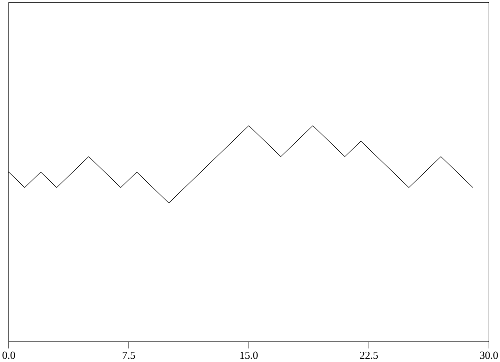
RANDOM WALKS ON SOME NONCOMMUTATIVE SPACES

Philippe Biane

Vietri Sul Mare, 01/09/2009

Classical random walk



$$S_n = X_1 + \ldots + X_n$$

$$X_k = \pm 1$$

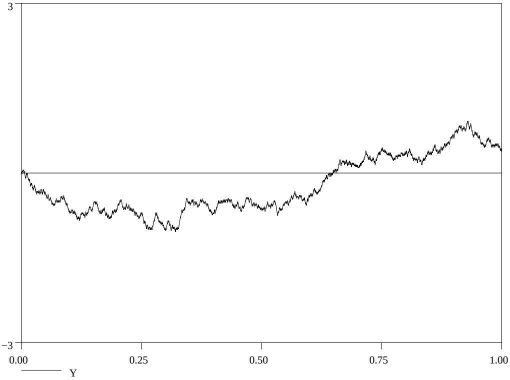
Brownian motion

Scale by ε in time and $\sqrt{\varepsilon}$ in space.

$$\sqrt{\varepsilon} \quad \uparrow$$

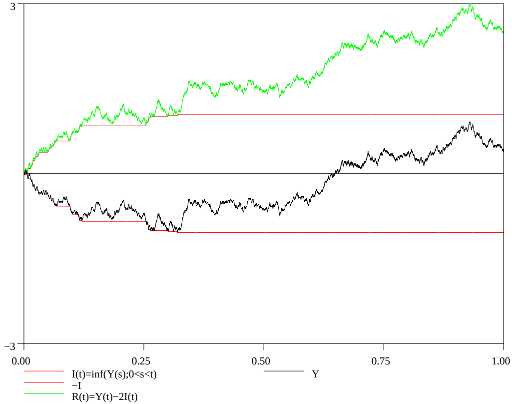
$$\longrightarrow$$

$$\varepsilon$$



PITMAN THEOREM (1975)

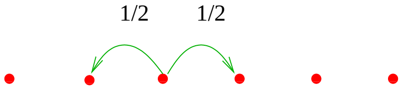
$B_t; t \geq 0$ Brownian motion; $I_t = \inf_{0 \leq s \leq t} B_s$
$R_t = B_t - 2I_t; t \geq 0$ is distributed as the norm of a three dimensional Brownian motion(=Bessel 3 process)



Explained by considering a random walk in a non-commutative space.

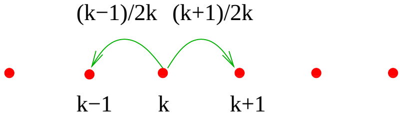
DISCRETE VERSION

$$X_i = \pm 1; \quad S_n = X_1 + X_2 + \ldots + X_n$$



$R_n = S_n - 2\min_{0\le k\le n} S_k$ is a Markov chain(=discrete Bessel 3 process)

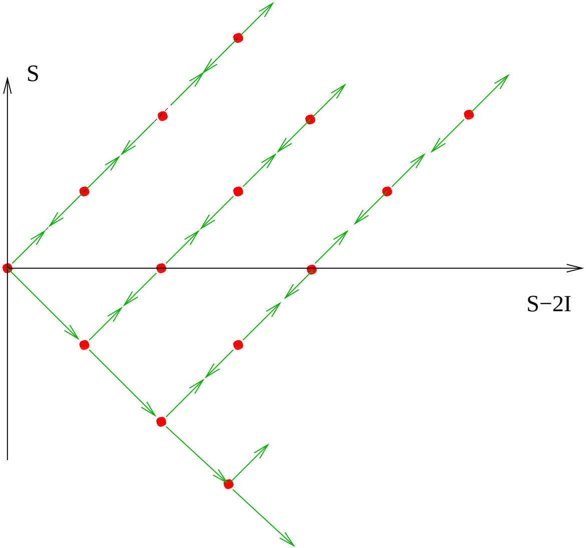
$$P(R_{n+1} = k+1 | R_n = k) = \frac{k+1}{2k}$$
$$P(R_{n+1} = k-1 | R_n = k) = \frac{k-1}{2k}$$



