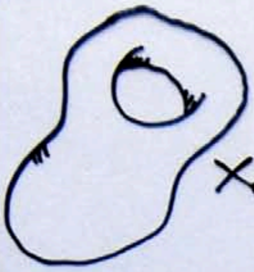
# THE SIGNATURE FORMULA

## 1. CLOSED MANIFOLDS



$X_0$

$\dim X = 4k$

- TOPOLOGICAL SIGNATURE $\sigma(X_0)$

$$(\,,\,): H^{2k}(X_0;\mathbb{R}) \times H^{2k}(X;\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(x,y) := \langle x \cup y, [X_0]\rangle$$

- DE RHAM SIGNATURE $\sigma_{dR}(X_0)$

$$H^{2k}_{dR}(X_0;\mathbb{R}) \times H_{dR}(X_0;\mathbb{R}) \longrightarrow \mathbb{R}$$
$$([\omega],[\phi]) := \int_{X_0} \omega \wedge \phi$$

- HODGE SIG. $\sigma_{Hodge}(X_0)$

$$\mathcal{H}^{2k} \times \mathcal{H}^{2k} \longrightarrow \mathbb{R}$$
$$(\omega,\phi) := \int \omega \wedge \phi$$

- ANALYTICAL SIGNATURE $\sigma_{an}(X_0)$

$$\sigma_{an}(X_0) := \operatorname{ind}\left(D^{sign,+}\right)$$

POSITIVE PART OF THE CHIRAL SIGNATURE OPERATOR

$$\Lambda T^* X_0 \longrightarrow X_0$$

$$\sigma(X_0) = \sigma_{dR}(X_0) = \sigma_{Hoge}(X_0) = \sigma_{an}$$

$$\sigma(X_0) = \int_{X_0} L(X_0)$$

Hirzebruck (Thom Cobordism)

Atiyah-Singer (K-theory, analysis, Hodge th.)

## 2. GALOIS COVERINGS

$$\begin{array}{c} \Gamma \curvearrowright \tilde{X} \\ \downarrow \\ X \end{array}$$

$L^2(\tilde{X})$ is a $\Gamma$-Hilbert module (on the Von Neumann group algebra $\mathcal{W}(\Gamma)$.

There's a natural notion of $\Gamma$-DIMENSION.

$$\sigma_\Gamma(\underbrace{\tilde{X}}_{\text{Total space}}) = \int_X L(\underbrace{X}_{\text{compact base}})$$

ATIYAH $L^2$-SIGNATURE FORMULA

## 3 MEASURED FOLIATIONS → NON COMMUTATIVE GEOMETRY

ALAIN CONNES '78

$$\boxed{\sigma(X,\mathcal{F}) = \langle L(T\mathcal{F}), [C_\Lambda] \rangle}$$

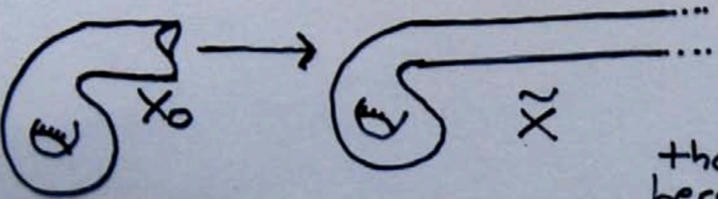
FOLIATIONS → VON NEUMANN ALGEBRAS

transverse measures → Weights (traces)

## MANIFOLDS WITH BOUNDARY (CYLINDRICAL ENDS)

### 1' COMPACT CASE

ATIYAH PATODI SINGER '75




$\sigma(X_0)$

the bilinear product becomes non degenerate on the image

Relative cohomology ⟶ absolute cohomology

the same of range $\left( H^{2k}_0(\tilde{X}) \longrightarrow H^{2k}(\tilde{X}) \right) = \mathcal{H}^{2k}(\tilde{X})$

harmonic forms

$$\sigma(X_0) = h^+ - h^- = ind_{L^2}(D^{sign,+}) = \int_{X_0} L(X_0, \nabla) \oplus \frac{\eta(D^{sign}_{\partial X_0})}{2}$$

