Analysis on singular spaces, Lie manifolds, and non-commutative geometry IV

Applications

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Noncommutative geometry and applications
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Abstract of series

We study **Analysis and Index Theory** on singular and non-compact spaces using **exact sequences** (foliations algebras, but also AS and APS frameworks):

$$0 \to I \to A \to Symb \to 0 \,.$$

- A is a suitable **algebra of operators** that describes the analysis on a given (class of) singular space(s). Was **constructed** using Lie algebroids and Lie groupoids.
- the **ideal** $I = A \cap \mathcal{K}$ of compact operators (to describe). **Will be determined using the representation theory of groupoids.**

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The contents of the four talks

1. Motivation: Index Theory (a) Exact sequences and index theory (b) The Atiyah-Singer index theorem (c) Foliations (d) The Atiyah-Patodi-Singer index theorem (e) More singular examples. **No new results.**
2. Lie Manifolds: (a) Definition (b) The APS example (c) Lie algebroids (d) Metric and connection (e) Fredholm conditions (f) Examples: Lie manifolds and Fredholm c.
3. Pseudodifferential operators on groupoids: (a) Groupoids (b) Pseudodifferential operators (c) Principal symbol (d) Indicial operators (e) **Groupoid C^*-algebras** and Fredholm conditions
4. Applications: (a) The index problem and periodic cyclic cohomology (b) Essential spectra (c) An index for Callias-type operators (d) Hadamard well posedness on polyhedral domains

Collaborators

- Bernd Ammann (Regensburg),
- Catarina Carvalho (Lisbon),
- Alexandru Ionescu (Princeton),
- Robert Lauter (Mainz ...),
- Bertrand Monthubert (Toulouse)

Table of contents

Definition of Lie manifolds using Lie algebroids

Recall that a vector bundle $A \to \overline{M}$ is a **Lie algebroid** if $\Gamma(A)$ is a Lie algebra over $\mathcal{C}^\infty(\overline{M})$ and there is given a structural map (anchor) $\varrho : A \to T\overline{M}$ that induces a morphism of Lie algebras.

If $\overline{M}$ is a compact manifold with corners and the anchor map $\varrho : A \to T\overline{M}$ is an isomorphism over $M := \overline{M} \smallsetminus \partial\overline{M}$, then we say that the pair $(\overline{M}, A)$ is a **Lie manifold.**

A metric on A gives a **metric** on $M : \overline{M} \smallsetminus \partial\overline{M}$. Ex: cylindrial ends, asymptotically hyperbolic and euclidean spaces.

The **isotropy** $\ker(\varrho_x : A_x \to T_x\overline{M})$ is a Lie algebra. (Injective ϱ_x for foliations. For Lie manifolds, isomorphism for $x \in M := \overline{M} \smallsetminus \partial\overline{M}$, the interior, not injective over $\partial\overline{M}$.)

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Fredholm operators

Assume $(\overline{M}, A)$ is a "nice" Lie manifold (recall $\mathcal{V} = \Gamma(A)$).

- Denote $(Z_\alpha)_{\alpha \in I}$ the family of **orbits** of $\mathcal{V}$ on $\partial\overline{M}$.
- Let G_α be a family of Lie groups that integrates the corresponding isotropies, $\alpha \in I$.

Theorem. (Lauter-Monthubert-V.N.) *We can choose the groups G_α and we can associate to each $P \in \mathrm{Diff}(\mathcal{V})$ a family of G_α-invariant operators P_α on $Z_\alpha \times G_\alpha$ such that:*

P is Fredholm $\Leftrightarrow$ P is elliptic and all P_α are invertible.

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Lie groupoids: definitions

Definition. A **Lie groupoid** is a groupoid $\mathcal{G}$ such that $\overline{M} = \mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, $\overline{M}$ is Hausdorff, the domain and the range maps are submersions and all the other structural maps are smooth.

