Quantum Complex Projective Spaces

Fredholm modules, K-theory, spectral triples

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Introduction

A case study: the standard Podleś sphere $S^2_q = SU_q(2)/U(1)$ (here $0 < q < 1$).

L. Dąbrowski – A. Sitarz
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- Prescribed Hilbert space + $SU_q(2)$ equivariance = unique real spectral triple (modulo equivalences).
- $\mathrm{Spectrum}(D) = \{\pm[n]_q\}_{n\geqslant 1}$ with $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$.
- The spectrum of D diverges exponentially $\rightsquigarrow$ the resolvent $(D^2 + m^2)^{-1}$ of the Laplacian is of trace class.

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- The representation in DS spectral triple is the direct sum of two copies of the left regular representation.
- Generators of $\mathcal{U}_q(\mathfrak{su}(2))$ are (external) derivations on S^2_q. With these one constructs D.
- D^2 is proportional to the Casimir of $\mathcal{U}_q(\mathfrak{su}(2))$: this explains why eigenv. diverge exponentially.

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- On S^2_q the tadpole diagram – the only basic divergence of ϕ^4 theory in 2D – becomes finite at $q \neq 1$.

$\rightsquigarrow$ Reason: the propagator $(D^2 + m^2)^{-1}$ is of trace class.

- Regularization of QFT with quantum groups symmetries: what about higher dimensional spaces?

Outline

1. Preliminary definitions
   - The quantum $SU(\ell+1)$ group
   - The QUEA $\mathcal{U}_q(\mathfrak{su}(\ell+1))$
   - $S_q^{2\ell+1}$ and $\mathbb{CP}_q^\ell$
2. K-theory and K-homology
   - K-theory
   - K-homology
3. Antiholomorphic forms and real spectral triples
   - The quantum Grassmann algebra
   - The algebra of forms
   - Vector fields and the Dolbeault operator
4. From GDAs to spectral triples
   - Reality and the first order condition
   - A family of spectral triples

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The quantum $SU(\ell+1)$ group

Let $\ell > 1$. For $G := SU(\ell+1)$, the functions $u^i_j : G \to \mathbb{C}$, $\boxed{u^i_j(g) := g^i_j}$, generate a Hopf $*$-algebra $\mathcal{A}(G)$. As abstract $*$-algebra it is defined by the relations

(1) $$u^i_j u^k_l = u^k_l u^i_j \,, \qquad \sum\nolimits_{p \in S_{\ell+1}} (-1)^{||p||} u^1_{p(1)} u^2_{p(2)} \dots u^{\ell+1}_{p(\ell+1)} = 1 \,,$$

where $||p||$ = length of the permutation $p \in S_{\ell+1}$, and with $*$-structure

(2) $$(u^i_j)^* = (-1)^{j-i} \sum\nolimits_{p \in S_\ell} (-1)^{||p||} u^{k_1}_{p(n_1)} u^{k_2}_{p(n_2)} \dots u^{k_\ell}_{p(n_\ell)}$$

where $\{k_1, \dots, k_\ell\} = \{1, \dots, \ell+1\} \smallsetminus \{i\}$ and $\{n_1, \dots, n_\ell\} = \{1, \dots, \ell+1\} \smallsetminus \{j\}$ (as ordered sets). Coproduct, counit and antipode are of 'matrix type'

$$\Delta(u^i_j) = \sum\nolimits_k u^i_k \otimes u^k_j \,, \qquad \varepsilon(u^i_j) = \delta^i_j \,, \qquad S(u^i_j) = (u^j_i)^* \,.$$

Similarly coproduct, counit and antipode of $\mathcal{A}(G_q)$, $\boxed{0 < q < 1}$, are given by the same formulas above, while (1) and (2) becomes:

$$R^{ij}_{kl}(q) u^k_m u^l_n = u^j_l u^i_k R^{kl}_{mn}(q) \,, \qquad \sum\nolimits_{p \in S_{\ell+1}} (-q)^{||p||} u^1_{p(1)} u^2_{p(2)} \dots u^{\ell+1}_{p(\ell+1)} = 1 \,,$$

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The QUEA $\mathcal{U}_q(\mathfrak{su}(\ell+1))$

