Net Cohomology and the Construction of Physical Models

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08.31.2009

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Dedicated to John E. Roberts
on the occasion of his 70th birthday

... yesterday there was something to learn
today something to understand,
perhaps...

Outline

Introduction

The Streater and Wilde model

- Global gauge group and superselection structure
- Cohomology and Completeness of Superselection Sectors

Nets on S^1 associated to a lattice (loop group)

- Lattices, torus, loop groups and S^1-extensions
- Local net associated to a lattice
- Net cohomology of Lattice models

Conclusions and Outlook

Introduction

Aims

- Net cohomology is a cohomological theory of a partially ordered set $\mathcal{P}$ with coefficients in a net of (non Abelian) von Neumann algebras $\mathcal{N}_{\mathcal{P}}$ indexed by $\mathcal{P}$
- In another way: it is a strong tool of AQFT for the description of different QFTs, on different Spacetimes, with different Superselection Criteria, for different Models
- Introduced by J. Roberts around '76 with new ideas about Abstract Non Abelian Cohomology based on *n*-categories
- and physical ideas from the description of (global and local) Quantum Field Theory also proposed by R. Haag

Results

- Equivalent description of DHR Superselection theory [Roberts '90]
- Trivial Sectors of the Massless scalar free field in $1 + 3$ dimensions [Buchholz, Doplicher, Longo, Roberts '92]
- Classification of subsystems for the $1 + 3$-dimensional theories [Carpi, Conti, Doplicher, Roberts]
- Massless scalar free field in $1 + 1$ dimensions $\equiv$ The Streater and Wilde model [C. '05, '08]
- Description of QFT Topological Sectors [Brunetti, Ruzzi, Franceschini, Moretti '08]
- Superselection Theory on Curved Spacetimes and approaches to the description of electromagnetic and QED charges [Roberts 70's; Roberts, Ruzzi, Vasselli]
- ...

- Very New
  Net Representations and a New Superselection Criterion (generalizing DHR) for the non trivial topological nature of the spacetime [Brunetti, Ruzzi]

DHR	$\pi\restriction\mathcal{O}^\perp \cong \iota\restriction\mathcal{O}^\perp$	$\mathcal{O}$ Minkowski double cone
BF	$\pi\restriction\mathcal{C}^\perp \cong \iota\restriction\mathcal{C}^\perp$	$\mathcal{C}$ Minkowski spacelike cone
BR	$\{\pi, \Psi\}\restriction\mathcal{O}^\perp \cap N$	
	$\cong \{\iota, \Psi\}\restriction\mathcal{O}^\perp \cap N$	$\mathcal{O}$ diamond on curved spacetime M Ψ family of unitaries $N \subset M$ simply connected sub-spacetime

Streater and Wilde model

- net $\mathcal{A}$: theory of the (potential of the) massless scalar free field in 1+1 dimensions in time zero formulation [Hislop, Longo '82] modular structure and split property
- non-regular representation of the field net $\mathcal{F}$ on a non-separable Hilbert space: way to bypass the unphysical space with non-definite metrics
- global compact gauge group obtained as the Bohr compactification of $\mathbb{R}^2$
- time-zero formulation versus chiral formulation and their equivalence by d'Alembert formula
- classification of the sectors of the net of observables as restrictions of solitonic (twisted) and non solitonic (untwisted) automorphisms

Reference:

Massless scalar free Field in 1+1 dimensions I: Weyl algebras Products and Superselection Sectors.
Reviews in Mathematical Physics **21**, No. 6 (2009), 735-780.