Let $\mathcal{G}_x = d^{-1}(x)$ and

$$T_d\mathcal{G} = \cup T\mathcal{G}_x = \ker(d_* : T\mathcal{G} \to T\overline{M}),$$

the ***vertical* tangent bundle.** Then the **Lie algebroid** $A(\mathcal{G})$ associated to $\mathcal{G}$ is

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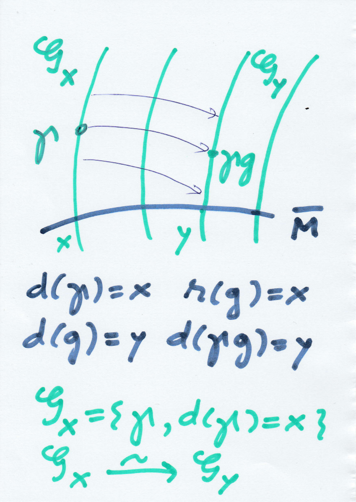
Right invariance



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The Lie algebroid of a Lie groupoid

A vector field X on $\mathcal{G}$ is called ***d*-vertical** X is tangent to all the submanifolds $\mathcal{G}_x$). Equivalently, $d_*(X) = 0$.

Let us denote by $\mathcal{V}$, the set of **right invariant, *d*-vertical vector fields on** $\mathcal{G}$. It is naturally a Lie algebra for the Lie bracket.

The space $\Gamma(A(\mathcal{G}))$ of sections of $A(\mathcal{G})$ identifies with $\mathcal{V}$ and hence the "Lie algebroid of $\mathcal{G}$," namely $A(\mathcal{G})$, is indeed a **Lie algebroid**.

If $A \to \overline{M}$ is a Lie algebroid and $\mathcal{G}$ is a Lie groupoid with units $\overline{M}$ such that $A(\mathcal{G}) \simeq A$, then we say that $\mathcal{G}$ **integrates** $A \to \overline{M}$.

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Lie's third theorem for Lie groupoids

A major technical ingredient is **to find** of a Lie groupoid $\mathcal{G}$ such that $\Gamma(A(\mathcal{G})) = \mathcal{V}$, that is to integrate the Lie algebroid defining our Lie manifold.

This amounts to a **Lie's third theorem** for $\mathcal{V}$ (or A).

The Lie's third theorem is **not valid** for any Lie algebroid, but **is valid** for those arising from Lie manifolds!
(Androulakis-Skandalis, Crainic-Fernandez, Debord, V.N., Pradines.)

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Example zero

$\overline{M} = M$ is compact without boundary (or corners) and $A = TM$.

The smallest groupoid integrating $(M, \mathcal{V})$ is then $\mathcal{G} = M \times M$, the **pair groupoid** of M. Let us check this.

The space of units (or objects) of $\mathcal{G} := M \times M$ is M. We set $d(x, y) = y$, $r(x, y) = x$, and $(x, y)(y, z) = (x, z)$. Hence there exists *exactly* one arrow (or morphism) between any two units.

Then $\mathcal{G}_x := d^{-1}(x) = M \times \{x\}$. A **family of pseudodifferential operators** $(P_x)_{x \in \overline{M}}$ on $\mathcal{G}_x$ is then simply a family of pseudodifferential operators $(P_x)_{x \in \overline{M}}$ on M. This family is *right inviariant* if, and only if, it is in fact constant: $P_x = P_y = P$. It is then automatically smooth and will have compact support.

Therefore $\Psi^m(\mathcal{G}) \simeq \Psi^m(M)$. **We recover the AS-framework!**

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The actions of $\Psi^\infty(\mathcal{G})$ and of $\mathrm{Diff}(\mathcal{V})$ on $\mathcal{C}_c^\infty(M)$ and $L^2(M)$ are the standard ones.

IMPORTANT: the C^*-algebra of $M \times M$ is the algebra of compact operators on $L^2(M)$.

Other groupoids will also integrate (M, TM), however, and we will see that they lead to **quite different** analysis properties.