Symmetries are described by the Hopf $*$-algebra $\mathcal{U}_q(su(\ell+1))$, generated by $\{K_i, K_i^{-1}, E_i, F_i\}_{i=1,\dots,\ell}$, with $K_i = K_i^*$, $F_i = E_i^*$, and relations (a_{ij} = Cartan matrix)

$$[K_i, K_j] = 0\,, \qquad K_i E_j K_i^{-1} = q^{a_{ij}/2} E_j\,, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}$$

$$E_i E_j^2 - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i-j| = 1\,, \qquad [E_i, E_j] = 0 \quad \text{if } |i-j| > 1\,.$$

Coproduct, counit and antipode are given by (with $i = 1, \dots, \ell$)

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$\rightsquigarrow$ With $K_i = q^{H_i}$, at the 0-th order in $\hbar := \log q$ one gets Serre's presentation of $\mathcal{U}(\mathfrak{su}(\ell+1))$.

The Hopf $*$-subalgebra with generators $\{K_i, E_i, F_i\}_{i=1,2,\dots,\ell-1}$ is $\mathcal{U}_q(\mathfrak{su}(\ell))$; its commutant is generated by $K_1 K_2^2 \dots K_\ell^\ell$. We enlarge the algebra with

$$\hat{K} := (K_1 K_2^2 \dots K_\ell^\ell)^{\frac{2}{\ell+1}}\,,$$

and its inverse, and call $\mathcal{U}_q(\mathfrak{u}(\ell))$ the algebra generated by $\mathcal{U}_q(\mathfrak{su}(\ell))$, $\hat{K}$ and $\hat{K}^{-1}$.

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$\mathcal{A}(SU_q(\ell+1))$ as a function algebra

The set of linear maps $\mathcal{U}_q(\mathfrak{su}(\ell+1)) \to \mathbb{C}$ is a Hopf $*$-algebra with operations dual to those of $\mathcal{U}_q(\mathfrak{su}(\ell+1))$. For f, g two such linear maps we define the product by

$$(f \cdot g)(x) := f(x_{(1)})g(x_{(2)}) \qquad \forall\, x \in \mathcal{U}_q(\mathfrak{su}(\ell+1)) \;,$$

the unity is the map $1(x) := \varepsilon(x)$, coproduct, counit, antipode and $*$-involution are

$$\Delta(f)(x,y) := f(xy) \;, \qquad \varepsilon(f) := f(1) \;, \qquad S(f)(x) := f(S(x)) \;, \qquad f^*(x) := \overline{f(S(x)^*)} \;.$$

The Hopf $*$-subalgebra generated by the matrix elements of type 1 irreps is $\mathcal{A}(SU_q(\ell+1))$.

The algebra $\mathcal{A}(SU_q(\ell+1))$ is a $\mathcal{U}_q(\mathfrak{su}(\ell+1))$-bimodule $*$-algebra for the actions:

$$(x \triangleright f)(y) := f(yx) \qquad \text{and} \qquad (f \triangleleft x)(y) := f(xy) \;.$$

If $\mathcal{L}_x f := f \triangleleft S^{-1}(x)$, the two left actions $\triangleright$ and $\mathcal{L}$ are unitary w.r.t. the inner product $\langle\,,\,\rangle$ associated to the Haar state $h : \mathcal{A}(SU_q(\ell+1)) \to \mathbb{C}$:

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$S_q^{2\ell+1}$ and $\mathbb{CP}_q^\ell$

The algebra of 'functions' on $S_q^{2\ell+1}$ is the left $\mathcal{U}_q(\mathfrak{su}(\ell+1))$-module $*$-algebra

$$\mathcal{A}(S_q^{2\ell+1}) := \mathcal{A}(SU_q(\ell+1))^{\mathcal{U}_q(\mathfrak{su}(\ell))} .$$

It is generated by $\boxed{z_i := u^{\ell+1}_{\ell+1-i}}$, $i = 0, \dots, \ell$, with relations

$$\begin{aligned}
z_i z_j &= q^{-1} z_j z_i && \forall\, 0 \leqslant i < j \leqslant \ell , \\
z_i^* z_j &= q z_j z_i^* && \forall\, i \neq j , \\
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[z_\ell^*, z_\ell] &= 0 , \\
z_0 z_0^* + z_1 z_1^* &+ \dots + z_\ell z_\ell^* = 1 .
\end{aligned}$$

The algebra of 'functions' on $\mathbb{CP}_q^\ell$ is the left $\mathcal{U}_q(\mathfrak{su}(\ell+1))$-module $*$-algebra

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It is generated by the matrix entries $\boxed{P_{ij} := z_i^* z_j}$ of a projection.

