Upper triangular forms for some classes of infinite dimensional operators

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Schur's upper triangular forms of matrices

Thm. (Schur)

Every element of $T \in M_n(\mathbb{C})$ is unitarily conjugate to an upper triangular matrix, i.e. there is some unitary matrix U such that

$$U^{-1}TU = \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ \vdots & & 0 & \lambda_{n-1} & * \\ 0 & \dots & \dots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T listed according to algebraic multiplicity.

- If T is a normal matrix, then Schur's decomposition is the spectral decomposition of T.

Relation to invariant subspace problem

- Schur decomposition for operators is related to fundamental invariant subspace problems in operator theory and operator algebras.
- If $\{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathbb{C}^n$ and P_k, $1 \leq k \leq n$ is the orthogonal projection onto the subspace spanned by $\{e_1, e_2, \ldots, e_k\}$, then a matrix $T \in M_n(\mathbb{C})$ is upper-triangular with respect to this basis if and only if T leaves invariant each of the subspaces $P_k(\mathbb{C}^n)$, $1 \leq k \leq n$.
- Equivalently, $P_k T P_k = T P_k$ for every P_k in the nest of selfadjoint projections $0 = P_0 < P_1 < \ldots < P_n = 1$
- or, T belongs to the associated nest algebra, that is to $\mathcal{A} := \{A \in M_n(\mathbb{C}) : (1 - P_k) A P_k = 0; k = 1, \ldots, n\}$.
- Thus, Schur decomposition involves an appropriate notion of upper triangular operators and operators that have sufficiently many suitable invariant subspaces.

Important Corollaries of Schur's Theorem

- The Schur decomposition of the matrix T allows one to write $T = N + Q$ where

$$N = \sum_{k=1}^{n} (P_k - P_{k-1}) T (P_k - P_{k-1})$$

  is a normal matrix (that is, a diagonal matrix in some basis) with the same spectrum as T.
- Observe that N is the conditional expectation $\mathrm{Exp}_{\mathcal{D}}(T)$ onto the algebra $\mathcal{D}$ generated by $\{P_k\}_{k=1}^n$.
- The operator $Q = T - N$ is nilpotent (i.e. $Q^n = 0$ for some $n \in \mathbb{N}$).
- From the Schur decomposition one easily obtains that the trace of an arbitrary matrix is equal to the sum of its eigenvalues.

How can Schur's decomposition be generalized to operators?

- Projection P is said to be T-invariant if $PTP = TP$.
- An analogue of Schur's decomposition in the setting of an operator algebra $\mathcal{M}$ (typically, a von Neumann algebra) can be stated in terms of invariant projections:

Problem 1

We look for a decomposition $T = N + Q$, where N is normal and belongs to the algebra generated by some nest of T-invariant projections and where Q is upper triangular with respect to this nest of projections and is, in some sense, spectrally negligible.

- This version would require that T has (many) invariant subspaces.
- This is not a problem when T is a matrix
- Whether every bounded operator T on a separable (infinite-dimensional) Hilbert space H has a nontrivial invariant subspace is not known and is called the Invariant Subspace Problem.

Ringrose Theorem

- The existence of a nontrivial invariant subspace for a compact operator allowed Ringrose in 1962 [6] to establish a Schur decomposition for compact operators.

Theorem (Ringrose)

For a compact operator T there is a maximal nest of T-invariant projections P_λ, $\lambda \in [0,1]$ and $T = N + Q$, where
$\star$ N is a normal operator and belongs to the algebra generated by this nest
$\star$ Q is upper triangular with respect to this nest and which is a quasinilpotent $(\mathrm{spec}(Q) = \{0\})$ compact operator.

- Observe that N has the same spectrum (and multiplicities) as T.
- Compact operators have a discrete spectrum composed of eigenvalues that can be listed and naturally associated with invariant subspaces.
- The task becomes much harder for a non-compact operator whose spectrum is generally a closed subset of $\mathbb{C}$.

Brown measure

- In 1986 Lawrence G. Brown, made a pivotal contribution to operator theory by introducing his spectral distribution measure (Brown measure) associated to an operator in a finite von Neumann algebra.
- In general, the support of the Brown measure of an operator T is a subset of the spectrum of T.
- we think of Brown measure as a sort of spectral distribution measure for T.
- If $T \in M_n(\mathbb{C})$ and if $\lambda_1, \dots, \lambda_n$ are the eigenvalues (listed according to algebraic multiplicity), then it's Brown measure ν_T is given by $\nu_T = \frac{1}{n}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_n})$.
- Let $\mathcal{M}$ be a finite von Neumann algebra with normal faithful tracial state τ. If $N \in \mathcal{M}$ is normal operator (i.e., $N^*N = NN^*$), then $\nu_N = \tau \circ E_N$, where E_N is a spectral measure of the operator N.