Twisted crossed product of Weyl algebras

(V_a, σ_a) and (V_f, σ_f) symplectic spaces of the observables and (putative) fields and $\mathcal{W}(V)$ the respective Weyl algebras

$$\mathcal{W}(V_f, \sigma_f) = \mathcal{W}(V_a, \sigma_a) \otimes \mathcal{W}(N, \sigma_N) \rtimes_z \mathcal{U}(C)$$

where z is the 2-cocycles of the extension

Hence we obtain the following simple currents extension

$$\begin{array}{ccccc}
 & & \mathcal{A} \otimes \mathcal{Z}_b & & \mathcal{B} \rtimes \mathcal{U}(C) \\
 & & \| & & \| \\
\mathcal{A} = \mathcal{Q}^{\mathcal{G}_q} & \subset & \mathcal{B} = \mathcal{E}^{\mathcal{G}_q} = \mathcal{C}^{\mathcal{G}_c} = \mathcal{F}^{\mathcal{G}} & \subset & \mathcal{C} = \mathcal{F}^{\mathcal{G}_q} \\
\cap & & \cap & & \cap \\
\mathcal{Q} & \subset & \mathcal{E} = \mathcal{F}^{\mathcal{G}_c} & \subset & \mathcal{F} \\
\| & & \| & & \| \\
\mathcal{A} \rtimes \mathcal{U}(Q) & & \mathcal{B} \rtimes \mathcal{U}(Q) & & \mathcal{B} \rtimes \mathcal{U}(Q) \rtimes \mathcal{U}(C)
\end{array}$$

Cohomology and Completeness of Sectors

- accurate definition of the nets on different index sets of the time zero line $\mathbb{R}$:
  - $\mathcal{I}$ open bounded intervals
  - $\mathcal{D}$ open double intervals $E = I_1 \cup I_2$, with $I_1 \perp I_2$
- study relevant properties (additivity, locality, Haag duality) of nets $\mathcal{A}$ and $\mathcal{F}$ both on $\mathcal{I}$ and $\mathcal{D}$
- anyonic commutation rules (generalization of $\mathbb{Z}_2-$Bose/Fermi alternative)
  the net $\mathcal{F}$ is graded by $\mathbb{R} \times \mathbb{R}$, i.e. $\mathbb{R} \times \mathbb{R}$ acts on $\mathcal{F}$, giving a $\mathbb{R} \times \mathbb{R}$-decomposition of any $\mathcal{F}(I)$ and a covariant action of the space inversion S
- for $\mathcal{F}$, locality and duality are effected by the $\mathbb{R} \times \mathbb{R}$-grading
- $\mathcal{F}$ is non-separably represented ($\Rightarrow$ no split property), but is possible to compute its 1-cohomology $Z^1_{\mathcal{P}}(\mathcal{F})$ both for $\mathcal{P} = \mathcal{I}, \mathcal{D}$, that results to be trivial, i.e. absence of non trivial sectors
- The result hold for a large class of Weyl algebras models

Relevant non trivial superselection condition

- for $E' = J_l \cup I_3 \cup J_r$ the causal complement of E

$$\mathcal{A}_{\mathcal{D}}^{\mathcal{I}}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subsetneq \mathcal{A}_{\mathcal{D}}^{\mathcal{I}d}(E) := \big(\mathcal{A}(J_l) \vee \mathcal{A}(I_3) \vee \mathcal{A}(J_r)\big)'$$
$$\equiv \mathcal{A}_{\mathcal{D}}(E) := \pi_a\big(\mathcal{W}\big(V_a(E)\big)\big)''$$

- the larger algebras $\mathcal{A}_{\mathcal{D}}^{\mathcal{I}d}(E)$ contains non-trivial charge carrying observable operators supported on $I_1 \cup I_2$ that are not present in $\mathcal{A}_{\mathcal{I}}(I_1) \vee \mathcal{A}_{\mathcal{I}}(I_2)$. This implies the non triviality for the superselection of $\mathcal{A}$
- similar considerations are done in Conformal AQFT, see [Kawahigashi, Longo, Müger '01]
- completeness of superselection theory of $\mathcal{A}$ is obtained via the triviality of $Z^1_{\mathcal{P}}(\mathcal{F})$, using general theorems for the change of index set [Roberts '04] and through the action of the global gauge group $\mathcal{G}$
- $\mathrm{Rep}^{\perp}\,\mathcal{A}_{\mathcal{I}}$, the braided category of DHR representation of $\mathcal{A}_{\mathcal{I}}$, is the composition of two full symmetric subcategories, labeled by the the charges C and Q

Reference:
Massless scalar free Field in 1+1 dimensions II:
Net Cohomology and Completeness of Superselection Sectors.
arXiv:0811.4673

Nets on S^1 associated to a lattice (loop group)

- Example of Vertex Operator Algebras (VOA)
- Related to the representation theory of Lie groups
- Building block of conformal quantum field models, e.g. Moonshine VOA
- Rational model, i.e. finite number of Superselection Sectors
- Approached in AQFT by the theory of Subfactors [Kawahigashi, Longo, Müger, Carpi, Rehren, ...]

