Algebraic models for equivariant cohomology of noncommutative spaces

Lucio Simone Cirio

(Max Planck Institute for Mathematics - Bonn, Germany)

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Outline

- Equivariant cohomology: classical models, equiv. HKR thm
- i) Deformation of symmetries by Drinfeld twists (arXiv:[math.QA]0706.3602v3)
- ii) Drinfeld-Jimbo quantum groups (with C.Pagani, A.Zampini)

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- $W_{\mathfrak{g}} = Sym(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ as algebraic model for $\Omega(EG)$:

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- **Cartan model**: by $\Phi = \exp\{\theta^a \otimes i_a\} \in Aut^{\mathfrak{g}}(W_{\mathfrak{g}} \otimes \Omega(\mathcal{M}))$

  $((W_{\mathfrak{g}} \otimes \Omega(\mathcal{M}))_{|bas}, d \otimes 1 + 1 \otimes d) \xrightarrow{\Phi} ((Sym(\mathfrak{g}^*) \otimes \Omega(\mathcal{M}))^{\mathfrak{g}}\,,\; d_G)$

  with induced differential $(d_G\alpha)(\xi) = d(\alpha(\xi)) - i_\xi(\alpha(\xi))\,, \xi \in \mathfrak{g}$

- To define a Weil model for nc symmetries we need:
  - a deformation of $\mathfrak{U}(\mathfrak{g})$: Drinfeld twists, Drinfeld-Jimbo ...
  - a nc differential calculus Γ on a Hopf module algebra
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Thm (*Atiyah&Segal*): $H^i_G(X, \mathbb{Q}) \cong (K^i_G(X) \otimes \mathbb{Q})_{\mathfrak{I}}$ as $R(G)$ modules ($\mathfrak{I}$ the ideal of repr's of (virtual) dimension 0, $i \in \mathbb{Z}_2$).

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- Equiv. cohom. for (nc) algebras as $HC(A \rtimes \mathcal{H})$ or $HC_{\mathcal{H}}(A)$

¿ there exist a 'geometric' description (via nc deloc equiv diff forms) of $HC_{\mathcal{H}}(A)$ which, localized, reduces to a nc Weil model ?

The Cartan calculus: $G \circlearrowright \mathcal{M}$ via operators (i, L, d)

- for $\mathfrak{g}$ given by $[e_a, e_b] = f_{ab}{}^c e_c$ introduce the super Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ generated by $\{\xi_a, e_a, d\}$ and relations

$$\begin{array}{lll} [e_a, e_b] = f_{ab}{}^c e_c & [e_a, \xi_b] = f_{ab}{}^c \xi_c & [e_a, d] = 0 \\ [\xi_a, \xi_b] = 0 & [\xi_a, d] = e_a & [d, d] = 0 \end{array}$$

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- The category of (left) Hopf-module algebras ${}_{\mathcal{H}}Alg$:

- monoidal category: $h \triangleright (A \otimes B) = (h_{(1)} \triangleright A) \otimes (h_{(2)} \triangleright B)$
- $(\mathcal{H}, R)$ quasitriang. ➥ braiding $\Psi_{A;B} = \tau \circ R : A \otimes B \to B \otimes A$
- braided tensor product $A \underline{\otimes} B$: the algebra structure is

$$\begin{aligned} (a_1 \otimes b_1) \cdot (a_2 \otimes b_2) &= (a_1 \otimes 1) \cdot [\Psi_{B;A}(b_1 \otimes a_2)] \cdot (1 \otimes b_2) \\ &= a_1 (R^{(2)} \triangleright a_2) \otimes (R^{(1)} \triangleright b_1) b_2 \end{aligned}$$

Deformed symmetries (Drinfel'd-Jimbo QEA, Drinfel'd twists ...) by covariance 'generate' nc geometries (and vice versa!)

deformation of $\mathcal{H}$ as Hopf algebra	**covariance** $\Longleftrightarrow$	deformation of $\cdot_A$ in ${}_{\mathcal{H}}Alg$

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The Drinfel'd twist of a Hopf algebra $\mathcal{H}$

Let $\chi = \chi^{(1)} \otimes \chi^{(2)}$ be an invertible element of $\mathcal{H} \otimes \mathcal{H}$, satisfying

- $(1 \otimes \chi)(id \otimes \triangle)\chi = (\chi \otimes 1)(\triangle \otimes id)\chi$ (cocycle condition)
- $(id \otimes \epsilon)\chi = (\epsilon \otimes id)\chi = 1$ (counitality)

Then it is possible to define $\mathcal{H}^\chi = (\mathcal{H}, \triangle^\chi, \epsilon, S^\chi)$ with