Brown measure in matrix algebra

- If $A \in M_n(\mathbb{C})$ and if $\lambda_1, \dots, \lambda_n$ are its eigenvalues, then

$$\log(\det(|A-\lambda|)) = \sum_{k=1}^{n} \log(|\lambda - \lambda_k|).$$

- It is a standard fact that applying the Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\lambda = x + iy$ and dividing by 2π, we have

$$\frac{1}{2\pi}\nabla^2\Big(\lambda \to \log(\det(|A-\lambda|))\Big) = \sum_{k=1}^{n} \delta_{\lambda_k}.$$

- Thus, if $f(\lambda) = \frac{1}{n}\log(\det(|A-\lambda|))$, in case of matrices the Brown measure can be defined by

$$\nu_A = \frac{1}{n}\frac{1}{2\pi}\nabla^2\Big(\lambda \to \log(\det(|A-\lambda|))\Big) = \frac{1}{2\pi}\nabla^2 f.$$

- To define the Brown measure in general we recall the notion of Fuglede-Kadison determinant.

Fuglede-Kadison determinant

- Let $\mathcal{M}$ be a finite von Neumann algebra with normal faithful tracial state τ.
- Consider the mapping $\Delta : \mathcal{M} \to \mathbb{R}_+$ defined by the setting

$$\Delta(T) = \exp(\tau(\log(|T|))), \quad T \in \mathcal{M}$$

  and $\Delta(T) = 0$ when $\log(|T|)$ is not a trace class operator.
- Fuglede and Kadison proved that

$$\Delta(ST) = \Delta(S)\Delta(T), \quad S, T \in \mathcal{M}.$$

- If $(\mathcal{M}, \tau) = (M_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr})$, then $\Delta(A) = (|\det(A)|)^{1/n}$ for every $A \in \mathcal{M}$, and therefore

$$\log \Delta(A - \lambda) = \frac{1}{n} \log(\det(|A - \lambda|)).$$

Definition of Brown measure

Let $\mathcal{M}$ be a finite von Neumann algebra with normal faithful tracial state τ.

Definition of Brown measure

The Brown measure ν_T of $T \in \mathcal{M}$ is a Borel probability measure on $\mathbb{C}$;

- $f(\lambda) = \log \Delta(T - \lambda)$ is subharmonic and

$$\nu_T = \frac{1}{2\pi} \nabla^2 f$$

  in the sense of distributions.

- $$\log(\Delta(T - \lambda)) = \int_{\mathbb{C}} \log |z - \lambda| \, d\nu_T(z), \quad \lambda \in \mathbb{C}$$

- In fact, $\operatorname{supp}(\nu_T) \subseteq \operatorname{spec}(T)$ with equality in some cases.

Haagerup–Schultz invariant projections

- A tremendous advance in construction invariant subspaces was made recently by Uffe Haagerup and Hanne Schultz.
- Using free probability, they have constructed invariant subspaces that split Brown's spectral distribution measure.

Theorem 1 (Haagerup–Schultz) [5]

Let $\mathcal{M}$ be a finite von Neumann algebra with faithful tracial state τ. For every operator $T \in \mathcal{M}$, there is a family $\{p_B\}_{B\subset\mathbb{C}}$ of T-invariant projections indexed by Borel subsets of $\mathbb{C}$ such that

- $\tau(p_B) = \nu_T(B)$
- if $\nu_T(B) > 0$, then the Brown measure of Tp_B (in the algebra $p_B\mathcal{M}p_B$) is supported in B.
- if $\nu_T(B) < 1$, then the Brown measure of $(1 - p_B)T$ (in the algebra $(1 - p_B)\mathcal{M}(1 - p_B)$) is supported in $\mathbb{C}\backslash B$.

- The projection p_B is called the Haagerup-Schultz projection.

s.o.t.-quasinilpotent operators

- Now we are ready to explain in what sense the operator Q in Problem 1 should be spectrally negligible.
- To keep the analogy of our result with the results of Schur and Ringrose, the operator Q should have Brown measure ν_Q supported on $\{0\}$.
- Haagerup and Schultz proved that Brown measure ν_Q supported on $\{0\}$ if and only if $\lim_{n\to\infty}|Q^n|^{1/n}=0$ in the strong operator topology.

Definition s.o.t.-quasinilpotent

$Q \in \mathcal{M}$ is *s.o.t.-quasinilpotent* if any of the following equivalent conditions hold:

(i) $\nu_Q = \delta_0$

(ii) $\lim_{n\to\infty}|Q^n|^{1/n}=0$ in the strong operator topology.