when $n \to \infty$ $S_{[nt]}/\sqrt{n} \to_{n\to\infty}$ Brownian motion

$R_{[nt]}/\sqrt{n} \to_{n\to\infty}$ norm of 3D-Brownian motion

PROOF OF PITMAN'S THEOREM



Quantum Bernoulli random walks

We "quantize" the set of increments of the random walk $\{\pm 1\}$ to obtain $M_2(\mathbf{C})$.
The subset of hermitian operators in $M_2(\mathbf{C})$ is a four dimensional real subspace, generated by the identity matrix I as well as the three matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. They satisfy the commutation relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z; \quad [\sigma_y, \sigma_z] = 2i\sigma_x; \quad [\sigma_z, \sigma_x] = 2i\sigma_y \qquad (1)$$

The random walk

For ω a state on $M_2(\mathbf{C})$, in $(M_2(\mathbb{C}),\omega)^{\otimes\mathbb{N}}$ we put

$$x_n = I^{\otimes(n-1)}\otimes\sigma_x\otimes I^{\otimes\infty},\ y_n = I^{\otimes(n-1)}\otimes\sigma_y\otimes I^{\otimes\infty},\ z_n = I^{\otimes(n-1)}\otimes\sigma_z\otimes I^{\otimes\infty}$$

x_n is a commuting family of operators, a sequence of independent Bernoulli random variables.

$$X_n = \sum_{i=1}^{n} x_i; \quad Y_n = \sum_{i=1}^{n} y_i \quad Z_n = \sum_{i=1}^{n} z_i$$

are Bernoulli random walks.
They do not commute but obey

$$[X_n, Y_m] = 2iZ_{n\wedge m} \tag{2}$$

as well as the similar relations obtained by cyclic permutation of X, Y, Z.
$(X_n, Y_n, Z_n); n \geq 1$ is a *quantum Bernoulli random walk*.

The spin process

Let $S_n = \sqrt{I + X_n^2 + Y_n^2 + Z_n^2}$

Proposition For all n, m one has

$$[S_n, S_m] = 0$$

Thus we have a commutative process and we can try to compute its distribution.

Theorem

Let ω be the tracial state $\frac{1}{2}Tr$, then S_n is distributed as a Markov chain on the positive integers, with probability transitions

$$p(k,k+1)=\frac{k+1}{2k};\quad p(k,k-1)=\frac{k-1}{2k}.$$

Random walks on groups

W = abelian group	$\hat{W}$ = dual group
	$\xi \in \hat{W}$ = character of W
$F(W)$=algebra of functions on W	$A(\hat{W})$=group algebra of $\hat{W}$
$F(W) \to F(W \times W)$	$\Delta : A(\hat{W}) \to A(\hat{W}) \otimes A(\hat{W})$
$f(x) \to f(x+y)$	$\Delta(\xi) = \xi \otimes \xi$
$\mu : F(W) \to C$	ϕ=positive definite function on $\hat{W}$
=probability measure on W	$\phi(\xi) = \int_W \xi(x) d\mu(x)$
	state ω on $A(\hat{W})$

$\Omega = (W, \mu)^\infty$

$M = \otimes^\infty (A(\hat{W}), \omega)$

$Y_n = w_1 + \ldots + w_n$

$f \rightarrow f(w_1 + \ldots + w_n)$

$j_n : A(\hat{W}) \rightarrow M$

$j_{n+1} = (\Delta \otimes I^{\otimes(n+1)}) \circ I \otimes j_n$

Markov operator

$\Phi(f)(x) = \int_W f(x+y) d\mu(y)$ $\quad \Phi(f) = (I \otimes \omega) \circ \Delta$

Random walks on duals of compact groups

Replace $\hat{W}$ by a compact group G.

