $\bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \omega_n)$ type I, hence α is an irreducible projective representation!

Projective representations and twists

Twisting between von Neumann algebraic quantum groups:

$$\Delta_Q : Q := (Q_{ij}) := \begin{pmatrix} P & N \\ O & M \end{pmatrix} \rightarrow \begin{pmatrix} P \otimes P & N \otimes N \\ O \otimes O & M \otimes M \end{pmatrix} \subseteq Q \otimes Q.$$

Theorem

Let $\mathbb{G}$ be a compact quantum group. Then there is a one-to-one correspondence between

1. *'outer conjugacy classes' of irreducible projective representations for $\mathbb{G}$, and*
2. *isomorphism classes of twists for $(\mathscr{L}^\infty(\mathbb{G}), \Delta)$.*

Duality theory for twists

Proposition (Duality theory)

To any linking structure (Q, Δ_Q), *there corresponds a dual object* $(\widehat{Q}, \Delta_{\widehat{Q}})$ *with*

1. $\widehat{Q} = \oplus \widehat{Q}_{ij} = \widehat{P} \oplus \widehat{O} \oplus \widehat{N} \oplus \widehat{M}$ *a direct sum of vN algebras, and*
2. $\Delta_{\widehat{Q}} : \widehat{Q}_{ij} \rightarrow (\widehat{Q}_{i1} \otimes \widehat{Q}_{1j}) \oplus (\widehat{Q}_{i2} \otimes \widehat{Q}_{2j})$.

Structure of the dual

Proposition

Let $\mathbb{G}$ be a compact quantum group, and Q a linking structure between $\mathscr{L}^\infty(\mathbb{G})$ and some von Neumann algebraic quantum group (P, Δ_P). Then $\widehat{O}$ (and $\widehat{N}$) is atomic (i.e. $\widehat{O} = \oplus B(\mathscr{H}_i)$ for certain Hilbert spaces $\mathscr{H}_i$).

Warning: The Hilbert spaces are not necessarily finite dimensional.

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Warning: The Hilbert spaces are not necessarily finite dimensional.

Remark: if (N, Δ_N) is a twist for $\mathscr{L}^\infty(\mathbb{G})$, then the 'adjoint coaction' of $\mathscr{L}^\infty(\mathbb{G})$ on any of the summands of $\widehat{O}$ is an *irreducible* projective representation.

A characterization

Theorem

Let $\mathbb{G}$ be a compact quantum group. The following two statements are equivalent:

1. $(\mathscr{L}^\infty(\mathbb{G}), \Delta)$ *can be twisted into a non-compact locally compact quantum group* (P, Δ_P).
2. $\mathbb{G}$ *allows an infinite-dimensional irreducible projective representation.*

Open question

Question (Vaes)

Let $\mathbb{G}$ be a compact matrix quantum group. Can it have an infinite-dimensional irreducible projective representation?