Reference: [Dong, Xu '06]

Positive even lattice and its dual

- Q positive definite even lattice
  free Abelian group of finite rank d with a
  $\mathbb{Z}$-valued bilinear form $\langle \cdot\,, \cdot \rangle$ such that for $\alpha,\ \beta \in Q$
  symmetric: $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$
  positive definite: $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq 0$
  non-degenerate: $\langle \alpha, \beta \rangle = 0$ for all $\beta \in Q \Rightarrow \alpha = 0$
  even: $\langle \alpha, \alpha \rangle \in 2\,\mathbb{N}$
- $\{\alpha_i\}_{i \in 1,\dots d}$ base of Q $\alpha = \sum_{i=1}^{d} n_i \alpha_i\,, \quad n_i \in \mathbb{Z}$
- $\mathbb{R}Q$ real vector space on Q
- assume $\langle \cdot\,, \cdot \rangle$ enlarges to $\mathbb{R}Q$
- dual lattice of Q
  $Q^* := \{\beta \in \mathbb{R}Q : \langle \beta, \alpha \rangle \subset \mathbb{Z} \text{ for all } \alpha \in Q\}$

$$\Rightarrow Q \subseteq Q^* \subseteq \mathbb{R}Q \cong \mathbb{R}Q^*$$

The torus of a lattice and its Loop group

- $T := \mathbb{R}Q/Q$ torus of Q
- abstract presentation: $T = \{e^{2\pi i\alpha}, \alpha \in \mathbb{R}Q\}$
- $e^{2\pi i\alpha} = 1$ iff $\lambda \in Q$
- $LT := C^{\infty}(S^1, T)$ Loop group of the group T
- $e^{2\pi i f} \in LT$ where $f : S^1 \to \mathbb{R}Q$
- $f(\theta) = \Delta_f\,\theta + f_0 + f_1(\theta)$ with $\theta \in [0, 1]$
  $f(1) - f(0) =: \Delta_f \in Q$ the winding number of f
  $f_0 := \int_{S^1} f d\theta \in T$ the zero mode of f
  $f_1(\theta) = \sum_{n\neq 0} a_n\, e^{2\pi i n\theta}$ the rest of the Fourier expansion
  with $a_n \in T$

S^1-extension of the loop group LT

S^1 is used as spacetime and as the Abelian group of the extension

- If $f, g \in LT$

$$\int_{S^1} \langle f, g \rangle \, d\theta \ := \ \int_0^1 \langle f, g \rangle \, d\theta$$

The central extension $\mathcal{L}T$ of LT by S^1 is the set $LT \times S^1$ [Pressley, Segal; Dong, Xu, ...] with multiplication

$$(e^{2\pi i f}, x_1) \cdot (e^{2\pi i g}, x_2) = (e^{2\pi i (f+g)},\ e^{\pi i \, \sigma(f,g)} x_1 x_2)$$

- the symplectic form defines the 2-cocycle of the extension

$$\sigma(f, g) \ := \ \int \langle f', g \rangle \, d\theta \ - \ \langle f(1), \Delta_g \rangle \ + \ \frac{1}{2} \langle \Delta_f, \Delta_g \rangle$$

is antisymmetric, non degenerate if $\langle \cdot \, , \cdot \rangle$ is

Exact sequences

- The following is an exact sequence of groups

$$1 \mapsto S^1 \mapsto \mathcal{L}T \mapsto LT \mapsto 1$$

- The Heisenberg group of Q is the S^1-extension of the subgroup of LT

$$W_0 := \{e^{2\pi i f} \in LT : f_0 = \Delta_f = 0\}$$

  is called the Heisenberg group of Q and indicated by $\mathcal{U}(W_0, \sigma_0)$
- i.e. also the following sequence of groups is exact