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Let $\tilde{M} \to M$ be the universal cover of M and $\Gamma := \pi_1(M)$. The biggest (d-connected) groupoid integrating $(M, \mathcal{V})$ is

$$\mathcal{G} = \mathcal{P}M = (\tilde{M} \times \tilde{M})/\Gamma \,,$$

the *path groupoid* of M. Then $\mathcal{G}_x = d^{-1}(x) \simeq \tilde{M}$. The isotropy

$$\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x) \simeq \pi_1(M)$$

is then nontrivial. The operator P_x on $\mathcal{G}_x$ must be again idependent of $x \in M$ but also **invariant** with respect to the isotropy $\Gamma := \pi_1(M)$ and hence

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$\overline{M}$ is compact with boundary and $\mathcal{V} = \mathcal{V}_b$.

The smallest groupoid integrating $(\overline{M}, \mathcal{V}_b)$ is

$$\mathcal{G} = M \times M \cup \partial\overline{M} \times \partial\overline{M} \times \mathbb{R}\,.$$

The biggest is

$$\mathcal{G} = \mathcal{P}(M) \cup \mathcal{P}(\partial\overline{M}) \times \mathbb{R}\,.$$

There is at most one smooth structure on these disjoint unions and at least one in the simply-connected case (V.N.)

This recovers the *b*-calculus of Melrose (Monthubert, V.N.-Weinstein-Xu). Related to APS and Debord's construction.

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Foliation example

If $(M, \mathcal{F})$ is a foliated manifold, then the smallest Lie groupoid integrating $\mathcal{F}$ is the holonomy groupoid. The largest one is the path groupoid. The resulting algebras are Connes' algebras of pseudodifferential operators.

This example is important because the general case of Lie algebroids (not treated in my talk), would have to incorporate the case of foliations as well.

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Definition. $\Psi^m(\mathcal{G})$ consists of families (P_x), $P_x \in \Psi^m(\mathcal{G}_x)$ for all $x \in \overline{M}$, satisfying

- right invariant,
- smooth,
- with support in a compact neighborhood of the units of $\mathcal{G}$.

For instance, $\Psi^{-\infty}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G})$ with the convolution product.

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- right invariant,
- smooth,
- with support in a compact neighborhood of the units of $\mathcal{G}$.

For instance, $\Psi^{-\infty}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G})$ with the convolution product.

(Connes, Melrose, Monthubert, V.N.-Weinstein-Xu)

Action on functions on M and $\overline{M}$

We see that the **first order, differential** operators in $\Psi^\infty(\mathcal{G})$ coincide with $\mathcal{V}$ (right invariant, vertical vector fields on $\mathcal{G}$).

It follows that

$$\Psi^\infty(\mathcal{G}) \cap \mathrm{Diff} = \mathrm{Diff}(\mathcal{V})$$

We have seen that $\Psi^m(\mathcal{G})$ **acts** on $\mathcal{C}_c^\infty(M)$, and hence $\mathcal{V}$ and $\mathrm{Diff}(\mathcal{V})$ will act on $\mathcal{C}_c^\infty(M)$.

$\mathcal{V} \to \Gamma(T\overline{M})$ **is NOT injective, in general.** INJECTIVE for foliations, Lie manifolds.

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Principal symbol

Let $P = (P_x) \in \Psi^m(\mathcal{G})$. Then $\sigma_m(P_x) \in \mathcal{C}^\infty(S^*\mathcal{G}_x)$.

Hence globally, $\sigma_m(P) \in \mathcal{C}^\infty(S_d^*\mathcal{G})$, the cosphere bundle of the vertical tangent bundle (to the fibers of $d : \mathcal{G} \to \overline{M}$).

The right invariance of (P_x) gives the **right invariance** of $\sigma_m(P) \in \mathcal{C}^\infty(S_d^*\mathcal{G})$, so a function

$$\sigma_m(P) \in S_{\mathrm{cl}}(A^*)$$

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We continue to have the multiplication formula

$$\sigma_{m+m'}(PP') = \sigma_m(P)\sigma_{m'}(P').$$

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Indicial operators

Let again $P = (P_x) \in \Psi^m(\mathcal{G})$ and $H \subset \overline{M}$ be a (hyper)face.

