Noncommutative geometry of CP_q^ℓ

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The algebra of antiholomorhic forms on the quantum projective space $\mathbb{C}\mathrm{P}_q^\ell$ is constructed. The associated Dolbeault-Dirac operator defines a 0-dimensional $U_q(\mathfrak{su}(\ell+1))$-equivariant spectral triple. (Based on [F. D'Andrea, L. Dąbrowski, arXiv:0901.4735]).

'QG' match 'NCG'

QG is q-deformed matrix group à la Woronowicz (and dual QUEA), or its homogeneous space.
NCG is à la Connes:

$$\textit{riemannian, spin manifold} \longleftrightarrow \textit{spectral triple } (\mathcal{A}, \mathcal{H}, D),$$

where $\mathcal{A}$ is a $*$-algebra, $\mathcal{H}$ is a Hilbert space representation of $\mathcal{A}$ and D is an operator on $\mathcal{H}$ s.t. $D = D^\dagger$, $(D-z)^{-1} \in \mathcal{K}$ and $[D, \mathcal{A}] \subset B(\mathcal{H})$.

(Here compact M & unital $\mathcal{A}$ only).

Even if $\exists\ \gamma$ s.t. $\gamma^2 = 1$, $a\gamma = \gamma a$, $\forall a \in \mathcal{A}$, $D\gamma = -\gamma D$; so $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$.

Prototype: given a spin manifold M with a Riemannian metric g:
$\mathcal{A} = C^\infty(M, \mathbb{C})$ (commutative), $\mathcal{H} = L^2(\Sigma, vol_g)$, and $D = \not{D}$.
Even ($\gamma =$ "γ_5") iff dim$M$ is even.
It satisfies further 7 conditions which permit reconstruction [Connes08].

Two crucial properties directly pass to NC(diff)G of Connes

$(D - z)^{-1} \in \mathcal{K} \quad \Rightarrow \quad D|_{\mathcal{H}_\pm}$ coupled to gauge connections, vector bundles, projections, or their classes in the K-theory, are Fredholm $\rightsquigarrow$ 'topological' invariants via index computations.

Instead $[D, \mathcal{A}] \subset B(\mathcal{H})$ serves for 'locality'.

The tension between them + (some of) 7 other 'axioms', Connes formulated for NC algebras, permit spectacular calculations.
Among the axioms is *dimension* given by the assymptotics of spec$D$.
Another one is *reality*, c.f. the talk by G. Dossena.

• How to construct NC spectral triples (S.T.) on QG ?
Our strategy: equivariance under QUEA and 'harmonic analysis'.
E.g. 'isospectral' D, c.f. [CP03a, DLS05, DLPS05, DDLW07].
But on the standard Podleś quantum sphere $S^2_{q0} \equiv \mathbb{C}\mathrm{P}^1_q$, $\exists$ also D [DS03] with exponential spectrum (and a 0^+-summable S.T.)
This D has elegant description [SW04] in terms of $U_q(\mathfrak{su}(2))$ which act as derivations on S^2_{q0}, and so $[D, a]$ are bounded.

Along this line, [K04] constructed D on q-flag manifolds, e.g. $\mathbb{C}\mathrm{P}^\ell_q$.
However, spec$D$ & $(D-z)^{-1} \in \mathcal{K}$ are not addressed.
We completed this task [DD09]: constructed 'Dirac-Dolbeaut' operator D with derivation property (thus $[D, a] \subset B(\mathcal{H})$) and by relating D^2 to a Casimir element control spec$D$.
It is exponentially growing (so $(D-z)^{-1} \in \mathcal{K}$) and yields a 0^+-dim ST.

I'll recall $\mathbb{C}\mathrm{P}^1_q$, describe $\mathbb{C}\mathrm{P}^2_q$ (only spin^c) [DDL08-2] and mention $\mathbb{C}\mathrm{P}^\ell_q$.

- $$\mathbb{C}\mathrm{P}^1_q \quad = SU_q(2)/U(1) = S^3_q/U(1), \qquad 0 < q < 1.$$

The symmetry Hopf $*$-algebra is $U_q(\mathfrak{su}(2))$, generated by K, K^{-1}, E, F

$$KE = qEK \ , \qquad [E,F] = (q - q^{-1})^{-1}(K^2 - K^{-2})$$

(+ coproduct, counit and antipode).