TOP or de Rham

Harmonic forms on $\tilde{X}$

HODGE SIGNATURE

ANALYTICAL

ORIENTATION

$\eta$ INVARIANT OF THE BOUNDARY OPERATOR

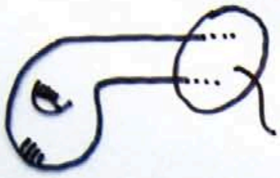
~~Dim~~ Proof

index formula of A.P.S (cylindrical version of the formula) * + · $L^2$ Hodge theory · Lefschetz duality

## 2' GALOIS COVERINGS

$$\begin{array}{c} \Gamma \curvearrowright \tilde{X} \\ \downarrow \\ X \end{array}$$

There's a notion of $(\Gamma)$ normalized signature (Hodge $\Gamma$-signature) $(L^2)$




- $\sigma_{\mathrm{Hodge},\Gamma}(\tilde{X}) = \int_X L(x,\nabla) + \frac{\eta_\Gamma}{2}\left(D^{\mathrm{sign},+}\big|_{\partial\tilde{X}}\right)$

Boris Vaillant ('97)

$$\| \\ \sigma_{an,(2)}(\tilde{X})$$

W. Luck and T. Schick define

- $\sigma_{top,(2)}(\tilde{X})$
- $\sigma_{dR,(2)}(\tilde{X})$

$\Big\}$ Using $\Gamma$-Hilbert modules

they prove:

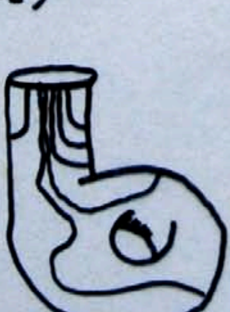
$$\boxed{\sigma_{dR,(2)}(\tilde{X}) = \sigma_{top,(2)}(\tilde{X}) = \sigma_{\mathrm{Hodge}}(\tilde{X})}$$

VERY DIFFICULT STEP

- $L^2$-Hodge theory
- Weakly exact $L^2$ long sequence
- boundary value problems on non compact $\partial$-manifolds
- elliptic regularity

## 3' FOLIATIONS (NORMAL TO THE BOUNDARY)

?



ONLY

THERE EXISTS AN INDEX THEOREM IN THE SPIRIT OF THE A.P.S BOUNDARY VALUE PROBLEM (RAMACHANDRAN '93)

- NO CYLINDRICAL INTERPRETATION
- NO SIGNATURE OPERATOR

WE WANT TO PROVE

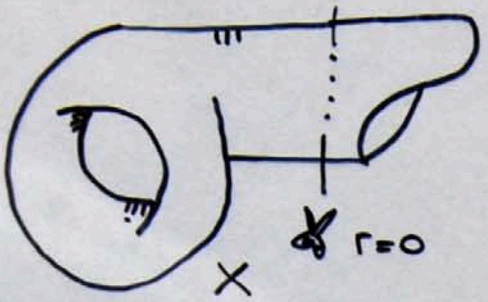
1. A PURELY CYLINDRICAL INDEX FORMULA
2. . GIVE DIFFERENT TYPES OF DEFINITION OF $\sigma$
   . PROVE AN HIRZEBRUCK TYPE SIGNATURE FORMULA

PLAIN OF THE TALK

1. EXPLAIN A.P.S IN THE CYLINDRICAL CASE
2. THE THEORY OF NON COMMUTATIVE GEOMETRY OF FOLIATIONS
3. THE INDEX FORMULA FOR CYLINDRICAL MEASURED FOLIATIONS
4. THE SIGNATURE OF A MEASURED CYLINDRICAL FOLIATION

# THE ATIYAH · PATODI · SINGER INDEX FORMULA






metric and geometric structures of "Dirac" are of PRODUCT TYPE in some collar.