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Outline

K-theory

Some notations:

$$[0]_q! := 1\ , \qquad [n]_q! := [n]_q \cdot [n-1]_q! \quad \forall\, n \geqslant 1\ ,$$
$$[j_0, \dots, j_n]_q! := \frac{[j_0 + \dots + j_n]_q!}{[j_0]_q! \dots [j_n]_q!} \quad \forall\, j_0, \dots, j_n \geqslant 0\ .$$

For $N \geqslant 0$ let $\Psi_N = (\psi^N_{j_0,\dots,j_\ell})$ be the vector-valued 'function' on $S^{2\ell+1}_q$ with components

$$\psi^N_{j_0,\dots,j_\ell} := [j_0, \dots, j_\ell]_q!^{\frac{1}{2}}\, q^{-\frac{1}{2}\sum_{r<s} j_r j_s} (z_\ell^{j_\ell} \dots z_0^{j_0})^*\ , \qquad \forall\, j_0 + \dots + j_\ell = N\ ,$$
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For all $N \in \mathbb{Z}$, $\Psi_N^\dagger \Psi_N = 1$. Thus, $P_N := \Psi_N \Psi_N^\dagger$ are projections with entries in $\mathcal{A}(\mathbb{CP}^\ell_q)$.

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K-theory

Some notations:

$$[0]_q! := 1\ , \qquad [n]_q! := [n]_q \cdot [n-1]_q! \quad \forall\, n \geqslant 1\ ,$$
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K-homology I: representations

Here $1 \leqslant n \leqslant \ell$. Let $\underline{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and, for $0 \leqslant i < k \leqslant n$, let

$$\underline{\varepsilon}_i^k := (\overbrace{0, 0, \ldots, 0}^{i \text{ times}}, \overbrace{1, 1, \ldots, 1}^{k-i \text{ times}}, \overbrace{0, 0, \ldots, 0}^{n-k \text{ times}}) \,.$$

Let $\mathcal{H}_n := \ell^2(\mathbb{N}^n)$, with orth. basis $|\underline{m}\rangle$.

For any $0 \leqslant k \leqslant n$, a $*$-rep. $\pi_k^{(n)} : \mathcal{A}(S_q^{2n+1}) \to \mathcal{B}(\mathcal{H}_n)$ is defined as follows: $\pi_k^{(n)}(z_i) = 0$ for all $i > k \geqslant 1$, while for the remaining generators (with notation $m_0 := 0$)

$$\pi_k^{(n)}(z_i)\,|\underline{m}\rangle = q^{m_i}\sqrt{1 - q^{2(m_{i+1} - m_i + 1)}}\,\big|\underline{m} + \underline{\varepsilon}_i^k\big\rangle \,, \qquad \forall\, 0 \leqslant i \leqslant k-1 \,,$$

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on the subspace linear span of basis vectors $|\underline{m}\rangle$ satisfying the restrictions

$$0 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_k \,, \qquad m_{k+1} > m_{k+2} > \ldots > m_n \geqslant 0 \,,$$

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K-homology II: Fredholm modules

Since $K^0(\mathbb{CP}^\ell_q) \simeq \mathbb{Z}^{\ell+1}$ (and $K^1(\mathbb{CP}^\ell_q) = 0$) we look for $\ell + 1$ even Fredholm modules.

1st = pull-back of the unique non-trivial even Fred. mod. of $\mathbb{C}$, via the only char. of $\mathcal{A}(\mathbb{CP}^\ell_q)$.

Lemma For $|j - k| > 1$, and for all $a, b \in \mathcal{A}(S^{2n+1}_q)$, we have $\pi^{(n)}_j(a)\,\pi^{(n)}_k(b) = 0$.

As a corollary, we have two $*$-reps $\pi^{(n)}_\pm : \mathcal{A}(S^{2n+1}_q) \to \mathcal{B}(\mathcal{H}_n)$

$$\pi^{(n)}_+(a) := \sum_{\substack{0\leqslant k\leqslant n \\ k \text{ even}}} \pi^{(n)}_k(a)\,, \qquad \pi^{(n)}_-(a) := \sum_{\substack{0\leqslant k\leqslant n \\ k \text{ odd}}} \pi^{(n)}_k(a)\,.$$

Proposition

$\pi^{(n)}_+(a) - \pi^{(n)}_-(a) \in \mathcal{L}^1(\mathcal{H}_n)$ *for all* $a \in \mathcal{A}(\mathbb{CP}^\ell_q)$.

