Fiberwise products of C*-bundles

E. Blanchard (CNRS)

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C^*-bundles

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(c) $\forall\ (a_x)_{x \in X} \in A, \quad \mathbf{x \mapsto \|a_x\|_{A_x}}$ is **continuous.**

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upper semi-continuous (u.s.c.)

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Definition 1.3
A is a **continuous C*-bundle over** X **with fibres** A_x

iff

$\forall\, a \in A$, the function $x \mapsto \|a_x\|$ is $\boxed{\text{continuous}}$.

Tensor products of $C(X)$-algebras

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– A, B be two unital $C(X)$-algebras.
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N.B. Wrong if $X = \{x\}$

Corollary 2.5 *Let X be a* $\boxed{\textit{perfect}}$ *compact metric space and A a separable* $\boxed{\textit{continuous}}$ *$C(X)$-algebra.*

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if and only

for all short exact sequence

$$0 \to J \to B \to D \to 0$$

of continuous $C(X)$-algebras, the sequence

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is exact.

Proof of (3) $\Rightarrow$ (4) in Proposition 2.4

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b) If B is a C*-algebra, $\mathcal{B} := C(X;B)$ and $D := A \underset{C(X)}{\overset{m}{\otimes}} \mathcal{B}$, then

$$\begin{array}{ccccccc} 0 \to & C_x(X)\,D & \to & D & \to & D_x & \to 0 \qquad \boxed{\text{exact}} \\ & \| & & \| & & \| & \\ & C_x(X) A \overset{m}{\otimes} B & & A \overset{m}{\otimes} B & & A_x \overset{m}{\otimes} B & \end{array}$$

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$$\begin{array}{ccccccccc} 0 \to & C_x(X)\, D & \to & D & \to & D_x & \to 0 & & \boxed{\text{exact}} \\ & \| & & \| & & \| & & & \\ & C_x(X) A \overset{m}{\otimes} B & & A \overset{m}{\otimes} B & & A_x \overset{m}{\otimes} B & & & \end{array}$$

c) If C continuous $C(X)$-algebra and $\boxed{x \in X}$,

$$(A \underset{C(X)}{\overset{m}{\otimes}} C)_x \twoheadrightarrow (A_x \overset{m}{\otimes} C)_x \twoheadrightarrow A_x \overset{m}{\otimes} C_x$$

d) Let $K \triangleleft B$ and $d \in \ker\{A \overset{m}{\otimes} B \to A \overset{m}{\otimes} B/K\}$.

$$\begin{aligned} d_x &\in \ker\{(A \overset{m}{\otimes} B)_x \to (A \overset{m}{\otimes} B/K)_x\} & \\ &= \ker\{A_x \overset{m}{\otimes} B \to A_x \overset{m}{\otimes} B/K\} & \text{by } b) \\ &= A_x \overset{m}{\otimes} K & \text{by } a) \\ &= (A \overset{m}{\otimes} K)_x & \text{by } (3) \end{aligned}$$

Thus $d \in A \overset{m}{\otimes} K$.

Proof of (3) $\Rightarrow$ (4) in Proposition 2.4

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Amalgamated free product over $C(X)$

$$
\begin{tikzcd}
C(X) \arrow[r, "i_A"] \arrow[d, "i_B"'] & A \arrow[d, "j_A"'] \arrow[ddr, bend left, "\sigma_A"] & \\
B \arrow[r, "j_B"] \arrow[drr, bend right, "\sigma_B"'] & A \underset{C(X)}{\circledast} B \arrow[dr, dashed, "\sigma_A * \sigma_B"] & \\
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$$= C(X) \oplus A^\circ \oplus B^\circ \oplus \left(A^\circ \underset{C(X)}{\otimes} B^\circ\right) \oplus \left(B^\circ \underset{C(X)}{\otimes} A^\circ\right) \oplus \dots$$

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and $\quad v.w := v \otimes w \quad , \quad (v.w)^* = w^*.v^*$

Amalgamated free product over $C(X)$

Assume that the unital $C(X)$-algebras A and B are continuous.

Question 2 Is the $C(X)$-algebra $A \underset{C(X)}{\overset{\alpha}{*}} B$ continuous ?