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial_r + D_0 \\ \partial_r + D_0 & 0 \end{pmatrix}$$

$P := \chi_{[0,\infty)}(D_0) \longleftarrow$ PSEUDODIFFERENTIAL

$$\left\{ \begin{array}{l} D^+ : C^\infty(X, E^+) \longrightarrow C^\infty(X, E^-) \\ P(s|_{\partial X}) = 0 \quad \text{GENERALIZED B.V PROBLEM} \end{array} \right. \longrightarrow \text{a Fredholm Operator} \ldots$$

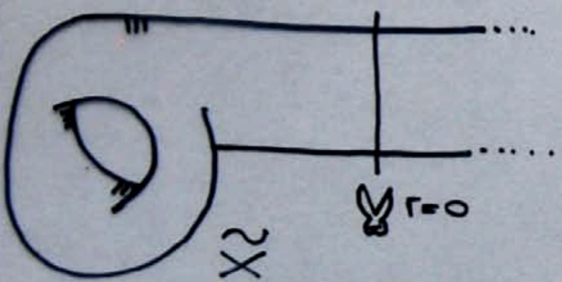
$$\boxed{\mathrm{ind}_{APS}(D^+) = \int_X \hat{A}(TX, \nabla)\,\mathrm{ch}(E) - \frac{h}{2} + \frac{\eta(0)}{2}}$$

$h$: dim Ker$(D_0)$

$\eta$ INVARIANT

$$\eta(s) := \sum_{\substack{\lambda \neq 0 \\ \lambda \in \mathrm{Spec}(D_0)}} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s}$$

Holomorphic Res (D0 Dirac) $\underline{\vee}\,-\frac{1}{2}$




ONE CAN PROVE

- $\mathrm{Ker}(D^+_{APS}) \cong \mathrm{Ker}_{L^2}(D^+_{\tilde{X}})$

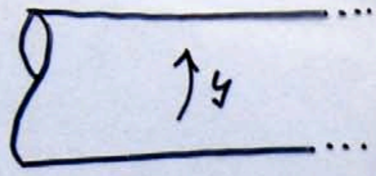
- $(D^+_{APS})^* = \overline{D^-, \text{ADJOINT BOUND. COND}} \rightsquigarrow$ Extended Solutions $\mathrm{Ext}(D^-)$

and

$$\boxed{\mathrm{ind}_{APS}(D^+) = \underbrace{\dim \mathrm{Ker}_{L^2}(D^+) - \dim \mathrm{Ker}_{L^2}(D^-)}_{\mathrm{ind}_{L^2}(D^+)\ \text{is not a Fredholm index}} - \underbrace{h_\infty(D^-)}_{\text{limiting Values of Extended Solutions}}}$$

$$\dim \mathrm{Ker}\,(D^+_{APS})^* = \dim \mathrm{Ker}_{L^2}(D^-) + h^-_\infty(D^-)$$

$$h^-_\infty(D^-) := \dim \underset{\text{values}}{\text{limiting}} \left\{ \mathrm{Ext}(D^-) \right\} \;*$$




$r \gg 0$

$s \in L^2_{loc} \quad s(y,r) = g(y,r) + s_\infty(y)$

$$\begin{cases} D_0 s_\infty = 0 \\ D^- s = 0 \end{cases}$$

$$h = h^+_\infty + h^-_\infty$$

$\dim \mathrm{Ker}(D_0)$

? $\oplus$

A conjecture of A.P.S proved by Melrose with the <u>b-calculus</u>

$$\boxed{\mathrm{ind}_{L^2}(D^+) = \int_X \hat{A}(x,\nabla)\,ch(E) + \tfrac{1}{2}\left\{ \eta(0) + h^-_\infty - h^+_\infty \right\}}$$

APS <u>CYLINDRICAL</u> VERSION

Vanishes on the cylinder

$$* \; \mathrm{Ext}(D^\pm) = \bigcap_{u>0} \mathrm{Ker}_{e^{ur}L^2}(D^\pm)$$

from the eigenfunction boundary expansion.