For $n = 1, \ldots, \ell$ a 1-summable Fredholm module is given on $\mathcal{H}_n \oplus \mathcal{H}_n$ by the (restriction of) the rep. $\pi_+ \oplus \pi_-$, grading $\gamma_n = 1 \oplus -1$, and $F_n(v \oplus w) := w \oplus v$.

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Pairing between K-theory and K-homology

Proposition

If μ_k, $0 \leqslant n \leqslant \ell$, *denotes the* n*-th Fredholm module previously introduced, we have*

$$\langle [\mu_n], [P_{-N}] \rangle := \mathrm{Tr}_{\mathcal{H}_n} (\pi_+^{(n)} - \pi_-^{(n)})(\mathrm{Tr}\, P_{-N}) = \binom{N}{n} ,$$

for all $N \in \mathbb{N}$*, where* $\binom{N}{n} := 0$ *when* $n > N$*.*

Proof.

In 3 steps:

1st) the pairing $\langle [\mu_n], [P_{-N}] \rangle$ as a function of q is continuous (it is given by a series that is absolutely convergent for $0 \leqslant q < 1$);

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The matrix M with entries $M_{ij} := \langle [\mu_i], [P_{-j}] \rangle$ is in $GL_{\ell+1}(\mathbb{Z})$, the inverse being

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Thus, the elements above are a basis of $\mathbb{Z}^{n+1}$ as a $\mathbb{Z}$-module, i.e. they generate $\mathbb{Z}^{n+1}$ as abelian group. □

Remark: the Fredholm module μ_ℓ can be realized as 'conformal class' of a regular spectral triple $(\mathcal{A}(\mathbb{CP}^\ell_q), \mathcal{H}_n \oplus \mathcal{H}_n, D)$ — i.e. $F_n := D|D|^{-1}$ — of any summability $d \in \mathbb{R}^+$.

Next: a 'geometric' Dirac operator (0^+-summable and real) is discussed in the following.

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Outline

Homogeneous vector bundles over $\mathbb{CP}^{\ell}_q$

Let $\sigma : \mathcal{U}_q(u(\ell)) \to \mathrm{End}(\mathbb{C}^n)$ be a $*$-representation. The set

$$\begin{aligned}\mathcal{M}(\sigma) &= \mathcal{A}(SU_q(\ell+1)) \boxtimes_\sigma \mathbb{C}^n \\ &:= \left\{ v \in \mathcal{A}(SU_q(\ell+1))^n \,\middle|\, \{\mathcal{L}_{x_{(1)}} \otimes \sigma(x_{(2)})\} v = \epsilon(x) v \;\; \forall\, x \in \mathcal{U}_q(u(\ell)) \right\},\end{aligned}$$

is an $\mathcal{A}(\mathbb{CP}^{\ell}_q)$-bimodule and a left $\mathcal{A}(\mathbb{CP}^{\ell}_q) \rtimes \mathcal{U}_q(\mathfrak{su}(\ell+1))$-module.
It is the analogue of (sections of) an **homogeneous vector bundle** of rank n over $\mathbb{CP}^{\ell}_q$.

A non-degenerate inner product is induced by the canonical one on $\mathcal{A}(\mathbb{CP}^{\ell}_q)^n$:

$$\langle v, w \rangle = \sum\nolimits_{i=1}^{n} h(v_i^* w_i)$$

Which σ gives the bimodule of antiholomorphic forms?

$\mathcal{U}_q(u(\ell))$ is a central extension of $\mathcal{U}_q(\mathfrak{su}(\ell))$ by $\hat{K}$ and $\hat{K}^{-1}$. Any irrep of $\mathcal{U}_q(u(\ell))$ is obtained from an irrep σ of $\mathcal{U}_q(\mathfrak{su}(\ell))$ by defining $\sigma(\hat{K})$ as a (non-zero) multiple of the identity transformation.

