Cyclic cohomology and local index theory for Lie groupoids

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1. Lie groupoids

2. Algebraic topology

3. Classical index theorem

4. Index theorem for improper actions

- *Adv. Math.* 246 (2013)
- arXiv :1401.0225 (2014)

1. Lie groupoids

Definition (Ehresmann)

A Lie groupoid $G \rightrightarrows B$ is given by

- *Two smooth manifolds G (space of arrows) and B (space of objects)*
- *Two submersions $r : G \to B$ and $s : G \to B$ (range and source maps)*

$$\begin{tikzcd} r(g) \bullet & \bullet\, s(g) \arrow[l, bend right, "g"'] \end{tikzcd}$$

- *A smooth associative composition law $G \times_{(s,r)} G \to G$*

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g_1"'] & \bullet \arrow[l, bend right, "g_2"'] \arrow[ll, bend right=60, "g_1 g_2"'] \end{tikzcd}$$

- *A smooth embedding $u : B \hookrightarrow G$ (units) and a diffomorphism $i : G \to G$ (inversion).*

Let $G \rightrightarrows B$ be a Lie groupoid. A **Haar system** on G is a smooth family of right-invariant measures on the fibers of the source map $s : G \to B$.

Definition

*Let dg be a smooth Haar system on G. The **convolution algebra** $C_c^\infty(G)$ is the vector space of compactly supported smooth functions on G, endowed with the associative product*

$$(a_1 a_2)(g) = \int_{g_1 g_2 = g} a_1(g_1)\, a_2(g_2)\, dg_2 \qquad a_1, a_2 \in C_c^\infty(G)$$

Remark

Up to isomorphism, the algebra $C_c^\infty(G)$ does not depend on the choice of Haar system.

2. Algebraic topology

K-theory

Definition (Grothendieck, Serre, Whitehead, Bass, ...)

The first two algebraic K-theory groups of an associative algebra $\mathscr{A}$ are

- $K_0(\mathscr{A}) =$ *group completion of the semigroup of equivalence classes of finitely generated projective modules over $\mathscr{A}$*
- $K_1(\mathscr{A}) =$ *abelianization of the group of invertible matrices $GL_\infty(\mathscr{A})$*

Theorem (Index map - Milnor 1971)

Any extension (short exact sequence) $0 \to \mathscr{B} \to \mathscr{E} \to \mathscr{A} \to 0$ leads to an exact sequence of K-theory groups :

$$K_1(\mathscr{B}) \to K_1(\mathscr{E}) \to K_1(\mathscr{A}) \xrightarrow{\mathrm{Ind}} K_0(\mathscr{B}) \to K_0(\mathscr{E}) \to K_0(\mathscr{A})$$

Cyclic cohomology

Definition (Connes)

A cyclic n-cocycle over an associative algebra $\mathscr{A}$ is a $(n+1)$-linear functional $\varphi : \underbrace{\mathscr{A} \times \ldots \times \mathscr{A}}_{n+1} \to \mathbb{C}$ with the properties

$$1) \quad \varphi(a_0, a_1, \ldots a_n) = (-1)^n \varphi(a_1, \ldots, a_n, a_0)$$

$$2) \quad \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n) = 0$$

Even/odd degree cyclic cocycles modulo equivalence relation assemble to even/odd cyclic cohomology groups $HP^0(\mathscr{A})$ and $HP^1(\mathscr{A})$.