$$1 \mapsto S^1 \mapsto \mathcal{U}(W_0, \sigma_0) \mapsto W_0 \mapsto 1$$

Local net associated to the lattice Q

Representations of the extension $\mathcal{L}T$ of LT

Proposition [Pressley, Segal; Dong, Xu]

i) $\mathcal{U}(W_0, \sigma_0)$ has a unique irreducible representation $\widetilde{\pi}_0$ with positive energy on a Hilbert space denoted by H_0

ii) the irreducible representations of $\mathcal{L}T$ with positive energy are of the form π_λ on the Hilbert space $\mathcal{H}_\lambda = \bigoplus_{\alpha\in\lambda+Q} H_\alpha$, for $\lambda \in Q^*$

iii) $H_\alpha \cong H_0$ and $\pi_\lambda \restriction \mathcal{U}(W_0, \sigma_0) = \pi_\alpha \cong \widetilde{\pi}_0$ for every $\lambda \in Q^*$, $\alpha \in Q$

iv) for any $\beta \in Q^*$ the operator $\pi_\lambda(e^{2\pi i\beta\theta})$ maps the Hilbert space H_α to the Hilbert space $H_{\alpha+\beta}$

(Observable) Net of von Neumann algebras associated to $\mathcal{L}T$

- reference Hilbert space
  $\mathcal{H}_0 = \oplus_{\alpha \in Q} H_\alpha$, i.e. $\mathcal{H}_\lambda$ with $\lambda = 0 \in Q^*$,
- index set of the open (bounded) intervals $\mathcal{I}$ of S^1
- the net $\mathcal{I} \to \mathcal{A}_Q$ is defined by

$$\mathcal{A}_Q(I) = \pi_0(\mathcal{L}T(I))'' := \left\{ \pi_0(e^{2\pi i f}), \,\mathrm{supp}\, f \subset I \right\}'', \quad I \in \mathcal{I}$$

Proposition [Dong, Xu]

i) $\mathcal{A}_Q$ is a local, conformal, strong additive and split net on the index set $\mathcal{I}$

ii) any $\lambda \in Q^*$ defines a DHR automorphisms of $\mathcal{A}_Q$ that corresponds to an irreducible representation of $\mathcal{L}T$, labeled by $\lambda \in Q^*/Q$

iii) $\mathcal{A}_Q$ is completely rational and for the μ-index of $\mathcal{A}_Q$ it holds

$$\mu_{\mathcal{A}_Q} = |Q^*/Q|$$

- Remember that for any double-interval $E = I_1 \cup I_2$ $I_1 \perp I_2$ in $\mathcal{I}$, the μ-index of the inclusion

$$[\widehat{\mathcal{A}}_Q(E) : \mathcal{A}_Q(E)]$$

  was defined by Kawahigashi, Longo, Müger and used in the subfactor approach of RCFT
- Q is said to be self-dual if $Q = Q^*$
- Q is self-dual if $Q = Q^*$ iff the rank of Q $d \equiv 0 \mathrm{mod}\, 8$ [Serre]
- In this case the superselection sector theory of $\mathcal{A}_Q$ is trivial, i.e. there is only the superselection sector π_0 up to isomorphism (holomorphic case)
- Dong and Xu studied the orbifold subnet of $\mathcal{A}_Q$ obtained by the action of isometries of the lattice Q

Net cohomology of Lattice models

More on the extension of LT

To describe $\mathcal{A}_Q$ in terms of group extension and net cohomology we observe

- $T_* := \mathbb{R}Q^*/Q^* \cong \mathbb{R}Q/Q^*$ torus of Q^*
  Because $Q \subset Q^* \Rightarrow T_* \subseteq T$
- $LT_* := C^\infty(S^1, T_*)$ Loop group of the group T_*

$$\begin{array}{ccccccc} LT & \cong & W_0 \oplus T \oplus Q & \subseteq & W_0 \oplus T \oplus Q^* & \cong & (LT)^* \\ & & & & \cup & & \\ & & & & LT_* & & \end{array}$$

- $(LT)^*$ is not a loop group and may be called the dual of LT
- Self dual case
  if $Q = Q^* \Rightarrow LT = LT_* = (LT)^* \cong W_0 \oplus T \oplus Q$