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Let $\sigma : \mathcal{U}_q(\mathfrak{u}(\ell)) \to \mathrm{End}(\mathbb{C}^n)$ be a $*$-representation. The set

$$\begin{aligned} \mathcal{M}(\sigma) &= \mathcal{A}(SU_q(\ell+1)) \boxtimes_\sigma \mathbb{C}^n \\ &:= \left\{ v \in \mathcal{A}(SU_q(\ell+1))^n \,\middle|\, \{\mathcal{L}_{x_{(1)}} \otimes \sigma(x_{(2)})\} v = \epsilon(x) v \ \ \forall\, x \in \mathcal{U}_q(\mathfrak{u}(\ell)) \right\} , \end{aligned}$$

is an $\mathcal{A}(\mathbb{CP}^\ell_q)$-bimodule and a left $\mathcal{A}(\mathbb{CP}^\ell_q) \rtimes \mathcal{U}_q(\mathfrak{su}(\ell+1))$-module.
It is the analogue of (sections of) an **homogeneous vector bundle** of rank n over $\mathbb{CP}^\ell_q$.

A non-degenerate inner product is induced by the canonical one on $\mathcal{A}(\mathbb{CP}^\ell_q)^n$:

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Fundamental representations of $\mathcal{U}_q(\mathfrak{su}(\ell))$

Let $k = 1, \ldots, \ell - 1$,

$$\Lambda_k := \left\{ \underline{i} = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k \,\middle|\, 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant \ell \right\} ,$$

and $W_k \simeq \mathbb{C}^{\binom{\ell}{k}}$ be the vector space with basis vectors labelled by elements in Λ_k, represented pictorially by **Young tableaux** (YT).

Generators of $\mathcal{U}_q(\mathfrak{su}(\ell))$ are represented by insertion/deletion operators:

- $K_j(\mathrm{YT}) = q^{\frac{1}{2}N_j}\,\mathrm{YT}$ where N_j = number of rows of length j in the Young tableau minus the number of rows of length $j+1$ (either 0 or ± 1);
- F_j add a slot to the row of length j, if any, otherwise it gives 0; E_j is the adjoint of F_j.

Example: $\ell = 3$ and $k = 2$.

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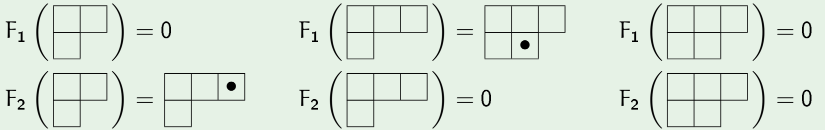

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Facts:

- W^k is the irrep with highest weight $\delta^k = (\overbrace{0, \ldots, 0}^{k-1 \text{ times}}, 1, \overbrace{0, \ldots, 0}^{\ell-k-1 \text{ times}})$;
- for $q = 1$, $W_k \simeq \wedge^k W_1 \;\forall\, 0 \leqslant k \leqslant \ell$ where $W_0 = W_\ell = \mathbb{C}$ is the rep. given by ε.
  Next point: to define an intertwiner $\wedge_q : W_j \otimes W_k \to W_{j+k}$ deformation of the wedge product, for all $j + k \leqslant \ell$.

The quantum Grassmann algebra, I

Elements of $S_{k,n-k} := (S_k \times S_{n-k}) \backslash S_n$ are called $(k, n-k)$-shuffles.

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The product $\wedge_q$ is surjective, associative, and a left $\mathcal{U}_q(\mathfrak{su}(\ell))$-module map.

Crucial to prove associativity is the factorization property above.

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$\rightsquigarrow$ *Let* $W_\bullet := \oplus_{k=0}^{\ell} W_k$. $\mathtt{Gr}_q^\ell := (W_\bullet, \wedge_q)$ *is a graded associative algebra – generated by* W_1 *– and a left* $\mathcal{U}_q(\mathfrak{su}(\ell))$*-module algebra.* $\dim_{\mathbb{C}} \mathtt{Gr}_q^\ell = 2^\ell$.

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The quantum Grassmann algebra, II

For $x \in W_1$, we define left/right 'exterior product' $\mathfrak{e}_x^{L,R} : W_k \to W_{k+1}$ as

$$\mathfrak{e}_x^L w = x \wedge_q w \,, \qquad \mathfrak{e}_x^R w = (-q)^k w \wedge_q x \,,$$

and left/right 'contraction' as the adjoint $\mathfrak{i}_x^{L,R}$ of $\mathfrak{e}_x^{L,R}$ w.r.t. the canonical inner product on $W_\bullet$.