The dual $\mathcal{A}(SU_q(2))$ is a $U_q(\mathfrak{su}(2))$-bimodule $*$-algebra w.r.t. $\triangleright$ and $\triangleleft$.

Let $\forall\ N \in \mathbb{Z}$, $$\Gamma_N = \left\{ a \in \mathcal{A}(SU_q(2)) \,|\, a \triangleleft K = q^{\frac{N}{2}} a \right\}.$$

$\mathcal{A} := \Gamma_0$ is just the $*$-algebra of $\mathbb{C}\mathrm{P}^1_q \ = \ S^2_{q0}$.
Each Γ_N is a $\mathcal{A}$-bimodule. Also, a $U_q(\mathfrak{su}(2))\triangleright$-module and as such

$$\Gamma_N \simeq \bigoplus_{n-|N| \in 2\mathbb{N}} V_n \ , \tag{1}$$

where V_n is the spin$\frac{1}{2}n$ irrep & $\simeq$ is unitary wrt the Haar state on Γ_N.

The Casimir

$$\mathcal{C}_q = (q - q^{-1})^{-2} (q^{\frac{1}{2}} K - q^{-\frac{1}{2}} K^{-1})^2 + FE$$

has spectrum

$$\mathcal{C}_q \Big|_{V_n} = [\tfrac{n+1}{2}]^2 \cdot id$$

with multiplicity $\dim V_n = n + 1$. (Here $[x] := \frac{q^x - q^{-x}}{q - q^{-1}}$).

Antiholomorphic forms: $\Omega = \Omega^0 \oplus \Omega^1$, w/ $\Omega^0 = \mathcal{A}$ and $\Omega^1 = \Gamma_2$.

The Dolbeault operator and its Hermitian conjugate are derivations

$$\begin{aligned} \bar{\partial} : \Omega^0 \to \Omega^1 \,, \qquad & a \mapsto a \triangleleft F \,, \\ \bar{\partial}^\dagger : \Omega^1 \to \Omega^0 \,, \qquad & a \mapsto a \triangleleft E \,. \end{aligned}$$

The Dolbeault-Dirac operator D on $\Omega^0 \oplus \Omega^1$ is

$$D(\omega_0, \omega_1) := (\omega_1 \triangleleft E, \omega_0 \triangleleft F) \ ,$$

and satisfies

$$D^2\omega = \omega \triangleleft (\mathcal{C}_q - [\tfrac{1}{2}]^2).$$

Using $\triangleleft \mathcal{C}_q = \mathcal{C}_q \triangleright$, the decomposition (1) of Γ_N, and

$$D\gamma + \gamma D = 0, \quad \text{where} \quad \gamma = 1 \oplus -1,$$

$$\mathrm{spec} D = \{0, \pm\sqrt{[k][k+1]} \,|\, k \in \mathbb{N} + 1\}$$

with multiplicity 1 (constant 0-forms) and $2k+1$, resp.

Real spinors? Tensor $\Omega^\bullet$ with Γ_{-1} $\quad$ ($=\sqrt{\text{canonical bundle}}$):

$$\mathcal{H} = (\Omega^0 \oplus \Omega^1) \otimes_{\mathcal{A}} \Gamma_{-1} \simeq \Gamma_{-1} \oplus \Gamma_1 \ .$$

Then the *real* $\not{D}$ is D twisted with the Grassmann connection of Γ_{-1}; it agrees [SW04] w/ [DS03] and $\exists$ a real structure J (= spin structure).

Byproduct: compute the Dolbeault cohomology $H^{\bullet}_{\bar\partial}(\mathbb{C}\mathsf{P}^2_q)$.
Define *harmonic* forms $\mathfrak{H}^n = \{\omega \in \Omega^{(0,n)} \,|\, D\omega = 0 (\equiv \bar\partial\omega = \bar\partial^\dagger\omega = 0)\}$.