Theorem (Excision : Wodzicki 1988, Cuntz-Quillen 1994)

Any extension $0 \to \mathscr{B} \to \mathscr{E} \to \mathscr{A} \to 0$ leads to an exact sequence of cyclic cohomology groups :

$$HP^1(\mathscr{B}) \leftarrow HP^1(\mathscr{E}) \leftarrow HP^1(\mathscr{A}) \xleftarrow{\mathrm{Exc}} HP^0(\mathscr{B}) \leftarrow HP^0(\mathscr{E}) \leftarrow HP^0(\mathscr{A})$$

Abstract index theory

Proposition (Connes 1981)

For any associative algebra $\mathscr{A}$ there exists a bilinear pairing

$$\langle\ ,\ \rangle\ :\ HP^i(\mathscr{A})\times K_i(\mathscr{A})\to\mathbb{C}\qquad i=0,1$$

Theorem (Nistor 1994)

For any extension $0\to\mathscr{B}\to\mathscr{E}\to\mathscr{A}\to 0$, the index and excision maps are adjoint with respect to Connes'pairing

$$\begin{tikzcd} K_0(\mathscr{B}) \arrow[d, dotted, dash, "\langle\ ,\ \rangle"'] & K_1(\mathscr{A}) \arrow[l, "\mathrm{Ind}"'] \arrow[d, dotted, dash, "\langle\ ,\ \rangle"] \\ HP^0(\mathscr{B}) \arrow[r, "\mathrm{Exc}"] & HP^1(\mathscr{A}) \end{tikzcd}$$

3. Classical index theorem

Pseudodifferential operators

Definition

- $\mathrm{CL}(M) = \bigcup_{m\in\mathbb{Z}} \mathrm{CL}^m(M)$ *algebra of classical (1-step polyhomogeneous) pseudodifferential operators on a smooth closed manifold M*

- $\mathrm{L}^{-\infty}(M) = \bigcap_{m\in\mathbb{Z}} \mathrm{CL}^m(M)$ *ideal of smoothing operators*

- $\mathrm{CS}(M) = \mathrm{CL}(M)/\mathrm{L}^{-\infty}(M)$ *algebra of formal symbols. Purely algebraic deformation quantization of the commutative algebra $C^\infty(T^*M)$*

Remark

One has an extension, with σ the leading symbol homomorphism

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{L}^{-\infty}(M) \arrow[r] & \mathrm{CL}^0(M) \arrow[r] & \mathrm{CS}^0(M) \arrow[r] \arrow[d, "\sigma"] & 0 \\ & \mathbb{C} \arrow[u, "\sim"', "\mathrm{Morita}"] & & C^\infty(S^*M) & \end{tikzcd}$$

Lemma

*The index map in K-theory associates to any **elliptic** pseudodifferential operator P its Fredholm index* $\mathrm{Ind}(P) = \dim \mathrm{Ker}(P) - \dim \mathrm{Coker}(P)$

$$\begin{tikzcd} K_0(\mathrm{L}^{-\infty}(M)) \arrow[d, equal] & K_1(\mathrm{CS}^0(M)) \arrow[l, "\mathrm{Ind}"'] \arrow[dl, dotted] \\ \mathbb{Z} & \end{tikzcd}$$

Theorem (Perrot, *Adv. Math.* 246 (2013))

There is a commutative diagram in cohomology

$$\begin{tikzcd} HP^0(\mathrm{L}^{-\infty}(M)) \arrow[r, "\mathrm{Exc}"] \arrow[d, equal] & HP^1(\mathrm{CS}^0(M)) \\ \mathbb{C} \arrow[r, dotted, "\partial"] & H_{\mathrm{odd}}(S^*M) \arrow[u, "\sigma^*"'] \end{tikzcd}$$

where $\partial[\mathrm{Tr}] = [S^*M] \cap \mathrm{Td}(TM \otimes \mathbb{C})$

Sketch of proof

- **Step 1 :** compute the excision map of the pseudodifferential extension

$$0 \to \mathrm{L}^{-\infty}(M) \to \mathrm{CL}(M) \to \mathrm{CS}(M) \to 0$$

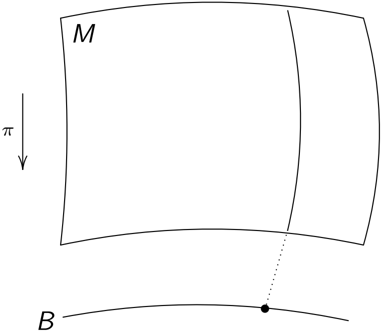
$$\begin{array}{ccc} [\mathrm{Tr}] \in HP^0(\mathrm{L}^{-\infty}(M)) & \xrightarrow{\mathrm{Exc}} & [\varphi] \in HP^1(\mathrm{CS}(M)) \\ \text{operator trace} & & \text{residue cocycle (Radul)} \end{array}$$