ϕ=continuous positive definite functions on G, with $\phi(e) = 1$.
=state ν on $A(G)$.
ν= distribution of the increments.

$$\Phi_\nu : \mathcal{A}(G) \to \mathcal{A}(G)$$

$$\Phi_\nu = (I \otimes \nu) \circ \Delta$$

is a completely positive map. It generates a semigroup $\Phi_\nu^n; n \geq 1$.

$$(\mathcal{N}, \omega) = (\mathcal{A}(G), \nu)^{\infty}$$

$j_n : \mathcal{A}(G) \to \mathcal{N}$ defined by $j_n(\lambda_g) = \lambda_g^{\otimes n} \otimes I$

The morphisms $(j_n)_{n \geq 0}$, define a random walk on the noncommutative space dual to G, with Markov operator.

$$\Phi_\nu : \mathcal{A}(G) \to \mathcal{A}(G)$$

$$\Phi_\nu = (I \otimes \nu) \circ \Delta$$

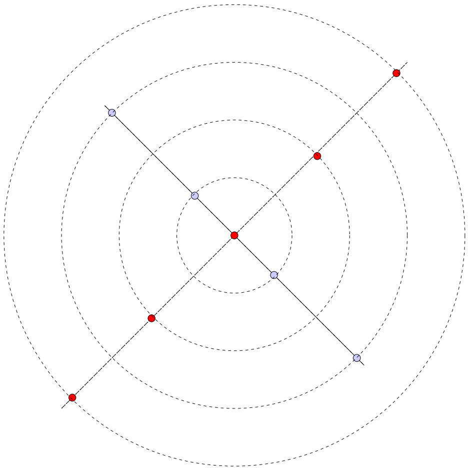
The quantum Bernoulli random walk is obtained for $G = SU(2)$, and ν the tracial state associated with the 2-dimensional representation.

The dual of $SU(2)$ as a noncommutative space

$G = SU(2) =$ unitary 2×2 matrices with determinant 1.
$\hat{G} = \{1, 2, 3, \ldots\}$

$$\mathcal{A}(SU(2)) = \oplus_{n=1}^{\infty} M_n(\mathbb{C})$$

is the noncommutative space dual to $SU(2)$.
The Pauli matrices belong to the Lie algebra $su(2)$, they define unbounded operators X, Y, Z, on $L^2(SU(2))$.
They generate oneparameter sugbroups isomorphic to $U(1)$. This is true also of any linear combination $xX + yY + zZ$ with $x^2 + y^2 + z^2 = 1$.



Noncommutative space underlying $\mathcal{A}(SU(2))$

If you are in this space and measure your coordinate in some direction (x, y, z) using the operator $xX + yY + zZ$, and you will always find an integer.

You cannot measure coordinates in two different directions at the same time.

The operator $D = \sqrt{I + X^2 + Y^2 + Z^2} - I$ is in the center of the algebra $\hat{\mathcal{A}}(SU(2))$, and therefore can be measured simultaneoulsy with any other operator.
Its eigenvalues are the nonnegative integers $0, 1, 2 \ldots$, and its spectral projections are the identity elements of the algebras $M_n(\mathbb{C})$

$$D = \sum_{n=1}^{\infty} (n-1) I_{M_n(\mathbb{C})}$$

$M_n(\mathbb{C})$ is a kind of "noncommutative sphere of radius $n-1$".
Looking at the eigenvalues of the operators $xX + yY + zZ$ the coordinate on this "radius" can only take the $n+1$ values $n, n-2, n-4, \ldots, -n$.

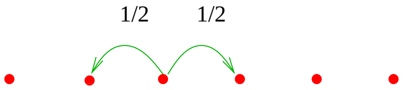
Construction of the quantum Bernoulli random walk

ω=state on $M_2(\mathbb{C})$, $\nu = \omega^{\otimes\infty}$ on $\mathcal{N} = \otimes_1^\infty M_2(\mathbb{C})$.
Construct morphisms $j_n : \mathcal{A}(SU(2)) \to \mathcal{N}$ by

$$j_n(\lambda_g) = \rho_2(g)^{\otimes n} \otimes I^{\otimes\infty}$$

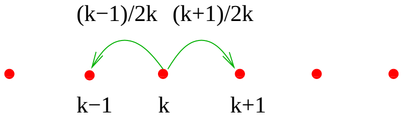
The family of morphisms $(j_n)_{n\geq 1}$ is a stochastic noncommutative process wih values in the dual of $SU(2)$.
This is just the iterated tensor product of the basic representation viewed as a random walk.