Prop.[hodge] For all n, there is an orthogonal decomposition

$$\Omega^{(0,n)} = \mathfrak{H}^n \oplus \bar\partial\Omega^{(0,n-1)} \oplus \bar\partial^\dagger\Omega^{(0,n+1)} \,. \tag{2}$$

$\exists!$ harmonic form in each cohomology class:

$$H^n_{\bar\partial}(\mathbb{C}\mathsf{P}^1_q) \simeq \mathfrak{H}^n = \ker D\Big|_{\Omega^{(0,n)}} \,.$$

Pf. As classically.

Corr.

$$H^0_{\bar\partial}(\mathbb{C}\mathsf{P}^1_q) = \mathbb{C} \,, \qquad H^1_{\bar\partial}(\mathbb{C}\mathsf{P}^1_q) = 0 \,.$$

- $\mathbb{C}\mathrm{P}_q(2)$

The symmetry Hopf $*$algebra $U_q(\mathfrak{su}(3))$, $0 < q < 1$.
Generators K_i, K_i^{-1}, E_i, F_i , $i = 1, 2$, with $K_i = K_i^*, F_i = E_i^*$,
& relations

$$[K_i, K_j] = 0 \,, \quad K_i E_i K_i^{-1} = q E_i \,, \quad [E_i, F_i] = (q - q^{-1})^{-1}(K_i^2 - K_i^{-2})$$
$$K_i E_j K_i^{-1} = q^{-1/2} E_j \,, \qquad [E_i, F_j] = 0 \,, \qquad \text{if } i \neq j \,,$$

+ (Serre)

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \qquad \forall\, i \neq j \,. \tag{3}$$

With the q-commutator $[a, b]_q := ab - q^{-1} ba$, (3) can be rewritten as

$$[E_i, [E_j, E_i]_q]_q = 0, \quad \text{or,} \quad [[E_i, E_j]_q, E_i]_q = 0 \,.$$

(+ coproduct, counit and antipode).

$*$-irreps of $U_q(\mathfrak{su}(3))$ are classified by $n_1, n_2 \in \mathbb{N}^2$, have dim.

$$\looparrowright \qquad \tfrac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2)\;, \tag{4}$$

and act on $V(n_1, n_2)$ with on. basis $|n_1, n_2, j_1, j_2, m\rangle$, where

$$j_i = 0, 1, 2, \ldots, n_i\;, \qquad \frac{j_1+j_2}{2} - |m| \in \mathbb{N}\;, \tag{5}$$

as (explicit) multiplication (K's) or weighted shift operators (E, F's). (The h.w.v. with weight $(q^{n_1/2}, q^{n_2/2})$ being $|n_1, n_2, n_1, 0, \frac{1}{2}n_1\rangle$).

$\exists$ Casimir operator [R-P91] (add $H := (K_1 K_2^{-1})^{2/3}$ and H^{-1})

$$\begin{aligned}\mathcal{C}_q = (q-q^{-1})^{-2}\Big((H+H^{-1})\big\{(qK_1K_2)^2+(qK_1K_2)^{-2}\big\}+H^2+H^{-2}-6\Big)\\ +(qHK_2^2+q^{-1}H^{-1}K_2^{-2})F_1E_1+(qH^{-1}K_1^2+q^{-1}HK_1^{-2})F_2E_2\\ +qH[F_2,F_1]_q[E_1,E_2]_q+qH^{-1}[F_1,F_2]_q[E_2,E_1]_q\end{aligned} \tag{6}$$

$\mathcal{C}_q = \mathcal{C}_q^*$ is central and $\mathcal{C}_q|_{V_{(n_1,n_2)}}$ is

$$\looparrowright \qquad [\tfrac{1}{3}(n_1-n_2)]^2+[\tfrac{1}{3}(2n_1+n_2)+1]^2+[\tfrac{1}{3}(n_1+2n_2)+1]^2\ \times \mathrm{id}\,. \tag{7}$$

$\mathcal{A}(SU_q(3))$ is generated by 9 elements u^i_j $(i,j=1,...,3)$ & relations

$$\begin{aligned} &u^i_k u^j_k = q u^j_k u^i_k \,, && u^k_i u^k_j = q u^k_j u^k_i \,, && \forall\, i<j \,, \\ &[u^i_l, u^j_k] = 0 \,, && [u^i_k, u^j_l] = (q-q^{-1}) u^i_l u^j_k \,, && \forall\, i<j,\ k<l \,, \end{aligned}$$

$$\textstyle\sum_{\pi\in S_3} (-q)^{|\pi|} u^1_{\pi(1)} u^2_{\pi(2)} u^3_{\pi(3)} = 1 \,.$$

(+ $*$-structure, coproduct, counit and antipode).