- **Step 2 :** show that in $HP^1(\mathrm{CS}^0(M))$, the Radul residue cocycle is cohomologous to

$$\sigma^*\big([S^*M] \cap \mathrm{Td}(TM \otimes \mathbb{C})\big)$$

4. Index theorem for improper actions

Lie groupoids acting on submersions



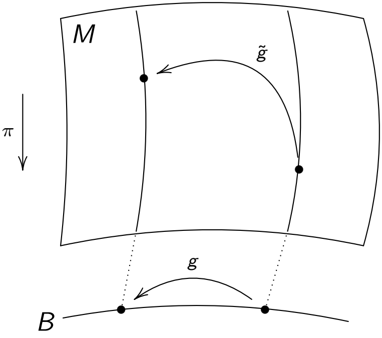
$\pi : M \to B$
surjective submersion

$\mathrm{CL}_\pi(M) = \bigcup_{m\in\mathbb{Z}} \mathrm{CL}^m_\pi(M)$

families of longitudinal pseudodifferential operators

Remark

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{L}^{-\infty}_\pi(M) \arrow[r] & \mathrm{CL}^0_\pi(M) \arrow[r] & \mathrm{CS}^0_\pi(M) \arrow[r] \arrow[d, "\sigma"] & 0 \\
 & C^\infty_c(B) \arrow[u, "\sim"', "\text{Morita}"] & & C^\infty_c(S^*_\pi M) &
\end{tikzcd}
$$



$\pi : M \to B$
surjective submersion

$G \rightrightarrows B$
Lie groupoid
acting on M

$\Rightarrow$ action groupoid $M \rtimes G$

Remark

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{L}_\pi^{-\infty}(M) \rtimes G \arrow[r] & \mathrm{CL}_\pi^0(M) \rtimes G \arrow[r] & \mathrm{CS}_\pi^0(M) \rtimes G \arrow[r] \arrow[d, "\sigma"] & 0 \\
 & C_c^\infty(G) \arrow[u, "\sim"', "\text{Morita}"] & & C_c^\infty(S_\pi^* M \rtimes G) &
\end{tikzcd}$$

Definition

*A **geometric cocycle** for a Lie groupoid* $G \rightrightarrows B$ *is a triple* $[E, \Gamma, \omega]$ *where*

- $E \to B$ *is a submersion with proper* G*-action*
- Γ *is an oriented étalification of the action groupoid* $E \rtimes G$
- $\omega \in H^\bullet_\delta(\mathrm{B}\Gamma)$ *is a (twisted) differentiable cohomology class of the classifying space of* Γ

$[E, \Gamma, \omega]$ *has parity* $\dim(B) + \deg(\omega)$ *modulo 2. We denote* $Z^i_{\mathrm{geo}}(G)$ *the set of geometric cocycles of parity* i*.*

Proposition

For any Lie groupoid G *one has a map (of sets)*

$$Z^i_{\mathrm{geo}}(G) \to HP^i(C^\infty_c(G))$$

Example (Etale groupoids)

*Suppose that $G \rightrightarrows B$ is **étale** and the submersion $E \to B$ has contractible fibers. Then one recovers Connes' characteristic map*

$$H^\bullet(\mathrm{B}G) \to HP^\bullet(C_c^\infty(G))$$

Example (Semisimple Lie groups)

Let G be a connected semisimple Lie group and K a maximal compact subgroup. Taking $E = K\backslash G$ one gets a characteristic map from the relative Lie algebra cohomology

$$H^\bullet(\mathfrak{g}, K) \to HP^\bullet(C_c^\infty(G))$$

Example (Differentiable cohomology)