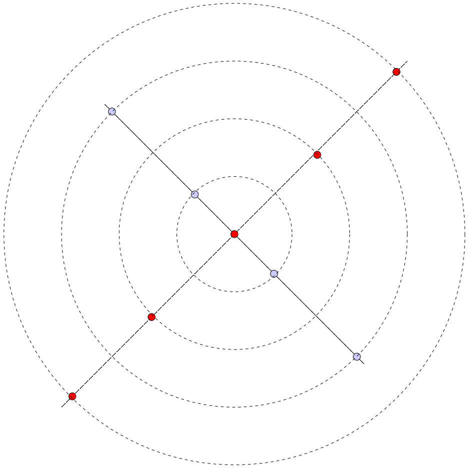
Restriction to a one parameter subgroup gives a Bernoulli random walk.



The spin process (radial part) is obtained by restriction of j_n to the center of the group algebra.
The restriction of the completely positive map Φ to this center can be computed by the Clebsch Gordan formula

$$\rho_2 \otimes \rho_k = \rho_{k-1} \oplus \rho_{k+1}$$



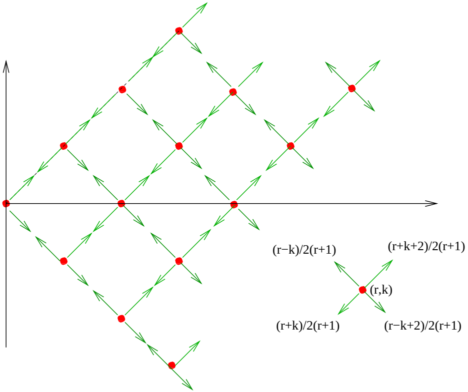


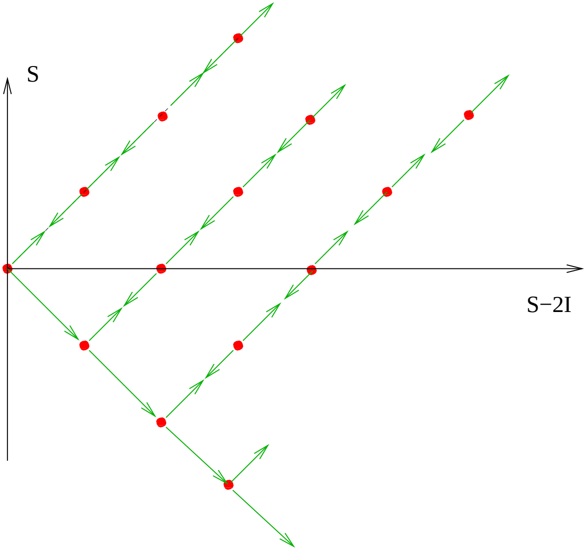
RESTRICTION TO A MAXIMAL ABELIAN ALGEBRA

Restrict the Markov operator to the maximal abelian subalgebra generated by the center and a one parameter subgroup.

In the decomposition $\mathcal{A}(SU(2)) = \oplus M_n(\mathbb{C})$ this is the algebra of diagonal operators.