$\mathcal{A}(SU_q(3))$ is a $U_q(\mathfrak{su}(3))$-bimodule $*$-algebra, which we turn into a left $U_q(\mathfrak{su}(3))\triangleright \otimes U_q(\mathfrak{su}(3))\blacktriangleright$-module $*$-algebra. As such, by Peter-Weyl

$$\mathcal{A}(SU_q(3)) \simeq \textstyle\bigoplus_{(n_1,n_2)\in\mathbb{N}^2} V_{(n_1,n_2)} \otimes V_{(n_1,n_2)},$$

where $\simeq$ is isometry wrt. the Haar state.

Important:

$$\mathcal{C}_q \triangleright = \mathcal{C}_q \blacktriangleright \ . \tag{8}$$

View [Meyer], [Welk] $\mathbb{C}\mathrm{P}^2_q = S^5_q/U(1)$ and in turn $S^5_q = SU_q(3)/SU_q(2)$. Dually,

$$\mathcal{A}(S^5_q) := \Big\{ a \in \mathcal{A}(SU_q(3)) \,\Big|\, h \blacktriangleright a = \epsilon(h)a,\ \forall\, h \in U_q(\mathfrak{su}(2)) \Big\}$$

(the $*$-subalgebra generated by $z_i := u^3_i$ and [VS] relations), where $U_q(\mathfrak{su}(2))$ is generated by $\{K_1, K_1^{-1}, E_1, F_1\}$.

$\mathcal{A}(S^5_q)$ is a module over $(U_q(\mathfrak{su}(2)) \blacktriangleright)' = U_q(\mathfrak{su}(3)) \triangleright \otimes U_q(\mathfrak{u}(1)) \blacktriangleright$, where $U_q(\mathfrak{u}(1))$ is generated by $\{K_1K_2^2, K_1^{-1}K_2^{-2}\}$, and decomposes

$$\mathcal{A}(S^5_q) \simeq \bigoplus\nolimits_{(n_1,n_2)\in\mathbb{N}^2} V_{(n_1,n_2)} \,,$$

where $$K_1K_2^2 \blacktriangleright|_{V_{(n_1,n_2)}} = q^{n_2-n_1}\ \mathrm{id}.$$

In turn

$$\mathcal{A}(\mathbb{C}\mathsf{P}^2_q) := \Big\{ a \in \mathcal{A}(S^5_q) \,\Big|\, K_1 K_2^2 \blacktriangleright a = a \Big\}$$
$$= \Big\{ a \in \mathcal{A}(SU_q(3)) \,\Big|\, h \blacktriangleright a = \epsilon(h) a \ , \ \forall\, h \in U_q(\mathfrak{u}(2)) \Big\} \ ,$$

where $U_q(\mathfrak{u}(2))$ is generated by $U_q(\mathfrak{su}(2))$ and $U_q(\mathfrak{u}(1))$.

The $*$-algebra $\mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$ is generated by $\quad p_{ij} := (u_i^3)^* u_j^3, \quad (p_{ij})^* = p_{ji},$
$+$ relations $\quad P^2 = P, \quad P = P^*, \quad \mathrm{tr}_q(P) := q^4 p_{11} + q^2 p_{22} + p_{33} = 1,$
in terms of $\{P\}_{ij} = p_{ij}$.