Assume H is also invariant for $\mathcal{G}$, that is, $d^{-1}(H) = r^{-1}(H)$. (True if G_x are all connected).

Then the restriction $\mathcal{G}_H := d^{-1}(H) = r^{-1}(H)$ groupoid is also a Lie groupoid. The restriction map

$$\Psi^m(\mathcal{G}) \ni P = (P_x)_{x \in \overline{M}} \to (P_x)_{x \in H} \in \Psi^m(\mathcal{G}_H)$$

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Nice Lie manifolds

Let $(M, \mathcal{V})$ be a Lie manifold. Let Z_α be the orbits of $\mathcal{V}$ on $\partial \overline{M}$. We assume them to be open in their closures. (Locally closed.) We assume G_α integrate the isotropies of the orbits and are amenable. A Lie manifold $(M, \mathcal{V})$ is **'nice'** if we can put a smooth, Hausdorff structure on the disjoint union

$$\mathcal{G} := M \times M \cup \cup_\alpha \big(Z_\alpha \times Z_\alpha \times G_\alpha\big) \, ,$$

and the resulting C^*-algebra is faithfully represented on $L^2(M)$.

We thus see that $\mathcal{K} = C^*(M \times M) \subset C^*(\mathcal{G})$ by the theory of groupoid C^*-algebras and we can **characterize the quotient** using the complement $\overline{M} \smallsetminus M$. **Fredholm conditions.**

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We shall need also the groupoids $Z_\alpha \times Z_\alpha \times G_\alpha$, products of the pair groupoid and a Lie group. Then

$$\Psi^m(Z_\alpha \times Z_\alpha \times G_\alpha) \simeq \Psi^\infty(Z_\alpha \times G_\alpha)^{G_\alpha} .$$

This gives the following. Let then $x_1, \ldots, x_k$ be the defining functions of the hyperfaces of $\overline{M}$ and $f = x_1 \ldots x_k$. Then $\Psi^0(\mathcal{G}) \cap \mathcal{K} = f\Psi^{-1}(\mathcal{G})$ and *Symb* identifies with a subalgebra of

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An index problem

The exact sequence

$$0 \to f\Psi^{-1}(\mathcal{G}) \to \Psi^{0}(\mathcal{G}) \to Symb \to 0$$

gives rise to

$$\partial : K_1(Symb) \to K_0(I)\,.$$

The **Fredholm index problem** is in this case to compute

$$Tr_* \circ \partial : K_1(Symb) \to \mathbb{Z}\,.$$

Since $\phi_* \circ \partial = \psi_*$, where $\psi = \partial\phi \in HP^1(Symb)$, by Connes' results, the **Fredholm index problem** is equivalent to computing the class of ψ in periodic cyclic homology.

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Fifth example: foliated boundaries

- $\overline{M}$ = a compact manifold with **boundary.** (STILL)
- $\mathcal{V}$ = the space of vector fields on $\overline{M}$ that at the boundary are **tangent** to a foliation $\mathcal{F}$ of the boundary $\overline{M}$.
- A local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_k}$, $\partial_{y_{k+1}}$, $\ldots \partial_{y_n}$ in a local coordinate system adapted to the foliations (tangential and longitudinal coordinates).

There are no general Fredholm conditions **of the form** that we have seen before (but there could be others).

The resulting class of Riemann manifolds lead naturally to the study of foliation algebras.

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Asymptotically commutative Lie manifolds

Definition. (Carvalho-V.N) *We say that a be a Lie manifold* $(\overline{M}, \mathcal{W})$ *is* **asymptotically commutative** *if all vectors in* $\mathcal{W}$ *vanish on* $\partial\overline{M}$ *and all isotropy Lie algebras* $\ker(q_x)$ *are commutative.*

Let $x_1, x_2, \ldots, x_k$ be the defining functions of all the hyperfaces of $\overline{M}$ and $f = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$ for some positive integers a_j. Then

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defines an **asymptotically commutative** Lie manifold $(\overline{M}, \mathcal{W})$.