An antilinear map $J : W_k \to W_{\ell-k}$ is given by

$$(Jw)_{\underline{i}} = (-q^{-1})^{|\underline{i}|} q^{\frac{1}{4}\ell(\ell+1)} \, \overline{w_{\underline{i}^c}} \,,$$

where $|\underline{i}| := i_1 + \ldots + i_{\ell-k}$, $\underline{i}^c = (1, \ldots, \ell) \smallsetminus \underline{i}$, and $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$.

Proposition

The map J is equivariant (i.e. $x^* J = J\, S(x)$ for all $x \in \mathcal{U}_q(\mathfrak{su}(\ell))$), and has square

$$J^2 = (-1)^{\lfloor \frac{\ell+1}{2} \rfloor} \,.$$

Conjugating with J transforms the left exterior product into the right contraction:

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Antiholomorphic forms on $\mathbb{CP}^\ell_q$

For $N \in \mathbb{Z}$, the irrep. of $\mathcal{U}_q(\mathfrak{su}(\ell))$ with h.w. δ^k is lifted to a $\sigma^N_k : \mathcal{U}_q(\mathfrak{u}(\ell)) \to \mathrm{End}(W_k)$ by

$$\sigma^N_k(\hat{K}) = q^{k - \frac{\ell}{\ell+1}N} \cdot \mathrm{id}_{W_k} \;.$$

Since $\sigma^N_k \simeq \sigma^0_k \otimes \sigma^N_0$,

$$\Omega^k_N := \mathcal{M}(\sigma^N_k) \simeq \Omega^k_0 \otimes_{\mathcal{A}(\mathbb{CP}^\ell_q)} \Omega^0_N \;.$$

$\Omega^k_0 =$ antiholomorphic k-forms. $\Omega^0_N =$ 'sections of line bundles'.

If ℓ is odd: $\Omega^0_{\frac{1}{2}(\ell+1)} =$ square root of the canonical bundle, $\Omega^k_{\frac{1}{2}(\ell+1)} =$ chiral spinors.

An associative product $\wedge_q : \Omega^k_N \times \Omega^{k'}_{N'} \to \Omega^{k+k'}_{N+N'}$ is given by

$$\omega \wedge_q \omega' := a \cdot a'(v \wedge_q v') \;,$$

where $\omega = av \in \Omega^k_N$ and $\omega' = a'v' \in \Omega^{k'}_{N'}$, $a, a' \in \mathcal{A}(SU_q(\ell+1))$, $v \in W_k$, $v' \in W_{k'}$.

$\Omega^\bullet_0 := \bigoplus_{k=0}^{\ell} \Omega^k_0$ is a graded associative algebra.

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$$\sigma_k^N(\hat{K}) = q^{k - \frac{\ell}{\ell+1} N} \cdot \mathrm{id}_{W_k} \, .$$

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Ω_0^k = antiholomorphic k-forms. Ω_N^0 = 'sections of line bundles'.

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If ℓ is odd: $\Omega^0_{\frac{1}{2}(\ell+1)}$ = square root of the canonical bundle, $\Omega^k_{\frac{1}{2}(\ell+1)}$ = chiral spinors.

An associative product $\wedge_q : \Omega^k_N \times \Omega^{k'}_{N'} \to \Omega^{k+k'}_{N+N'}$ is given by

$$\omega \wedge_q \omega' := a \cdot a'(v \wedge_q v') \ ,$$

where $\omega = av \in \Omega^k_N$ and $\omega' = a'v' \in \Omega^{k'}_{N'}$, $a, a' \in \mathcal{A}(SU_q(\ell+1))$, $v \in W_k$, $v' \in W_{k'}$.

$\Omega^{\bullet}_0 := \bigoplus_{k=0}^{\ell} \Omega^k_0$ is a graded associative algebra.

Antiholomorphic forms on $\mathbb{CP}^\ell_q$

For $N \in \mathbb{Z}$, the irrep. of $\mathcal{U}_q(\mathfrak{su}(\ell))$ with h.w. δ^k is lifted to a $\sigma^N_k : \mathcal{U}_q(\mathfrak{u}(\ell)) \to \mathrm{End}(W_k)$ by

$$\sigma^N_k(\hat{K}) = q^{k - \frac{\ell}{\ell+1}N} \cdot \mathrm{id}_{W_k} \ .$$