- the same algebra structure and counity
- twisted coproduct $\triangle^\chi(h) = \chi\triangle(h)\chi^{-1}$
- twisted antipode $S^\chi(h) = US(h)U^{-1}$ where $U = \chi^{(1)}(S\chi^{(2)})$
- if $(\mathcal{H}, R)$ quasitriangular, $R^\chi = \chi^{op}R\chi^{-1}$

Theorem ($\mathcal{H}^\chi$-module structure)

Let A be an $\mathcal{H}$-module algebra, $\chi \in \mathcal{H} \otimes \mathcal{H}$ a twist element for $\mathcal{H}$. Then $\mathcal{H}^\chi$ acts covariantly on the deformed algebra $A_\chi = (A, \cdot_\chi)$

$$a \cdot_\chi b = \cdot(\chi^{-1} \triangleright (a \otimes b)) = ((\chi^{-1})^{(1)} \triangleright a) \cdot ((\chi^{-1})^{(2)} \triangleright b)$$

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A (graded)commutative $\Longleftrightarrow$ A_χ braided (graded)commutative

$$m \circ \tau_{A,A} = m \qquad\qquad m_\chi \circ \Psi_{A,A} = m_\chi$$

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- Cartan calculus on A_χ: repr of $\mathfrak{U}^\chi(\tilde{\mathfrak{g}})$, twisted derivations

$$L_X(a_1 \cdot_\chi a_2) := (L_{X_{(1)}} a_1) \cdot_\chi (L_{X_{(2)}} a_2)$$ and similar formula for i_X

- Examples: Moyal planes, toric isospectral deformations $(\mathbb{T}^n_\theta, S^n_\theta)$

Think of A_χ as $\Omega(\mathcal{M}_\theta)$. Note that the commutation relations of $\tilde{\mathfrak{g}} = (L, i, d)$ are not deformed; we say A_χ is a twisted $\tilde{\mathfrak{g}}$-da.

- Example: action of $\mathfrak{so}(5) = \{H_1, H_2, E_r\}$ on S^4_Θ

- twist $\mathfrak{U}(\mathfrak{so}(5))_{[[\Theta]]}$ with $\chi = \exp\left\{-\frac{i\Theta}{2}(H_1 \otimes H_2 - H_2 \otimes H_1)\right\}$
- $\triangle^\chi(H_i) = H_i \otimes 1 + 1 \otimes H_i$, $\triangle^\chi(E_r) = E_r \otimes \lambda_r^{-1} + \lambda_r \otimes E_r$
  where $\lambda_r = \exp\left\{-\frac{i\Theta}{2}(r_1 H_2 - r_2 H_1)\right\}$
- Lie derivative on S^4_Θ (same rules for $i_{H_{1,2}}$, i_{E_r}):

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\begin{aligned}
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- Example: action of $\mathfrak{so}(2n)$ on Moyal $\mathbb{R}^{2n}_\Theta$
  - now the relevant twist element is $\chi = \exp\left\{-\frac{i\Theta^{ab}}{2} P_a \otimes P_b\right\}$
  - the twisted symmetry is $\mathfrak{U}^\chi(\mathfrak{e}_{2n})$, with $\mathfrak{e}_{2n} = \mathbb{R}^{2n} \rtimes \mathfrak{so}(2n)$
  - from $[M_{\mu\nu}, P_a] = g_{\mu a} P_\nu - g_{\nu a} P_\mu$: $\triangle^\chi(M_{\mu\nu}) = \triangle(M_{\mu\nu}) + \frac{i\Theta^{ab}}{2}[(\delta_{\mu a} P_\nu - \delta_{\nu a} P_\mu) \otimes P_b + P_a \otimes (\delta_{\mu b} P_\nu - \delta_{\nu b} P_\mu)]$

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- $(\mathfrak{g}, B)$ quadratic; $\mathfrak{g}^B = \mathfrak{g}_{(ev)} \oplus \mathfrak{g}_{(odd)}$ as $\tilde{\mathfrak{g}}$, only $[\xi_a, \xi_b] = B_{ab}$.
- the twisted nc Weil algebra is $\mathcal{W}^\chi_\mathfrak{g} = \mathfrak{U}^\chi(\mathfrak{g}^B)$.
- Cartan calculus on $\mathcal{W}^\chi_\mathfrak{g}$ by (twisted) inner derivations:

$$\boxed{L_a = ad^\chi_{e_a} \ , \ i_a = ad^\chi_{\xi_a} \ , \ d = [\mathcal{D}, \ ]} \qquad (\mathfrak{g} = \{e_a\}, \mathcal{D} \text{ in } \mathcal{Z}(\mathcal{W}^\chi_\mathfrak{g}))$$