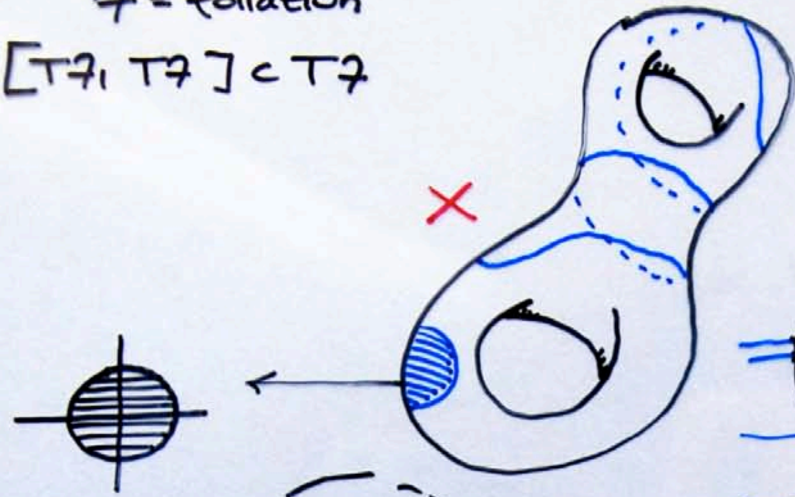
Weighted $L^2$ spaces

# FOLIATIONS AND NON COMMUTATIVE INTEGRATION THEORY

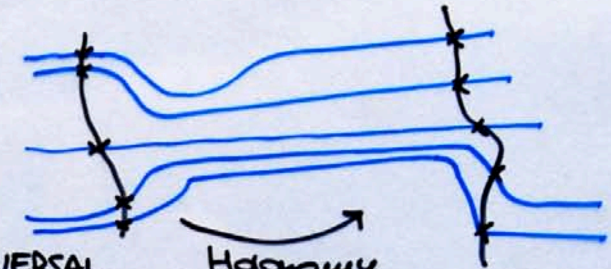
$\mathcal{F}$ = foliation

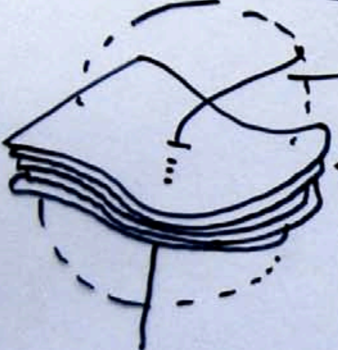
$[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$

Alain Connes '78



$\Lambda$ TRANSVERSE MEASURE





TRANSVERSAL

PLAQUE

Holonomy

(Poincaré return map)

$X/\mathcal{F}$: The space of the leaves is highly singular for

- topology
- measure theory

Foliation

$\downarrow$

$W(X,\mathcal{F})$

VON NEUMANN ALGEBRA $\rightarrow$ MEASURE THEORY ON THE SPACE OF THE LEAVES

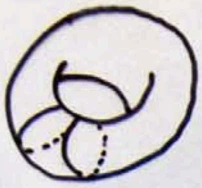
$L^\infty(X/\mathcal{F}) \cong Z(W(X/\mathcal{F}))$

it depends only on the measure class

DIMENSION THEORY (Murray, Von Neumann) $\rightarrow$ WEIGHTS

SEMIFINITE CASE (Ruelle-Sullivan) = HOLONOMY INVARIANT MEASURES = $\boxed{\text{TRACES}}$

As an example

•  $\mathbb{T}^2_\theta \quad \theta \in \mathbb{R} \setminus \mathbb{Q}$

$dy = \theta dx$

Factors = Ergodic foliations

└ every leaf is dense

• $\frac{X}{\mathcal{F}}$ Borel standard (Lebesgue) $\Longleftrightarrow$ $W(X, \mathcal{F})$ is type I

in the semifinite case it follows:

$$\underset{\tau}{\mathrm{ind}} := \tau(\operatorname{Ker} D^+) - \tau(\operatorname{Ker} D^-)$$

└→ ker and coker are projections in the algebra, the operator is BREUER-FREDHOLM

trace
- • semifinite
- • normal
- • faithful

th (Connes)

$$\boxed{\mathrm{ind}_\tau(D^+) = \langle \hat{A}(x)\,\mathrm{ch}(E), [C_\Lambda] \rangle}$$

tangential char. classes

foliated current of Ruelle-Sullivan

More generally in the non commutative integration realm one speaks about:

$\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{G}^{(0)}$

Borel groupoid

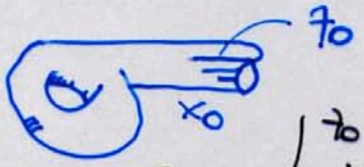
- · transverse measures
- · TRANSVERSE FUNCTIONS (longitudinal measures)
- Borel fields of Hilbert spaces
- square integrable representations of $\mathcal{G}$

RANDOM HILBERT SPACES

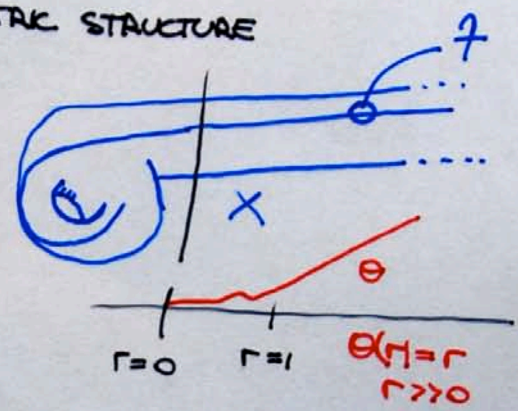
TRANSVERSE INTEGRATION OF FUNCTORS (RANDOM VARIABLES) ⟶ Formal dimensions and von Neumann algebras of Endom. of representations

# INDEX FORMULA FOR MEASURED CYLINDRICAL FOLIATIONS

Theorem (—)




- $X_0$ $\partial$-manifold with a foliation $\mathcal{F}_0$ $\left\{ \begin{array}{l} \mathcal{F}_0 \text{ ORIENTED} \\ \dim(T\mathcal{F}_0) = 2k \\ \mathcal{F}_0 \text{ TRANSVERSE TO THE BOUNDARY} \\ \text{(ACTUALLY IS NORMAL)} \end{array} \right.$
- EVERY LONGITUDINAL DIRAC GEOMETRIC STRUCTURE IS PRODUCT TYPE NEAR $\partial X_0$
- $X$ the elonged manifoed with every structure that's extended
- $\Lambda$ Holonomy invariant transveuse measure
- $R(X_0)$ $R(X)$ equivalence relations
- $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ Leafwise Dirac operator on the exted foliations on $X$ (miroel)




<u>Then</u>

- $\mathrm{Ker}_{L^2}(D^\pm)$ has finite trace in $\mathrm{End}_\Lambda(L^2(E^\pm))$
  the $L^2$-kernel is $\Lambda$-finite dimensionde
- One can define (Theory of formae dimensions of RANDOM HILBERT SPACES)

$$h^\pm_\Lambda := \dim_\Lambda \mathrm{Ext}(D^\pm) - \dim_\Lambda \mathrm{Ker}_{L^2}(D^\pm)$$

$$\dim_\Lambda(\mathrm{Ext}(D^\pm)) := \dim_\Lambda \left\{ \overline{\bigcap_{u>0} \mathrm{Ker}_{e^{u\theta}L^2}(D^\pm)}^{\,e^{u\theta}L^2} \right\}$$

and the formula:

$$\boxed{\mathrm{ind}_{L^2}(D^+) = \langle \hat{A}(X,\nabla)\,\mathrm{ch}(E/S), C_\Lambda \rangle + \tfrac{1}{2}\eta_\Lambda(D^{\mathcal{F}_0}) - \frac{h^+_\Lambda}{2} + \frac{h^-_\Lambda}{2}}$$

Ramachandran eta invariant of the boundary foliation.

- This formula is compatible with Ramachandran index formula ($\eta_\Lambda$ is the same)

- For the signoture operotor
  $\hat{A}(x)\, ch'(E) = L(x)$
  $\boxed{h^+_\Lambda = h^-_\Lambda}$