Remind on Extensions

- A group E is an extension of the group G by another group N if the sequence $1 \to N \to E \to G \to 1$ is exact
- The multiplication for $(n, s), (m, t) \in N \times G = E$

$$(n, s)(m, t) := (n\, \beta_s(m)\; y(s, t), s\; t)$$

$$z = (\beta, y) : (G, G \times G) \to (\mathrm{Aut}\, N, N)$$

is the non Abelian 2-cocycle of the extension valued in the 2-category $(\mathrm{Aut}\, N, N)$ satisfying the 2-cocycle equations

$$y(s, t) \in (\beta_{st},\, \beta_s \circ \beta_t), \quad s, t \in G$$

$$\beta_r(y(s, t))y(r, st) = y(r, s)y(rs, t), \quad r, s, t \in G$$

- the 2-cocycle z essentially defines the extension, up to a coboundary, and the extensions are classified by the 2-cohomology $Z^2(G, (\mathrm{Aut}\, N, N))$
- E results to be a twisted crossed product of G by N with the action β twisted by the function y, i.e. $E = G \rtimes_z N$

S^1-extension of LT

- For $e^{2\pi i f}, e^{2\pi i g} \in (LT)^*$ it holds

$$\sigma(f,g) = \sigma_0(f_1,g_1) + \langle \Delta_f, g_0 \rangle - \langle \Delta_g, f_0 \rangle$$

  and $\mathcal{L}T$ is a twisted crossed product decomposition of the (symbols) group $\mathcal{U}(W_0 \oplus T, \sigma_0)$ by the action of Q, twisted by a 2-cocycle $z_Q = (\beta, y)$

- The map y is trivial and $\beta_Q : Q \to \mathrm{Aut}\, S^1$ defines an action of Q on T by the (anti-)symmetric form of $\langle \cdot\,, \cdot \rangle$. Hence

$$\begin{aligned} \mathcal{L}T &= \left(\mathcal{U}(W_0 \oplus T) \rtimes_{z_0} S^1\right) \rtimes_{z_Q} \mathcal{U}(Q) \\ &= \mathcal{U}(W_0, \sigma_0) \times \left(\mathcal{U}(T) \rtimes_{z_Q} \mathcal{U}(Q)\right) \end{aligned}$$

- because $LT \subseteq (LT)^*$, it holds

$$\begin{aligned} \mathcal{L}T &= \mathcal{U}(W_0 \oplus T, \sigma_0) \rtimes_{z_Q} \mathcal{U}(Q) \\ &\subseteq \mathcal{U}(W_0 \oplus T, \sigma_0) \rtimes_{z_Q} \mathcal{U}(Q^*) =: (\mathcal{L}T)^* \end{aligned}$$

Algebraic Superselection Contents

By [C. '09], once such a decomposition and inclusion are given

- the group T is related to the (global gauge) symmetry of the model
- the group Q^* corresponds to its (dual) charges
- Interchanging the twisted crossed product of T and Q^*

$$\begin{aligned}(\mathcal{L}T)^* &= \mathcal{U}(W_0 \oplus Q^*, \sigma_0) \rtimes_{z_Q} \mathcal{U}(T) \\ &= \dot{\bigcup}_{t \in T} \mathcal{U}(W_0 \oplus Q^*, \sigma_0) \rtimes_{z_Q} \{t\}\end{aligned}$$

i.e. $(\mathcal{L}T)^*$ is a T-graded group with equivalent homogeneous components, from the point of view of the Superselection of $\mathcal{A}_Q$

Aim: to furnish a further method, together with VOA and subfactors theory, for the analysis of the superselection sectors of models on S^1

Counting the Cohomology for the net of a lattice

- Naturally define $\mathcal{A}_Q$ on double intervals $\mathcal{D} := \{E = I_1 \cup I_2\}$

$$\mathcal{A}_{Q,\mathcal{D}}(E) = \pi_0(\mathcal{L}T(E))'' := \left\{\pi_0(e^{2\pi i f}),\ \mathrm{supp}\, f \subset E\right\}''$$