- we use the braided monoidal structure of twisted $\tilde{\mathfrak{g}}$-da's (i.e. in ${}_{\mathfrak{U}^\chi(\tilde{\mathfrak{g}})}\mathcal{A}lg$) to consider $\mathcal{W}^\chi_\mathfrak{g} \underline{\otimes} A_\chi$ together with its Cartan calculus

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Twisted noncommutative Weil model

The nc equivariant cohomology of a twisted nc $\tilde{\mathfrak{g}}$-da A_χ is defined as the cohomology of the twisted nc Weil complex

$$\mathcal{H}_{\mathfrak{U}^\chi(\mathfrak{g})}(A_\chi) = ((\mathcal{W}^\chi_\mathfrak{g} \underline{\otimes} A_\chi)^G_{hor},\ d \otimes 1 + 1 \otimes d)$$

  - we also define a twisted nc Kalkman map to get a Cartan model

- Properties of twisted nc equivariant cohomology:
  - Basic cohomology ring: $\mathcal{H}_{\mathfrak{U}^{\chi}(\mathfrak{g})}(\mathbb{C}) = (\mathcal{W}^{\chi}_{\mathfrak{g}})^{\mathfrak{g}}_{hor} \cong (\mathfrak{U}(\mathfrak{g}))^{\mathfrak{g}}$

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Using dual Drinfeld twists on $Fun(G)$ we have

$$\boxed{\mathcal{H}_{\mathcal{U}^{\chi}(\mathfrak{g})}((Fun_{\gamma}(G))^{coP}) = \mathcal{H}_{\mathcal{U}^{\chi}(\mathfrak{p})}({}^{coG}(Fun_{\gamma}(G))) = \mathfrak{U}(\mathfrak{p})^{\mathfrak{p}}}$$

Example: $\mathcal{H}_{\mathcal{U}^{\chi}(\mathfrak{so}(5))}(S^4_{\theta}) = \mathfrak{U}(\mathfrak{so}(4))^{SO(4)} \cong Sym(\mathfrak{t}^2)^W$

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  - Note that abelian Drinfeld twists on T are trivial ➥ quite classical behaviour of $\mathcal{H}_{\mathfrak{U}^\chi(\mathfrak{g})}$. Consistent with $HC(A_\chi \rtimes \mathfrak{U}^\chi(\mathfrak{g}))$, since when χ is a 2-cocycle $A_\chi \rtimes \mathcal{H}^\chi \cong A \rtimes \mathcal{H}$ as algebras.

(joint work with C. Pagani, A. Zampini)

- We want to test our models on Drinfeld-Jimbo QEA's $\mathfrak{U}_q(\mathfrak{g})$.

As guiding example we take $\mathfrak{U}_q(\mathfrak{su}(2))$ and the q-Hopf fibration $U(1) \hookrightarrow SU_q(2) \to S^2_q$. Goal: to compute $\boxed{\mathcal{H}_{\mathfrak{U}_q(\mathfrak{su}(2))}(S^2_q)}$

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- What is different from the Drinfeld twists case:

 - if A is a $\mathfrak{U}(\mathfrak{g})$-mod algebra there is no canonical $\mathfrak{U}_q(\mathfrak{g})$-covariant differential calculus on A_q (cfr with Drinfeld twists and toric isospectral deformations $\Omega_\chi = (\Omega, \wedge_\chi)$)

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  - if A is a $\mathfrak{U}(\mathfrak{g})$-mod algebra there is no canonical $\mathfrak{U}_q(\mathfrak{g})$-covariant differential calculus on A_q (cfr with Drinfeld twists and toric isospectral deformations $\Omega_\chi = (\Omega, \wedge_\chi)$)
  - needed a q-deformed Cartan calculus along the generators of the symmetry (quantum tangent vectors). So far only for bicovariant differential calculi (Woronowicz) and induced calculi on quantum homogeneous spaces.

(joint work with C. Pagani, A. Zampini)

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- does (and should) the equivariant cohomology depend on the choice of a FODC on A_q? Compare with $HC(A_q \rtimes \mathfrak{U}_q(\mathfrak{g}))$. (More natural to consider twisted cyclic homology HC^σ?)

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- we get respectively a $2D$ and $3D$ calculus on S^2_q with different cohomologies. Which one (if any..) has to be regarded as the q-deformed de Rham cohomology?

- any relation with cyclic homology $HC(SU_q(2) \rtimes \mathfrak{U}(\mathfrak{u}(1)))$?