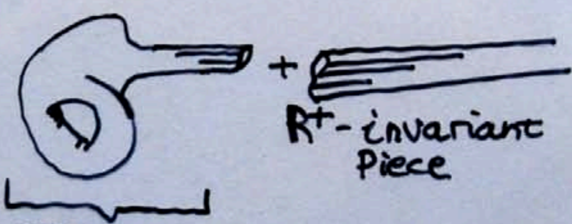
THE PROOF

BREUER FREDHOLM THEORY

$\Lambda$-ESSENTIAL SPECTRUM
$\mathrm{Spec}_{\Lambda,e}(\cdot)$

$\varepsilon,u$ - PERTURBATION.
$D^\pm_{\varepsilon,u}$

DECOMPOSITION PRINCIPLE



COMPACT PIECE

$R^+$-invariant Piece

NON COMMUTATIVE INTEGRATION THEORY

- Ext$(D^\pm)$ WELL DEFINED
- $\text{TRACE} = \int_X \underbrace{\bigcirc}_{\text{LONGITUDINAL MEASURE}} \underbrace{\ast}_{\text{TRANSVERSE MEASURE}}$

ELLIPTIC REGULARITY MEANS:

- THE KERNEL IS $\Lambda$-FINITE DIMENSIONAL
- BROWDER GÅRDING BOUNDARY
- ANALYS ON BOUNDED GEOMETRY MANIFOLDS
- FINITE PROPAGATION SPEED EXTIMATES CHEEGER-GROMOV-TAYLOR RELATIVE KERNEL EXTIMATES

$B_{\varepsilon,u} \longrightarrow B$ (boundary operator)

$D_{\varepsilon,u} \longrightarrow D$

# THE SIGNATURE FORMULA

We shall define:

- ANALYTICAL SIGNATURE $\sigma_{an,\Lambda}(X_0,\partial X_0)$
- $L^2$ de Rham-Signature $\sigma_{\Lambda,dR}(X_0,\partial X_0)$
- ($L^2$) Hodge Signature $\sigma_{\Lambda,Hodge}(X_0,\partial X_0)$

$X_0$

dim $X_0 = 4k$
+ precedent assumptions

Th(—)

All the defined signatures are equal:

$$\sigma_{an,\Lambda}(X_0,\partial X_0) = \sigma_{\Lambda,dR}(X_0,\partial X_0) = \sigma_{\Lambda,Hodge}(X_0,\partial X_0)$$

with the following Hirzebruch type Formula

$$\sigma_{dR,\Lambda}(X_0,\partial X_0) = \langle L(X), c_\Lambda \rangle + \frac{1}{2}\eta_\Lambda(D^{\partial})$$

depends on the metric $L(X,\nabla)$

exactly as in A.P.S is not a cohomological pairing

## ANALYTICAL SIGNATURE

$$\sigma_{an,\Lambda}(X_0,\partial X_0) := \mathrm{ind}_\Lambda\left(D^{sign,+}\right) \text{ on } X$$

$D^{sign} = d + d^*$

$n = 4k$

$$\tau := * \circ (-1)^{\frac{|\cdot|(|\cdot|-1)}{2}+k}$$ CHIRALITY

SELF DUAL    ANTI SELF DUAL

$$\Omega(T^*) = \Omega^+(T^*) \oplus \Omega^-(T^*)$$

Theorem

$\eta^+_\Lambda(D^{sign}) = \eta^-_\Lambda(D^{sign})$ then

$$\sigma_{an,\Lambda}(X_0,\partial X_0) = \langle L(T\mathcal{F}), C_\Lambda\rangle + \frac{1}{2}\eta_\Lambda(D^{sign}_{|\partial F})$$

Proof: by symmetries of the signature operator. Computations must be performed before $\varepsilon \to 0$ □

## ⊙ THE HODGE SIGNATURE

$\mathcal{R}$ — equivalence relation $\quad x \longmapsto \mathcal{H}^{2k}(L_x) =$ 2k Harmonic forms on the leaf $L_x$

Cylindrical elliptic regularity $\Longrightarrow$ $\left\{ \mathcal{H}^{2k}(L_x) \right.$ SQUARE INTEGRABLE REPRESENT. $\left. \right\}_{x\in X}$

A RANDOM HILBERT SPACE

$$\mathcal{H}^{2k}_x \times \mathcal{H}^{2k}_x \longrightarrow \mathbb{C}$$

$$S^\infty_\Lambda(\alpha,\beta) = \int_{L_x} \alpha\wedge\beta = \int_{L_x} (\alpha, \underset{\mathcal{F}}{*}\beta) = (\alpha, A_x\beta) \longleftarrow$$