$\mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$ is an $U_q(\mathfrak{su}(3))\triangleright$-module and decomposes

$$\mathcal{A}(\mathbb{C}\mathsf{P}^2_q) \simeq \bigoplus\nolimits_{n \in \mathbb{N}} V_{(n,n)} \ .$$

The Dolbeault complex

Think of $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q) \hookrightarrow \mathcal{A}(S^5_q)$ as a quantum principal $U(1)$-bundle, and sections of associated line bundle over $\mathbb{C}\mathrm{P}^2_q$ as $U(1)$-equivariant maps. For $N \in \mathbb{Z}$, define $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$-bimodules

$$\mathcal{L}_N := \Big\{ a \in \mathcal{A}(S^5_q) \,\Big|\, K_1 K_2^2 \blacktriangleright a = q^N a \,\Big\}; \tag{9}$$

or $\equiv$, being $\mathcal{A}(S^5_q) \hookrightarrow \mathcal{A}(SU_q(3))$ a quantum principal $SU_q(2)$-bundle,

$$\mathcal{L}_N = \Big\{ a \in \mathcal{A}(SU_q(3)) \,\Big|\, K_1 K_2^2 \blacktriangleright a = q^N a\,,\ h \blacktriangleright a = \epsilon(h) a\,,\ \forall\, h \in U_q(\mathfrak{su}(2)) \Big\}.$$

$(\mathcal{L}_0 = \mathcal{A}(\mathbb{C}\mathrm{P}^2_q))$.

Each $\mathcal{L}_N$ is $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$-bimodule and also a left $U_q(\mathfrak{su}(3))\triangleright$-module and decomposes

$$\mathcal{L}_N \simeq \textstyle\bigoplus_{n \in \mathbb{N}} V_{(n,n+N)}.$$

Motivated by the case $q = 1$ [GS99] (Kähler structure of $\mathbb{C}\mathrm{P}^2$) let

$$\Omega^{(0,0)} := \mathcal{L}_0 = \mathcal{A}(\mathbb{C}\mathrm{P}^2_q) \, , \qquad \Omega^{(0,2)} := \mathcal{L}_3 \, .$$

Instead using the spin 1/2 $*$-irrep $\tau : U_q(\mathfrak{su}(2)) \to \mathrm{Mat}_2(\mathbb{C})$

$$\tau(K_1) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} , \qquad \tau(E_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

we define $\Omega^{(0,1)}$ as the $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$-bimodule

$$\Omega^{(0,1)} := \Big\{ v = (v_+, v_-) \in \mathcal{A}(SU_q(3))^2 \,\Big| \\ \Big| \, K_1 K_2^2 \blacktriangleright v = q^{\frac{3}{2}} v, \; (h_{(1)} \blacktriangleright v)\tau(S(h_{(2)})) = \epsilon(h) v, \; \forall \, h \in U_q(\mathfrak{su}(2)) \Big\} \, . \quad (10)$$

It is not difficult to see that $\Omega^{(0,1)}$ has $U_q(\mathfrak{su}(3))\triangleright$-decomposition

$$\Omega^{(0,1)} \simeq \bigoplus\nolimits_{n\geq 1} V_{(n,n)} \oplus \bigoplus\nolimits_{n \geq 0} V_{(n,n+3)} \, .$$

Now Dolbeault cochain and dual chain complexes.

Prop. Set $\quad X := F_2F_1 - 2[2]^{-1}F_1F_2\,, \quad Y := E_2E_1 - 2[2]^{-1}E_1E_2.$
The maps

$$\bar\partial\;:\Omega^{(0,0)} \to \Omega^{(0,1)}\,, \qquad \bar\partial a := (X^* \blacktriangleright a\,,\, E_2 \blacktriangleright a)\,, \tag{11a}$$
$$\bar\partial\;:\Omega^{(0,1)} \to \Omega^{(0,2)}\,, \qquad \bar\partial v := -E_2 \blacktriangleright v_+ - Y \blacktriangleright v_-\,, \tag{11b}$$

with $v = (v_+, v_-)$, are well defined and $\bar\partial^2 = 0$. Similarly, the maps

$$\bar\partial^\dagger :\Omega^{(0,2)} \to \Omega^{(0,1)}\,, \qquad \bar\partial^\dagger b := (-F_2 \blacktriangleright b\,,\, -Y^* \blacktriangleright b)\,, \tag{11c}$$
$$\bar\partial^\dagger :\Omega^{(0,1)} \to \Omega^{(0,0)}\,, \qquad \bar\partial^\dagger v := X \blacktriangleright v_+ + F_2 \blacktriangleright v_-\,, \tag{11d}$$

are well defined and $(\bar\partial^\dagger)^2 = 0$.