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An index for asymptotically commutative Lie manifolds

$(\overline{M}, \mathcal{W})$ is **asymptotically commutative**, then

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is commutative, and hence *Symb* is also commutative. Its completion will be of the form $C(\Omega)$, as in the work of Cordes.

It is possible then to compute the index of Fredholm operators using classical invariants (Carvalho-V.N.). Then we can compute the index of Dirac operators coupled with potentials of the form $f^{-1}V_0$, where V_0 is invertible at infinit on any Lie manifold.

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The Poisson problem

Domain Ω an *open, bounded* subset of $\mathbb{R}^n$ with boundary $\partial\Omega$.

The "simplest" boundary value problem, the *Poisson problem*

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = 0. \end{cases}$$

(Dirichlet boundary conditions.)

We want to find the solution in *Sobolev spaces*.

Hadamard well posedness on smooth domains

Assume Ω is **smooth** and recall that restriction maps $H^s(\Omega)$ **onto** $H^{s-1/2}(\partial\Omega)$.

Theorem. *The Laplacian Δ defines an isomorphism*

$$\Delta : H^{s+2}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to H^s(\Omega), \quad s \geq -1.$$

(Basic Well Posedness result.)

A useful consequence (easy to contradict for non-smooth domains) is:

Corollary. If f, g, and $\partial\Omega$ are smooth, then u is also smooth (recall $-\Delta u = f$, $u|_{\partial\Omega} = g$).

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Review of the Lipschitz case

The **Basic Well Posedness Theorem** **does not extend** to $\partial\Omega$ **not smooth** (Babuska, Costabel, Dauge, Fabes, Jerison, Kenig, Kondratiev, Mitrea(s), Mazya, Nazarov, Plamenevski, Schwab, Taylor, Verchota, ...).

Example: Assume $\Omega = (0,1)^2$, $g = 0$, and u smooth. Then

$$\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0)$$

and hence $f(0,0) = \Delta u(0,0) = 0$ is necessary for $u \in \mathcal{C}^\infty(\overline{\Omega})$.

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Weighted Sobolev spaces

Assume now Ω is a **polyhedral domain.**

Key technical point: **Replace**

$$H^m(\Omega) = \{u,\ \partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq m\},$$

$m \in \mathbb{Z}_+$, with the **weighted** Sobolev space

$$\mathcal{K}_a^m(\Omega) = \{u,\ \rho^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq m\},$$

$\rho =$**distance to singular points of** $\partial\Omega$

Well posedness in weighted Sobolev spaces for polyhedral domains

Theorem. (Bacuta-Mazzucato-N.-Zikatanov) *Let $\Omega \subset \mathbb{R}^n$ be a bounded, polyhedral domain and $m \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that*

$$\Delta : \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to \mathcal{K}^{m-1}_{a-1}(\Omega),$$

is an isomorphism for all any $|a| < \eta$.

In 2D it is due to Kondratiev '67 ($n = 2$).

$\eta = \pi/\alpha_{MAX}$ if Ω is a polygon (using also a result of Kondratiev).

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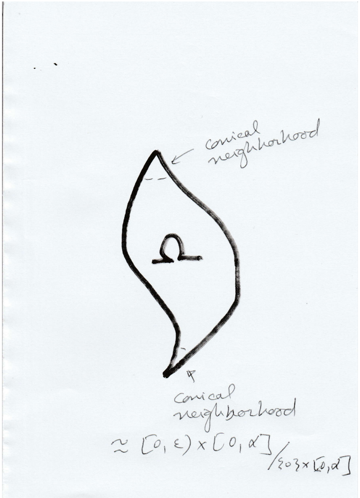
Lie manifolds and the proof

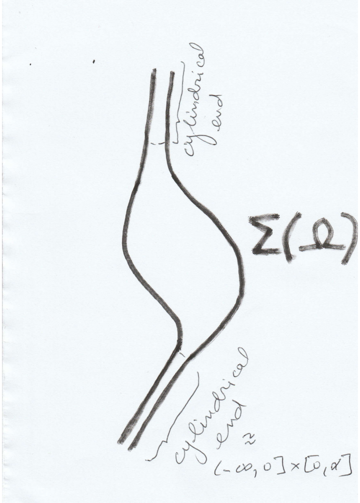
The proof is to study the properties of a **Lie manifold with boundary** $\Sigma(\Omega)$ canonically associated to Ω by a **blow-up** procedure.

The weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ can be shown to coincide with the usual Sobolev spaces associated to $\Sigma(\Omega)$.

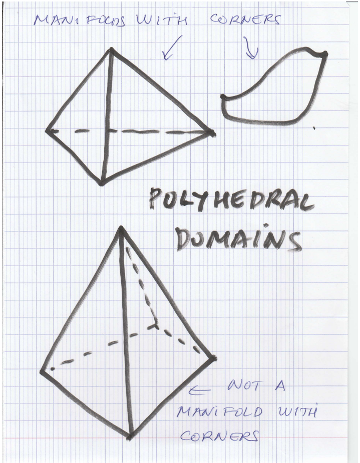
The blow-up procedure is an inductive procedure that consists in (roughly) replacing $CL := [0, \epsilon) \times L/(\{0\} \times L)$ with $[0, \epsilon) \times L$.

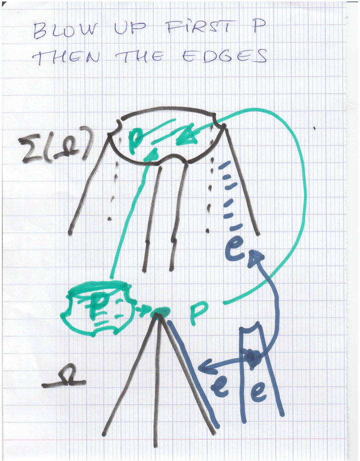
Blow up (Kondratiev transform)





Blow up in 3D


MANIFOLDS WITH CORNERS
POLYHEDRAL DOMAINS
NOT A MANIFOLD WITH CORNERS


BLOW UP FIRST P
THEN THE EDGES
Σ(Ω)
P
e
P
Ω
e
e

WARNING!!!

The well posedness does not hold for the **Neumann problem** (normal derivative at the boundary is zero).

The Neumann problem is significantly different. Two approaches

- Costabel-Dauge-Nicaise (based on Mazya)
- Li-Mazzucato-V.N. (based on APS-type index theory)

it is not yet clear what is the connection between the two approaches.

The case of singular spaces (no boundary) is **more like the Neumann problem.**

Applications to optimal rates of convergence in 3D.

Sixth example: manifolds with corners

- $\overline{M}$ = a compact manifold with **corners.** (NEW)
- $\mathcal{V}$ = the space of vector fields on $\overline{M}$ that are **tangent** to all hyperfaces of $\overline{M}$. This generalizes our first example.
- At the codimension k face $\{x_1 = \ldots = x_k = 0\}$, a local basis is

$$x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_{k+1}}, \ldots, \partial_{y_n}.$$

The manifolds Z_α are the open faces of $\overline{M}$. The group G_α is $\mathbb{R}^k$, with k the *codimension* of Z_α. (Also Melrose and Piazza.)

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Essential spectrum for the sixth example

Theorem.[Lauter-N]

$$\sigma(\Delta_M) = [0, \infty)$$

The proof is similar to that of lower rank.

The **complete characterization of the spectrum** (multiplicity of the spectral measure, discreteness of the point spectrum, absence of continuous singular spectrum) is **open**, in spite of its importance.

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The Dirac operator

Similarly, let $\not{D}$ be the Dirac operator associated to a Cliff(A)-bundle over $\overline{M}$. Then

Theorem.[N] The Dirac operator $\not{D}$ is invertible if, and only if, the Dirac operator $\not{D}_F$ associated to any open face F of $\overline{M}$ (including M), has no harmonic spinors (=zero kernel).

The proof uses the fact that the P_α are also Dirac operators.

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Thank you for your attention!