Suppose that $G \rightrightarrows B$ is any Lie groupoid and $E \to B$ has contractible fibers. There exists a canonical choice of étalification for $\Gamma = E \rtimes G$ giving rise to a characteristic map from (twisted) differentiable groupoid cohomology

$$H^\bullet_{\mathrm{diff}}(G) \to HP^\bullet(C_c^\infty(G))$$

Take any Lie groupoid $G \rightrightarrows B$ acting on a submersion $\pi : M \to B$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{L}_\pi^{-\infty}(M) \rtimes G \arrow[r] & \mathrm{CL}_\pi^0(M) \rtimes G \arrow[r] & \mathrm{CS}_\pi^0(M) \rtimes G \arrow[r] \arrow[d, "\sigma"] & 0 \\ & C_c^\infty(G) \arrow[u, "\sim"', "\mathrm{Morita}"] & & C_c^\infty(S_\pi^* M \rtimes G) & \end{tikzcd}$$

Theorem (Perrot 2014, arXiv :1401.0225)

The excision map of the above extension fits in a commutative diagram

$$\begin{tikzcd} HP^0(\mathrm{L}_\pi^{-\infty}(M) \rtimes G) \arrow[r, "\mathrm{Exc}"] & HP^1(\mathrm{CS}_\pi^0(M) \rtimes G) \\ Z^0_{\mathrm{geo}}(G) \arrow[u] \arrow[r, dotted, "\partial"] & Z^1_{\mathrm{geo}}(S_\pi^* M \rtimes G) \arrow[u, "\sigma^*"'] \end{tikzcd}$$

where $\partial[E, \Gamma, \omega] = [E \times_B S_\pi^* M \ , \ \Gamma \times_B S_\pi^* M \ , \ \mathrm{Td}(T_\pi M \otimes \mathbb{C}) \cup \omega]$

Sketch of proof

- **Step 1 :** compute the excision map of the pseudodifferential extension

$$0 \to \mathrm{L}_\pi^{-\infty}(M) \rtimes G \to \mathrm{CL}_\pi^0(M) \rtimes G \to \mathrm{CS}_\pi^0(M) \rtimes G \to 0$$

$$\underset{\text{geometric cocycle}}{[E, \Gamma, \omega] \in HP^0(\mathrm{L}_\pi^{-\infty}(M) \rtimes G)} \xrightarrow{\mathrm{Exc}} \underset{\text{residue cocycle}}{[\varphi] \in HP^1(\mathrm{CS}_\pi^0(M) \rtimes G)}$$

- **Step 2 :** show that in $HP^1(\mathrm{CS}_\pi^0(M) \rtimes G)$, the residue cocycle is cohomologous to

$$\sigma^*[E \times_B S_\pi^* M \ , \ \Gamma \times_B S_\pi^* M \ , \ \mathrm{Td}(T_\pi M \otimes \mathbb{C}) \cup \omega]$$

Examples (Old results)

- *Atiyah-Singer index theorem for families of elliptic operators on a submersion $M \to B$, with $G = B$.*
- *Connes-Skandalis index theorem for longitudinal elliptic operators on foliations, with G the holonomy groupoid.*
- *Connes-Moscovici index theorem for coverings, with G a discrete group and M a free, proper, cocompact G-manifold.*

Examples (New results)

- *Index theorem for families of equivariant elliptic operators under* ***improper*** *actions of Lie groupoids.*
- *Index theorem for families of* ***non-pseudodifferential operators*** *in the algebra* $\mathrm{CL}^0_\pi(M) \rtimes G$.
- *After refinement to the Connes-Moscovici hypoelliptic calculus, index theorem for equivariant* ***Heisenberg-elliptic operators*** *on foliated manifolds (joint work with R. Rodsphon).*

THANK YOU