Compare with quasinilpotent operators

Definition quasinilpotent

$Q \in B(H)$ is *quasinilpotent* if any of the following equivalent conditions hold:

(i) $\mathrm{spec}(Q) = \{0\}$

(ii) $\lim_{n\to\infty} |Q^n|^{1/n} = 0$ in the uniform norm topology.

- Every quasinilpotent operator is clearly s.o.t.-quasinilpotent.
- There exists s.o.t.-quasinilpotent operator Q with $\mathrm{spec}(Q) = \{z \in \mathbb{C} : |z| \leq 1\}$.

A Ringrose-type theorem on upper triangular forms in finite von Neumann algebras

Haagerup and Schultz's result allowed us in 2013 to prove the following result.

Main Theorem (Dykema, Sukochev, Zanin)[2]

Let $\mathcal{M}$ be a finite von Neumann algebra $\mathcal{M}$ equipped with a faithful tracial state τ. For every $T \in \mathcal{M}$, there exists a commutative von Neumann subalgebra $\mathcal{D}$ such that

- The conditional expectation $N = \mathrm{Exp}_{\mathcal{D}}(T)$ onto $\mathcal{D}$ is normal.
- $\nu_N = \nu_T$.
- $Q = T - N$ is s.o.t.-quasinilpotent.

Why not a full analogue of Ringrose's theorem? The von Neumann subalgebra $\mathcal{D}$ is generated by the nest of T-invariant projections, which is not necessarily maximal. However, if Brown measure of $\mathrm{spec}(T)$ does not have a discrete component, then this nest is maximal.

Construction of the algebra $\mathcal{D}$

- Let $\rho : [0,1] \to \big(\text{a disk containing } \mathrm{spec}(T)\big)$ be Peano curve, i.e. the continuous function from the unit interval to the unit square.
- For every $t \in [0,1]$, let $q_t := p_{\rho([0,t])}$ be the Haagerup-Schultz projection constructed in Theorem 1.
- $\mathcal{D}$ is the von Neumann algebra generated by $\{q_t\}_{t\in[0,1]}$.
- Similarly to the matrix case we set $N := \mathrm{Exp}_{\mathcal{D}}(T)$.
- It is immediate that N is a normal operator
- We prove that the operator $Q = T - N$ is s.o.t.-quasinilpotent (**this is the hard part**)
- and that $\nu_N = \nu_T$.

Why not quasinilpotent?

- Suppose that Main Theorem holds with quasinilpotent Q instead of s.o.t.-quasinilpotent.
- Take T to be an arbitrary s.o.t.-quasinilpotent.
- By the assumption, we have $T = Q + N$ with quasinilpotent Q and $\nu_N = \nu_T = \delta_0$.
- Since N is normal and $\nu_N = \delta_0$, it follows that $N = 0$.
  Indeed, recalling $\nu_N = \tau \circ E_N = \delta_0$, for every Borel subset $B \subseteq \mathbb{C}$, we have
  $$E_N(B) = \begin{cases} 0, & 0 \notin B \\ I, & 0 \in B \end{cases},$$
  where I is the identity operator. Hence, $N = 0$.
- Thus, $T = Q$, i.e. any s.o.t.-quasinilpotent operator is quasinilpotent operator, that is not true in general.

Holomorphic functional calculus

- The following result shows the stability of decomposition in [2] under holomorphic functional calculus.

Thm. (Dykema, Sukochev, Zanin)[3]

For $T \in \mathcal{M}$, let $T = N + Q$ be the upper triangular form from the previous result. Let h be a holomorphic function defined on a neighborhood of $\mathrm{spec}(T)$. Then $h(T) = h(N) + Q_h$, where Q_h is s.o.t.-quasinilpotent, $h(N)$ is normal and $\nu_{h(T)} = \nu_{h(N)}$.

- There also exists a multiplicative version of holomorphic calculus.

Thm. (Dykema, Sukochev, Zanin)[3]

Let $T \in \mathcal{M}$ and let h be holomorphic function such that $0 \notin \mathrm{supp}(\nu_{h(T)})$. We have $h(T) = h(N)(I + Q'_h)$, where Q'_h is s.o.t.-quasinilpotent.

Unbounded operators

- Let $\mathcal{M}$ be a finite von Neumann algebra $\mathcal{M}$ equipped with a faithful tracial state τ.
- Closed densely defined operator T is said to be affiliated with $\mathcal{M}$ if it commutes with every operator in the commutant $\mathcal{M}'$ of $\mathcal{M}$.
- The collection of all affiliated with $\mathcal{M}$ operators is denoted by $S(\mathcal{M}, \tau)$.
- The notions of the distribution function n_T, $T = T^*$ and the singular value function $\mu(T)$, $T \in S(\mathcal{M}, \tau)$ are defined as follows

$$n_T(t) := \tau(E_T(t, \infty)),\ t \in \mathbb{R} \quad \mu(t; T) := \inf\{s : n_{|T|}(s) \leq t\}, \quad t \geq 0,$$

where $E_T(t, \infty)$ is the spectral projection of the self-adjoint operator T corresponding to the interval (t, ∞).