Riesz

A field of sesquilinear forms (Borel field)

$$(A_x)_{x\in X}\,; \quad A = A^* \in \mathrm{End}_\Lambda(\mathcal{H}^{2k})$$

$$\boxed{\sigma_{\Lambda,\mathrm{Hodge}}(X_0,\partial X_0) := \dim_\Lambda V^+ - \dim_\Lambda V^- < \infty}$$

This is a statement of elliptic (b-elliptic) regularity...

$$V^+ := \chi_{(0,\infty]}(A) = \mathrm{Ker}_{L^2}(D^{sign,+})$$

$$V^- := \chi_{[-\infty,0]}(A) = \mathrm{Ker}_{L^2}(D^{sign,-})$$

$$A_{|_{2k}} = *_{|_{\Omega^{2k}}} = \tau_{|_{\Omega^{2k}}} \ldots$$

Theorem (—)

$$\boxed{\begin{array}{c} \tau_{an,\Lambda}(X_0, \partial X_0) = \tau_{Hodge,\Lambda}(X_0, \partial X_0) \\ \overset{?}{=} \\ \tau_{dR,\Lambda}(X_0, \partial X_0) \end{array}}$$

LAST PART: THE MOST DIFFICULT ONE.

## COMPLEXES OF RANDOM HILBERT SPACES & DE RHAM SIGNATURE

Random Hilbert space " = " square integrable representation of $\mathcal{R}_0 = \mathcal{R}(X_0)$

$L^0_x$ = generic leaf of $X_0$

$$\Omega^k_{x,d} := \Big\{ \omega \in C^\infty_0(\Lambda^k T^* L^0_x) : \underbrace{\omega_{|\partial L^0_x} = 0}_{\text{Dirichlet condition}} \Big\} \subset L^2 \xrightarrow{d} L^2$$

$$\Big( \underbrace{A^k_x(L^0_x, \partial L^0_x)}_{\text{minimal closure}}, \|\cdot\|_{graph} \Big)$$

$$\cdots \longrightarrow A^{k-1}_x \xrightarrow{d} A^k_x \xrightarrow{d} A^{k+1}_x \longrightarrow \cdots$$

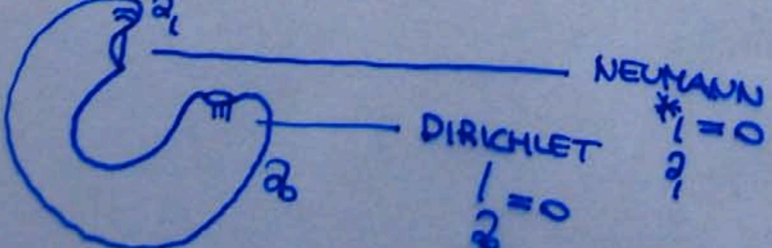
$Z^k_x(L^0_x ; \partial L^0_x)$ cicli

$B^k_x(L^0_x ; \partial L^0_x)$ boundaries

There are Various Hodge - de Rham decompositions with $L^2$ - harmonic forms with boundary conditions

$$\frac{Z^k_x}{B^k_x} \overset{!!}{=} H^{k,x}_{dR,(2)}(L^0_x ; \partial L^0_x)$$

$L^2$ - de Rham (Relative)



Other complexes of Random Hilbert spaces

- $L^{2,q}(\Omega^\bullet X_0, d)$ Unbounded differential

$$\cdots \longrightarrow L^2(\Lambda^k T^* L^\circ_x) \xrightarrow{d} L^2(\Lambda^{k+1} T^* L^\circ_x) \longrightarrow \cdots$$