As in the commutative case, is $\Omega^{(0,\bullet)}$ a graded algebra?
Since $\Omega^{(0,1)}$ and $\Omega^{(0,2)}$ are $\Omega^{(0,0)}$-bimodules and dim.≤ 2, we know how to multiply, except two 1-forms $v = (v_+, v_-)$ and $w = (w_+, w_-)$, for which we define

$$v \wedge_q w := \tfrac{2}{[2]}(q^{\frac{1}{2}} v_+ w_- - q^{-\frac{1}{2}} v_- w_+).$$

It can be seen that $v \wedge_q w \in \Omega^{(0,2)} = \mathcal{L}_3$ and associativity is OK!
This makes $\Omega^{(0,\bullet)}$ a graded algebra.
Moreover, $\Omega^{(0,\bullet)}$ is a left $U_q(\mathfrak{su}(3))\triangleright$ module $*$-algebra.
Using the (faithful) Haar state φ on $SU_q(3)$ we define a non-degenerate inner product

$$\langle \omega_1, \omega_2 \rangle := \varphi(a_1^* a_2 + v_{1+}^* v_{2+} + v_{1-}^* v_{2-} + b_1^* b_2) \,, \qquad (12)$$

w.r.t. which $U_q(\mathfrak{su}(3))\triangleright$ is unitary [DDL08] and the decomposition $\Omega^{(0,\bullet)} := \bigoplus_n \Omega^{(0,n)}$ is orthogonal.

The operators $\bar\partial$ and $\bar\partial^\dagger$, defined via $\blacktriangleright$ commute with $U_q(\mathfrak{su}(3))\triangleright$. Moreover, from [DDL08] follows $\forall v$ with entries in $\mathcal{A}(SU_q(3))$ that $h^* \blacktriangleright v = (h \blacktriangleright)^\dagger v$, wrt. $\langle\cdot,\cdot\rangle$ and so $\bar\partial^\dagger = (\bar\partial)^\dagger$.

Prop.[gd] $\bar\partial$ is graded-derivation: $\forall a, b \in \Omega^{(0,0)}$, $v \in \Omega^{(0,1)}$, $c \in \Omega^{(0,2)}$,

$$\bar\partial(ab) = a(\bar\partial b) + (\bar\partial a)b, \bar\partial(av) = (\bar\partial a) \wedge_q v + a(\bar\partial v), \bar\partial(va) = (\bar\partial v)a - v \wedge_q (\bar\partial a),$$

while $\bar\partial^\dagger$ satisfies:

$$[\bar\partial^\dagger, a]v = 2[2]^{-1}(F_2 \blacktriangleright a)v_- + q(X \blacktriangleright a)v_+, [\bar\partial^\dagger, a]c = -q^{\frac{3}{2}}(F_2 \blacktriangleright a\,,\ F_1F_2 \blacktriangleright a)c.$$

(Pf. omitted).

Hence $(\Omega^{(0,\bullet)}, \bar\partial)$ gives a left-covariant differential calculus; of dim. 2 (since ‘antiholomorphic’).

The spectral triple

On a Kähler spin$_c$ manifold $\exists$ a 'Dolbeault-Dirac' operator $\bar\partial + \bar\partial^\dagger$.
On $\mathbb{C}\mathrm{P}^2_q$, we need wider generality, for $s \in \mathbb{R}^+$ (a parameter) let

$$D(a, v, b) := (\bar\partial^\dagger v, \bar\partial a + s\bar\partial^\dagger b, s\bar\partial v) \ . \tag{13}$$

Let $\mathcal{H}_+ := \overline{\Omega^{(0,0)} \oplus \Omega^{(0,2)}}^{\langle\cdot,\cdot\rangle}$ and $\mathcal{H}_- = \overline{\Omega^{(0,1)}}^{\langle\cdot,\cdot\rangle}$.
Let $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ with grading $\gamma := 1 \oplus -1$.
Let the spinor rep. of $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$ be just the left multiplication.

They have few good properties

Our spinor rep. is bounded since $\mathcal{A}(\mathbb{C}\mathsf{P}^2_q) \subset \mathcal{A}(SU_q(3)) \subset \mathcal{A}(SU_q(3))^4$ and $\mathcal{H} \subset L^2(SU_q(3), \varphi)^4$, the left multiplication being bounded.