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Proposition 3.1 ($C(X)$-linear Ruan's theorem)
Let X be a compact Hausdorff space and
let W be a separable operator space which is

(1) a unital $C(X)$-module and
(2) such that $\forall\, w \in M_n(W)\,, \quad x \mapsto \|w_x\|$ is continuous.

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Then $\exists\, \Phi_x : W_x \to L(H)$ ($x \in X$) completely isometric maps s.t.

$$\forall\, w \in M_n(W)\,, \quad x \mapsto \Phi_x(w) \ *\text{-strongly continuous in } L(H).$$

(with $H = \ell^2(\mathbb{N})$)

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i.e. $\exists\, \Phi = (\Phi_x)_{x \in X} : W \to \mathcal{L}_{C(X)}(H \otimes C(X))$ such that ...

Suppose that $\phi : A \to C(X)$ and $\psi : B \to C(X)$ are continuous fields of $\boxed{\text{faithful states}}$ on the unital $C(X)$-algebras A and B.

Definition 3.2 (Voiculescu)
$(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$ is the **unique** unital $C(X)$-algebra with a continous field of states $\phi * \psi : C \to C(X)$ such that:

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– A and B are free in $(C, \phi * \psi)$.

Reduced amalgamated free product over $C(X)$

Proposition 3.3 (B.) If X is $\boxed{\text{perfect}}$ and ϕ cont.field of faithful states on the unital $C(X)$-algebra A, TFAE

(1) The unital continuous $C(X)$-algebra A is an $\boxed{\text{exact}}$ C*-alg.

(2) The $C(X)$-alg. $(C, \phi * \psi) := (A, \phi) \underset{C(X)}{\overset{r}{*}} (B, \psi)$ is $\boxed{\text{continuous}}$

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Proof. Let $D := A \underset{C(X)}{\overset{m}{\otimes}} B$ and $E := L^2(D, \phi \otimes \psi) \underset{C(X)}{\otimes} D$

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(Dykema-Schlyakhtenko)

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Toeplitz Cuntz-Pimsner C*-algebra

Definition 3.4

If D is a unital C*-algebra and F is a Hilbert D-bimodule,

– $\mathscr{F}_D(F) = D \oplus F \oplus F \otimes_D F \oplus \dots$ $\boxed{\text{full Fock Hilbert } D\text{-bimodule}}$

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– $\ell(\xi) \in \mathcal{L}_D(\mathscr{F}_D(F))$ $\boxed{\text{creation operator}}$ given by

$$\ell(\xi).d = \xi d \quad \text{and} \quad \ell(\xi)(\zeta_1 \otimes \ldots \otimes \zeta_k) = \xi \otimes \zeta_1 \otimes \ldots \otimes \zeta_k$$

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$$D \hookrightarrow \mathcal{L}_D(F) \quad \text{and} \quad f.\zeta = \zeta.f \text{ for } f \in C(X), \zeta \in F$$

then D is a continuous $C(X)$-algebra if and only if $\mathcal{T}_D(F)$ is a continuous $C(X)$-algebra with fibres isomorphic to $\mathcal{T}_{D_x}(F_x)$.

Proposition 3.6 (Pedersen)

If the unital separable continuous $C(X)$-algebras A and B are $\boxed{\text{nuclear}}$, then

$$A \overset{f}{\underset{C(X)}{*}} B \subset C(X;\mathcal{O}_2) \overset{f}{\underset{C(X)}{*}} C(X;\mathcal{O}_2)$$

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Proposition 3.7 (B.)

The $C(X)$-algebra $A \underset{C(X)}{\overset{f}{*}} B$ is **always** continuous.

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Sketch of proof.
If $d \in A \underset{C(X)}{\odot} B$,

$$\|d_x\|_h = \inf\left\{ \|\sum_i a_i a_i^*\|^{\frac{1}{2}} . \|\sum_i b_i^* b_i\|^{\frac{1}{2}} \ ; \ d_x = \sum_i a_i \otimes b_i \right\}$$

$$= \sup\left\{ \left| \langle \xi, \sum_i \pi(a_i).\sigma(b_i)\eta \rangle \right| \ ; \ \pi \, , \, \sigma \text{ unital } * -\text{rep.} \right\}$$