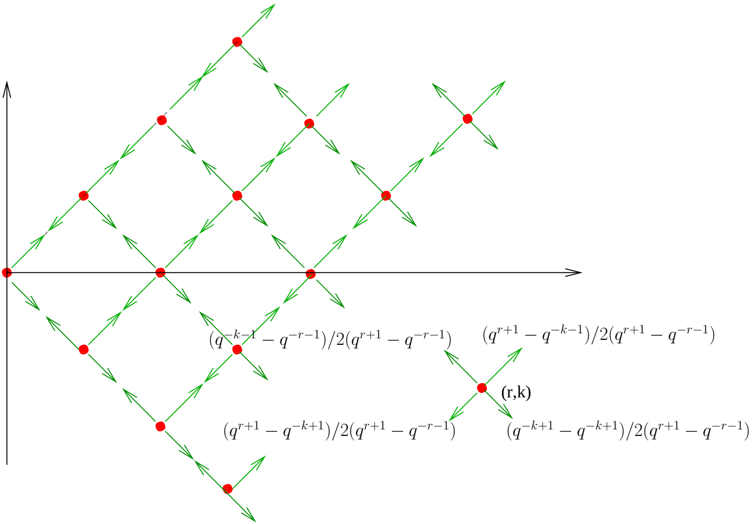
One gets probability transitions




S
S−2I

Kashiwara's crystallization

Replace $SU(2)$ by Drinfeld/Jimbo/Woronowicz $SU_q(2)$ then

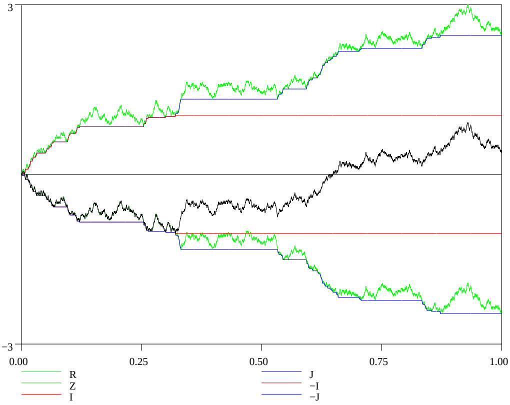


Let $q \to 0$ then one obtains Pitman's theorem.

We can generalize the preceding construction to the quantum groups $SU_q(n)$.

PITMAN OPERATORS

$Y : [0, T] \to \mathbf{R}, \qquad Y(0) = 0$



$$PY(t) = Y(t) - 2\inf_{0\leq s\leq t} Y(s)$$

For all t one has $PY(t) \geq 0$, in particular $PPY = PY$.

MULTIDIMENSIONAL PITMAN OPERATORS

V=real vector space, $\alpha \in V$, $\alpha^\vee \in V^*$ $\alpha^\vee(\alpha) = 2$.

$$P_\alpha Y(t) = Y(t) - \inf_{0 \le s \le t} \alpha^\vee(Y(s))\alpha$$

$$P_\alpha P_\alpha Y = P_\alpha Y$$

Braid relations

If the angle between α and β is $\theta = \pi/n$ then

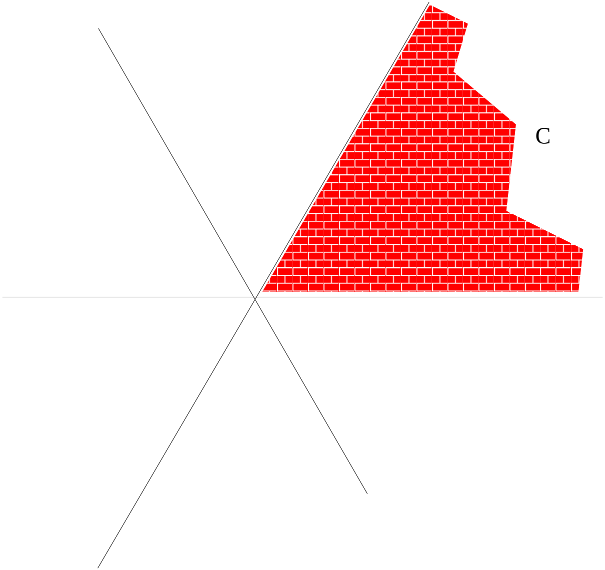
$$P_\alpha P_\beta P_\alpha \ldots = P_\beta P_\alpha P_\beta \ldots \qquad (n \text{ terms})$$

Corollary: Let (W, S)=Coxeter system on V and $\alpha, \alpha^\vee$=simple roots and coroots,
C=Weyl chamber. To each $s_\alpha \in S$ associate P_{s_α}. For each $w \in W$ with reduced decomposition $w = s_{\alpha_1} \ldots s_{\alpha_k}$ there exists

$$P_w = P_{s_{\alpha_1}} \ldots P_{s_{\alpha_k}}$$

If w_0=longest element then $P_{w_0}X$ takes values in C.

Example $W = S_3$



GENERALIZED PITMAN THEOREM

Let X be Brownian motion in V

then $P_{w_0}X$ is Brownian motion "conditioned to stay in C".