- Define $\mathcal{L}^1 := \{T \in S(\mathcal{M}, \tau) :\ \mu(T) \in L^1(0, \infty)\}$,
- The space $\mathcal{L}^1$ is a linear subspace of $S(\mathcal{M}, \tau)$ and the functional $T \longmapsto \|T\|_1 := \tau(|T|)$, $T \in \mathcal{L}^1$ is a Banach norm.

An appropriate class of unbounded operators

- We prove the decomposition result for a large class of unbounded operators affiliated with a finite von Neumann algebra $(\mathcal{M}, \tau)$.
- Note that the Brown measure plays an essential role in the solution of Problem 1 for bounded operators.
- Recall that the construction of Brown measure is based on the notion of Fuglede-Kadison determinant $\Delta(T) = \exp(\tau(\log(|T|)))$, $T \in \mathcal{M}$, which is well defined for bounded operators.
- Haagerup and Schultz [4] constructed the Fuglede-Kadison determinant and Brown measure for unbounded operators $T \in S(\mathcal{M}, \tau)$ with an additional assumption $\log(|T|)_+ \in \mathcal{L}_1$, where $\log(|T|)$ is defined due to functional calculus and $\log(|T|)_+$ is a positive part of $\log(|T|)$.

Main Theorem for unbounded operators (work in progress)

Theorem 2

Let $\log(|T|)_+ \in \mathcal{L}_1$. There exist operators N and Q such that

- $T = N + Q$
- N is normal and $\nu_N = \nu_T$
- Q is s.o.t.-quasinilpotent.
- $\log(|N|)_+ \in \mathcal{L}_1$ and $\log(|Q|)_+ \in \mathcal{L}_1$.

There are two key obstacles in comparison with the bounded case.

- There is no construction of Haagerup-Schultz projections for unbounded operators.
- Conditional expectation $\mathrm{Exp}_{\mathcal{D}}(T)$ is not defined when $T \notin \mathcal{L}_1$.

Brown's version of Lidskii formula

A deep result due to Lidskii allows us to compute a trace of a trace class operator $T \in B(H)$ in terms of eigenvalues $\lambda(k, T)$, $k \geq 0$.

Lidskii theorem

If $T \in B(H)$ is trace class operator, then

$$\mathrm{Tr}(T) = \sum_{k=0}^{\infty} \lambda(k, T),$$

In a finite von Neumann algebra $\mathcal{M}$ with tracial state τ, Brown proved the following analogue of Lidskii result in terms of the Brown measure.

Brown's theorem [1]

If $T \in \mathcal{M}$, then

$$\tau(T) = \int_{\mathbb{C}} z \, d\nu_T(z).$$

Logarithmic submajorization

Definition

Let T, S be such that $\log_+(|T|), \log_+(|S|) \in \mathcal{L}_1(\mathcal{M}, \tau)$. We say that S is logarithmically submajorized by T (written $S \prec\prec_{\log} (T)$) if

$$\int_0^t \log(\mu(s,S))ds \leq \int_0^t \log(\mu(s,T))ds, \quad t > 0.$$

Weyl estimates

Weyl (1949) proved that the following estimate

Theorem

If T is a compact operator, then

$$\prod_{k=0}^{n} |\lambda(k, A)| \leq \prod_{k=0}^{n} \mu(k, A), \quad n \geq 0.$$

A similar estimate holds in finite von Neumann algebras.

Theorem

Let $T = N + Q$ as in Theorem 2. We have $N \prec\prec_{\log} (T)$.

Spectrality of traces

The following Lidskii formula was proved in [7].

Theorem

Let $\mathcal{I}$ be an ideal in $B(H)$ which is closed with respect to the logarithmic submajorization. Let $T \in \mathcal{I}$. Then for every trace φ on $\mathcal{I}$, we have $\varphi(T) = \varphi(\lambda(T))$.

We prove Brown-Lidskii formula for traces on operator bimodules.

Theorem (work in progress)

Let $\mathcal{I}$ be an operator bimodule on a finite factor $\mathcal{M}$ which is closed with respect to the logarithmic submajorization. Let $T \in \mathcal{I}$. Then for every trace φ on $\mathcal{I}$, we have $\varphi(T) = \varphi(N)$, where N is ANY normal operator such that $\nu_N = \nu_T$.

In other words, the equality $\varphi(T) = \varphi(N)$ can be written as

$$\varphi(T) = \varphi\Big(\int_{\mathbb{C}} z\, dE_N(z)\Big).$$

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