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Let $\{e^i\}_{i=1,\ldots,\ell}$ be the canonical basis of $W_1 \simeq \mathbb{C}^\ell$ and

$$X_i := K_i K_{i+1} \ldots K_\ell \hat{K}^{-1} [E_i, [E_{i+1}, \ldots [E_{\ell-1}, E_\ell]_q]_q]_q \;, \qquad i = 1, \ldots, \ell \;,$$

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$$X = \textstyle\sum_i e^i X_i \in \mathcal{U}_q(\mathfrak{su}(\ell+1)) \boxtimes_\sigma W_1$$

(on the first leg we use the right adjoint action composed with S^{-1}.) By invariance of X,

$$\bar\partial := \textstyle\sum_{i=1}^{\ell} \mathcal{L}_{\hat{K} X_i} \otimes \mathfrak{e}^L_{e^i}$$

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Outline

Reality and the first order condition

The adjoint of $\bar{\partial}$ is

$$\bar{\partial}^\dagger = \sum_{i=1}^{\ell} \mathcal{L}_{X_i^* \hat{K}} \otimes i^L_{e^i}$$
$$\equiv -q^{N-k+1} \sum_{i=1}^{\ell} q^{-2i} \mathcal{L}_{S(X_i^* \hat{K})} \otimes i^R_{e^i} \;.$$

An equivariant antilinear map $\mathcal{J}_0 : \Omega^k_N \to \Omega^{\ell-k}_{\ell+1-N}$ is given by

$$\mathcal{J}_0 := (* \otimes J)(\mathcal{L}_{K_{2\rho}^{-1}} \otimes \mathrm{id}) \,,$$

where $K_{2\rho} = (K_1^\ell K_2^{2(\ell-1)} \dots K_j^{j(\ell-j+1)} \dots K_\ell^\ell)^2$.

Proposition

The antiunitary part $\mathcal{J}$ of $\mathcal{J}_0$ satisfies

(i) $\mathcal{J}^2 = (-1)^{\lfloor \frac{\ell+1}{2} \rfloor}$;

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A family of spectral triples

Data:

- $\mathcal{H}_N :=$ Hilbert space completion of $\Omega^\bullet_N$;
- $\gamma_N := +1$ on even forms and -1 on odd forms;
- $\mathcal{A}(\mathbb{CP}^\ell_q)$ acts by left multiplication on $\Omega^\bullet_N$ (completed to bounded operators);
- $\mathcal{J}$ extends to an (antilinear) isometry $\mathcal{H}_N \to \mathcal{H}_{\ell+1-N}$;
- an unbounded densely defined symmetric operator D_N on $\mathcal{H}_N$ is

$$D_N := q^{\frac{1}{2}(k-N)}\bar{\partial} + q^{\frac{1}{2}(k-N-1)}\bar{\partial}^\dagger \ ,$$

  and satisfies $\mathcal{J}D_N = D_{\ell+1-N}\mathcal{J}$ on $\Omega^\bullet_N$. For $q = 1$:
  - D_0 = Dolbeault-Dirac operator,
  - D_N = twist of D_0 with the Grassmannian connection Ω^0_N,
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Theorem

$(\mathcal{A}(\mathbb{CP}^\ell_q), \mathcal{H}_N, D_N, \gamma_N)$ is a 0^+-dimensional equivariant even spectral triple. If ℓ is odd and $N = \frac{1}{2}(\ell+1)$, the spectral triple is real with KO-dimension $2\ell \bmod 8$.

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Theorem

$(\mathcal{A}(\mathbb{CP}_q^\ell), \mathcal{H}_N, D_N, \gamma_N)$ is a 0^+-dimensional equivariant even spectral triple. If ℓ is odd and $N = \frac{1}{2}(\ell+1)$, the spectral triple is real with KO-dimension $2\ell \bmod 8$.

A family of spectral triples

Data:

- $\mathcal{H}_N :=$ Hilbert space completion of $\Omega^\bullet_N$;
- $\gamma_N := +1$ on even forms and -1 on odd forms;
- $\mathcal{A}(\mathbb{CP}^\ell_q)$ acts by left multiplication on $\Omega^\bullet_N$ (completed to bounded operators);
- $\mathcal{J}$ extends to an (antilinear) isometry $\mathcal{H}_N \to \mathcal{H}_{\ell+1-N}$;
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$$D_N := q^{\frac{1}{2}(k-N)}\bar{\partial} + q^{\frac{1}{2}(k-N-1)}\bar{\partial}^\dagger ,$$

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