By Prop.[gd], $[D, a]$ acts via left multiplication by elements of $\mathcal{A}(SU_q(3))$,

$$[D, a]\omega = \Big([\bar\partial^\dagger, a]v, [\bar\partial, a]b + s[\bar\partial^\dagger, a]c, s[\bar\partial, a]v \Big) \,, \qquad \forall\, \omega = (b, v, c) \,,$$

and is bounded for any $a \in \mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$.

Equivariance: since our $\mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$-modules of forms are equivariant and the operators $\bar\partial$ and $\bar\partial^\dagger$ are $U_q(\mathfrak{su}(3))$-invariant.

Still need: (essential) self-adjointness and $(D - z)^{-1} \in \mathcal{K}$?
We'll get both by diagonalizing D. For that

Lemma Fix $s = \sqrt{[2]/2}$, then

$$D^2 \omega = [2]^{-1} (\mathcal{C}_q - 2) \blacktriangleright \omega \; .$$

Pf. Direct verification on each grade.

Lemma[spec] The ker$D$ is the constant 0-forms, while non-zero eigenvalues, $\forall n \geq 1$, are

$$\pm \sqrt{\tfrac{2}{[2]}[n][n+2]} \qquad \text{with multiplicity } (n+1)^3 \; ,$$

$$\pm \sqrt{[n+1][n+2]} \qquad \text{with multiplicity } \tfrac{1}{2} n(n+3)(2n+3) \; .$$

Pf. Use $\mathcal{C}_q \blacktriangleright = \mathcal{C}_q \triangleright$ (8), decomposition of $\Omega^{(0,0)}$, $\Omega^{(0,1)}$, $\Omega^{(0,2)}$ into irreps of $U_q(\mathfrak{su}(3))\triangleright$ + symmetry wrt. 0 ($D\gamma + \gamma D = 0$).

Since the eigenvalues of D grow exponentially, $(D+i)^{-\epsilon}$ is trace class $\forall \epsilon > 0$ and the metric dim is 0^+, hence $(D-z)^{-1} \in \mathcal{K}$.
This justifies

Prop.[main] For $s = \sqrt{[2]/2}$ in (13), the datum $(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q), \mathcal{H}, D, \gamma)$ is a 0^+-dimensional $U_q(\mathfrak{su}(3))$-equivariant even spectral triple.

Byproduct:
As before $\mathfrak{H}^n = \ker D$ and Hodge decomposition works, and thus

$$H^0_{\bar\partial}(\mathbb{C}\mathsf{P}^2_q) = \mathbb{C}\,, \qquad H^1_{\bar\partial}(\mathbb{C}\mathsf{P}^2_q) = H^2_{\bar\partial}(\mathbb{C}\mathsf{P}^2_q) = 0\,.$$

In [DD09] we have generalized all that to $\mathbb{C}\mathrm{P}_q^\ell$: with much more involved combinatorics of the Grassman algebra (cf. the talk by F. D'andrea).

The bounded comutators are again assured using special algebraic properties of D, and the compactness by relatig D to an appropriate Casimir operator.

The main result is the construction of 0^+-dimensional $U_q(\mathfrak{su}(\ell+1))$-equivariant even spectral triple.

If ℓ is odd and $N = \frac{1}{2}(\ell+1)$, the spectral triple is a real with KO-dimension $2N \mod 8$.

On the basis of the low dim cases and now the whole series $\mathbb{C}\mathsf{P}_q^\ell$, $\ell \in \mathbb{N}$, the works of Chakraborty-Pal and the fascinating paper [Neshveyev-Tuset07] on QG, built on the impressive collection of papers by Lusztig, Kazhdan and Etingof on analytic generalization of the formal equivalence by Drinfeld (via the twist and associators) between the the categories of $U_h(g)$-modules and $U_g[[h]]$-modules,

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I can declare NCG and QG as matched

Now starts the party:
should clear the isospectral versus exponential spectrum (or intermediate?)
establish such properties of QG representations (without their explicit form), that assure $[D, a] \in \mathcal{B}(\mathcal{H})$ and $|D|^{-1} \in K(\mathcal{H})$
work out the other conditions of Connes.

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