- confront with the one defined by additivity on E

$$\mathcal{A}^{\mathcal{I}}_{Q,\mathcal{D}}(E) := \mathcal{A}_Q(I_1) \vee \mathcal{A}_Q(I_2)$$

Proposition For every double interval $E \in \mathcal{D}$

$$\mathcal{A}^{\mathcal{I}}_{Q,\mathcal{D}}(E) \subsetneq \mathcal{A}^{\mathcal{I}d}_{Q,\mathcal{D}}(E) := \left(\mathcal{A}_Q(I_1) \vee \mathcal{A}_Q(I_2)\right)^d = \mathcal{A}_{Q,\mathcal{D}}(E)$$

Proof. (involves Haag duality on (simple) intervals and calculations on the local loop group generators, using the symplectic form σ)

Observe

- $\mathcal{A}^{\mathcal{I}}_{Q,\mathcal{D}}$ do not satisfies (Haag) duality on $\mathcal{D}$
- all the charge transporter operators are actually contained in a larger (fields) algebra whose restriction to $\mathcal{H}_0$ gives the lacking elements of the proper inclusion $\mathcal{A}^{\mathcal{I}}_{Q,\mathcal{D}}(E) \subsetneq \mathcal{A}_{Q,\mathcal{D}}(E)$
- these elements are generated by $F = (F_1, F_0, \Delta_F) \in (LT)^*(I_1)$, $G = (G_1, G_0, \Delta_G) \in (LT)^*(I_2)$ such that $\Delta_F - \Delta_G \in Q$
- the 1-cohomology of $\mathcal{A}_Q$ is determined as a (braided) tensor category and

$$Z^1(\mathcal{A}_Q) \cong \mathrm{Rep}^{\perp}(\mathcal{A}_Q) \cong Q^*/Q$$

- It is possible to define a (putative) field net on $\mathcal{P} = \mathcal{I}, \mathcal{D}$ by

$$\mathcal{F}_Q(P) := \pi\big((\mathcal{L}T)^*(P)\big)'' =:= \Big\{\pi(e^{2\pi i F}), \,\mathrm{supp}\, f \subset P\Big\}''$$

Hence

Proposition It holds

i) $\mathcal{F}_{Q,\mathcal{I}}$ and $\mathcal{F}_{Q,\mathcal{D}}$ are non local but T-graded nets, whose identity components are local

ii) $\mathcal{F}_{Q,\mathcal{D}}(E) = \mathcal{F}^{\mathcal{I}}_{Q,\mathcal{D}}(E)$ for every double interval E

iii) $Z^1(\mathcal{F}_Q)$ is quasi-trivial, i.e. $\mathcal{F}_Q$ has no non trivial superselection sectors

Moreover

- there exists an Abelian compact group $\mathcal{G}$ given by $Q^*/Q \cong T/T_*$ with the compact-open topology
- $\mathcal{G}$ is the Pontryagin dual of the group Q^*/Q with the discrete topology
- $\mathcal{G}$ is the Bohr compactification of the continuous characters acting on Q^*/Q, and
- If $\mathcal{F}_Q(I)^{\mathcal{G}} = \mathcal{A}_Q(I)$ then $\mathrm{Rep}^{\perp}(\mathcal{A}_Q) \cong Z^0(\widehat{\mathcal{G}})$

Conclusions and Outlook

Other models

- Other simple currents extension model on the circle S^1: orbifolds, coset model and loop groups [...]
- Model on different spacetime: quantum fields on curved spacetime [Verch '01; Fewster, Verch '03]

Electromagnetic charges, QED and QCD on the lattice

- Stückelberg-Kibble QED_2 interacting model, in a non-regular representation. The charge symplectic subspace and its symplectic complement (local gauge symmetry) are spaces of test functions, [C., in preparation]
- [Kijowski, Rudolph, Thielmann, '97, '05] QCD and QED on the lattice, [Ruzzi, C., in preparation]

Algebraic theory of Superselection Sectors

- [Leyland, Roberts '78]: cohomology of classical nets over Minkowski spacetime computing 1- and 2-cohomology and the relation to the electric and magnetic charges
- Group Extension and Galois Theory