DOOB'S CONDITIONED BROWNIAN MOTION

$$\Psi(x) = \prod_{\beta \in R_+} \beta(x)$$

is a positive harmonic function on C

$$p_t^W(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, w(y))$$

is the fundamental solution of Laplacian on W with Dirichlet boundary conditions
(=transition probabilities for Brownian motion killed at the boundary of C).

$$q_t(x,y) = \frac{\Psi(y)}{\Psi(x)} p_t^W(x,y)$$

are the transition probabilities of Brownian motion conditioned to stay in C.

Fact:

when $W = S_n$ (i.e. Weyl group of type A_{n-1} then Brownian motion conditionned to stay in C is the same as the motion of eigenvalues

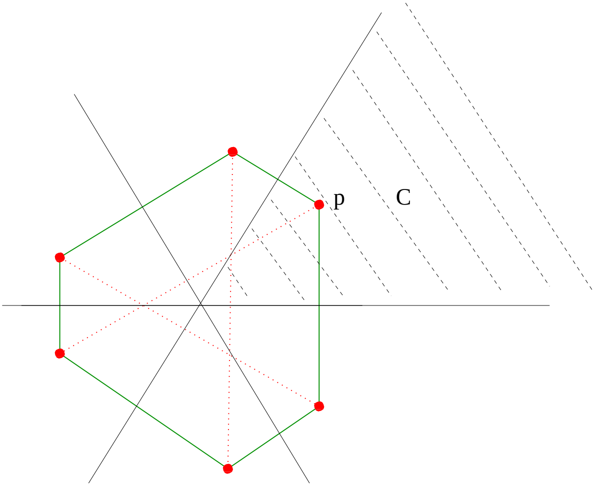
$$(\lambda_1(t)\lambda_2(t), \dots, \lambda_n(t))$$

of a Brownian traceless hermitian matrix.

$$\big(M_{ij}(t)\big)$$

CONVERSE THEOREM

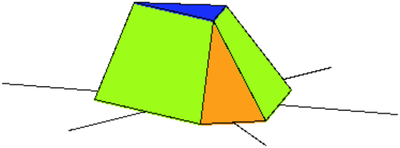
The conditional distribution of $X(t)$ knowing $P_{w_0}X(t) = p$ is the Duistermaat-Heckmann measure on the convex polytope with vertices $w(p); w \in W$.



Its Fourier transform is

$$\frac{1}{\prod_{\beta \in R} \beta(y)} \sum_{w \in W} \varepsilon(w) e^{i\langle p,y \rangle}$$

density is piecewise polynomial

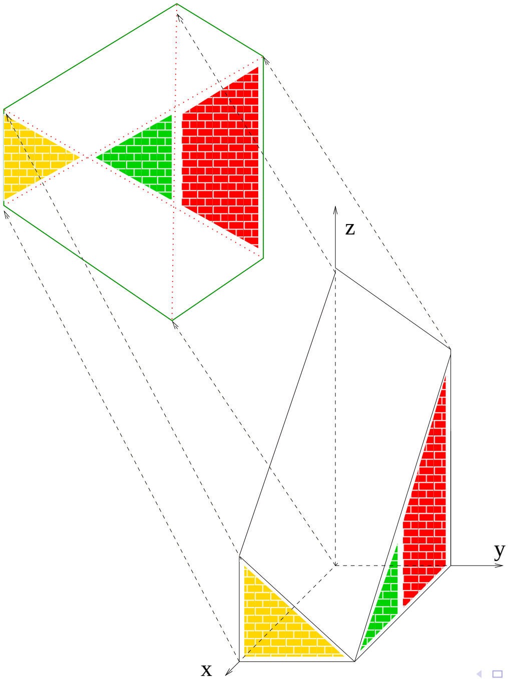


In order to recover X from $P_{w_0}X$ we need a positive real number x_i for each s_i in $P_{w_0} = P_{s_1} \dots P_{s_q}$.

Lemma Given $P_{w_0}X(t)$ the numbers $(x_1, \dots, x_q)$ belong to a certain convex polytope. Their distribution is the normalized Lebesgue measure on this polytope.

Cristallographic case: Berenstein-Zelevinsky polytopes

The Duistermaat-Heckman measure is the image of this measure by an affine map.



$0<x<a$

$0<y<b$

$0<z<(a-x)+(b-y)$