and its cohomology

$$\boxed{H^{\bullet,q}_{dR,(2)}(X_0)}$$

- $L^{2,q}((\Omega^\bullet X_0, \partial X_0), d)$ Relative cohomology

Dirichlet boundary conditions

$$\boxed{H^{\bullet,q}_{dR,(2)}(X_0,\partial X_0)}$$

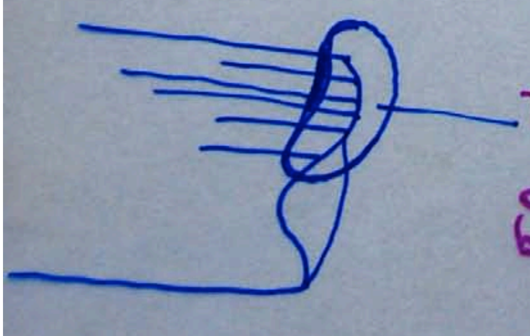
- The field of forms $S^\circ_x : A^{2k}_x(L^\circ_x ; \partial L^\circ_x) \times A^{2k}_x(L^\circ_x ; \partial L^\circ_x)$ passes to the Relative cohomology and factorizes through the natural field of mappings

$$H^{2k}_{dR,(2)}(L^\circ_x ; \partial L^\circ_x) \longrightarrow H^{2k}_{dR,(2)}(L^\circ_x).$$

The signature of the corresponding field of operators in $\mathrm{End}_\Lambda(H^{2k}_{dR,(2)}(X_0,\partial X_0))$ is by definition

$$\boxed{\sigma_{\Lambda,dR,(2)}(X_0,\partial X_0)}$$

in order to study the associated weakly exact long sequence we restrict the representations to



The foliation induced on the boundary

$$R_0\big|_{\partial X_0}$$

look at the field of short exact sequences
(consider the domains above defined)

$x_0 \in \partial X_0$

$$\boxed{0 \longrightarrow A^{k-1}_x(L^\circ_x ; \partial L^\circ_x) \xrightarrow{i} A^{k-1}_x(L^\circ_x) \xrightarrow{r} A^{k-1}_x(\partial L^\circ_x) \longrightarrow 0}$$

- The complexes are left $\Lambda$-Fredholm.
  to check one has to compute the exact domains of the relevant restrictions of the differentials (according to the definition of SPECTRAL DENSITY FUNCTION).
  As an example for the relative complex
  $$D(d) \cap \ker(d)^{\perp} = H^1_{Dir} \cap \overline{\delta^{k+1} C^\infty_0(\Lambda^{k+1} T^* L^\circ_x)}$$
  Using the various Hodge decompositions
- Traces of endomorphisms of restrictions of the representations to the boundary coincide with the natural trace of the boundary foliation.
  (this is a highly non trivial fact from the non commutative integration theory)
- The spectral density function is controlled by the spectral density function of the (Hort.) Laplacian
  Hence is controlled by the HEAT KERNEL

$$\chi_{[0,\mu]}(f^*f) = \underbrace{\chi_{[0,\mu]}(\Delta^{\perp}_k)\, e^{\Delta^{\perp}_k}\, \chi_{[0,\mu]}(\Delta^{\perp}_k)\, e^{-\Delta^{\perp}_k}}_{\Lambda\text{-TRACE CLASS}}$$

# $\Lambda$ FREDHOLMNESS $\Longrightarrow$ WEAKLY EXACT LONG SEQUENCES AT LEVEL OF THE RELEVANT VON NEUMANN ALGEBRAS

Theorem (–)

The long sequence ——— *

$$\cdots \longrightarrow H^{k,q}_{dR,(2)}(X_0,\partial X_0) \xrightarrow{i^*} H^{k,q}_{dR,(2)}(X_0) \xrightarrow{r^*}$$

$$\xrightarrow{r^*} H^{k,q}_{dR,(2)}(\partial X_0) \xrightarrow{\delta} H^{k-1}_{dR,(2)}(X_0,\partial X_0) \longrightarrow \cdots$$

is weakly exact at level of Von Neumann algebras of corresponding endos i.e. at level k

$$\overline{i_* H^{k,q}_{dR,(2)}} = \ker r_* \in \mathrm{End}_\Lambda\left(H^{k,q}_{dR,(2)}(X_0)\right).$$

in the sense of projections

thanks to this result some density results of Lück and Schick glue together in the Von N. Algebra and prove that

$$\boxed{\nabla_{\Lambda,dR}(X_0,\partial X_0) = \nabla_{\Lambda,\mathrm{Hodge}}(X_0,\partial X_0)}$$

* a Cheeger-Gromov type result