Analysis on singular spaces, Lie manifolds, and non-commutative geometry II

Lie manifolds

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Abstract of series

My four lectures are devoted to **Analysis and Index Theory** on singular and non-compact spaces. (Mostly the **analysis**.)

From a technical point of view, a central place in my presentation will be occupied by **exact sequences:**

$$0 \to I \to A \to Symb \to 0\,.$$

- A is a suitable **algebra of operators** that describes the analysis on a given (class of) singular space(s). Will be **constructed** using Lie algebroids and Lie groupoids.
- the **ideal** $I = A \cap \mathcal{K}$ of compact operators (to describe).
- **the algebra of symbols** $Symb := A/I$ needs to be described and leads to Fredholm conditions.

The contents of the four talks

1. Motivation: Index Theory (a) Exact sequences and index theory (b) The Atiyah-Singer index theorem (c) Foliations (d) The Atiyah-Patodi-Singer index theorem (e) More singular examples. **No new results.**
2. Lie Manifolds: (a) Definition (b) The APS example (c) Lie algebroids (d) Metric and connection (e) Fredholm conditions (f) Examples :Lie manifolds and Fredholm c.
3. Pseudodifferential operators on groupoids: (a) Groupoids, (b) Pseudodifferential operators, (c) Principal symbol, (d) Indicial operators, (e) **Groupoid C^*-algebras** and Fredholm conditions, (f) The index problem and homology.
4. Applications: (a) Well posedness on polyhedral domains (L2), (b) Essential spectrum (L3), (c) An index theorem for Callias-type operators (L4).

Collaborators

- Bernd Ammann (Regensburg),
- Catarina Carvalho (Lisbon),
- Alexandru Ionescu (Princeton),
- Robert Lauter (Mainz ...),
- Bertrand Monthubert (Toulouse)

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◇ Abstract index theorems

The exact sequence $0 \to I \to A \to Symb \to 0$ gives rise to

$$\partial : K_1(Symb) \to K_0(I)\,.$$

Let $\phi \in HP^0(I)$ (periodic cyclic cocycle). A **general (higher) index theorem** is then to compute

$$\phi_* \circ \partial : K_1(Symb) \to \mathbb{C}\,.$$

Since $\phi_* \circ \partial = \psi_*$, where $\psi = \partial\phi \in HP^1(Symb)$, the **higher index theorem** is equivalent to computing the class of ψ.

Typically in my talk, $I \subset \mathcal{K}$ and $\phi = Tr$.

Manifolds with cylindrical ends

We shall look now in some detail at the important example of **manifolds with cylindrical ends:** analysis and index theory.

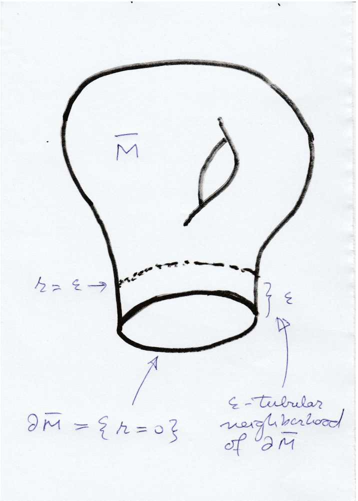
Let $\overline{M}$ be a manifold with **smooth boundary** $\partial\overline{M}$ to which we attach the semi-infinite cylinder

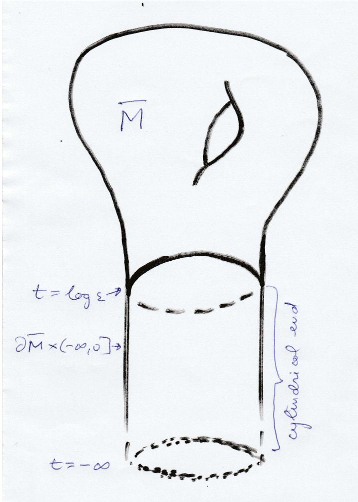
$$\partial\overline{M} \times (-\infty, 0]\,,$$

yielding a **manifold with cylindrical ends.** The metric is taken to be a product metric $g = g_{\partial\overline{M}} + dt^2$ far on the end.

Kondratiev's transform $r = e^t$ maps the cylindrical end to a tubular neighborhood of the boundary $g = g_{\partial\overline{M}} + (r^{-1}dr)^2$.

Kondratiev transform $t = \log r$





Differential operators for APS

We want differential operators with coefficients that extend to smooth functions even at infinity, that is on $\overline{M}$.

Important: ∂_t **becomes** $r\partial_r$.

In local coordinates (r, x') near the boundary $\partial\overline{M}$:

$$P = \sum_{|\alpha| \le m} a_\alpha(r, x')(r\partial_r)^{\alpha_1} \partial_{x'_2}^{\alpha_2} \ldots \partial_{x'_n}^{\alpha_n} .$$

totally characteristic differential operators. . Example:

$$\Delta = -(r\partial_r)^2 - \Delta_{\partial\overline{M}} .$$

Principal symbol

In suitable local coordinates near the bundary such that r is the distance to the boundary, shall write the resulting differential operators simply as

$$P = \sum_{|\alpha|\leq m} a_\alpha(r, x')(r\partial_r)^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n} = \sum_{|\alpha|\leq m} a_\alpha(r\partial_r)^{\alpha_1}\partial^{\alpha'} .$$

The right notion of principal symbol (near $\partial\overline{M}$) is then simply

$$\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \qquad \text{NO } r^{\alpha_1} .$$

(**It is not** $\sum_{|\alpha|=m} a_\alpha r^{\alpha_1}\xi^\alpha$ as one might think first!)

Indicial family

The **indicial family** of $P = \sum_{|\alpha|\le m} a_\alpha(r, x')(r\partial_r)^{\alpha_1}\partial^{\alpha'}$ is

$$\widehat{P}(\tau) := \sum_{|\alpha|\le m} a_\alpha(0, x')(\imath\tau)^{\alpha_1}\partial^{\alpha'}.$$

Note that $\widehat{P}(\tau)$ is a family of differential operators on $\partial\overline{M}$.

Theorem. *We have that $P : H^s(M; E) \to H^{s-m}(M; F)$ is Fredholm if, and only if, P is elliptic and $\widehat{P}(\tau)$* **is invertible for all $\tau \in \mathbb{R}$.**

Generalizes compact case, **model result.** (Lockhart-Owen, ...)

Exact sequences and the APS index formula

The index of a totally characteristic, **twisted Dirac operator** P is given by the **Atiyah-Patodi-Singer formula**, which expresses $\mathrm{ind}(P)$ as the sum of two terms:

1. The integral over $\overline{M}$ of an explicit form (local term, depends only on the principal symbol), as for AS.
2. A boundary contribution that depends only on $\widehat{P}(\tau)$, the "eta"-invariant, **not local.**

(Also Bismut, Carillo-Lescure-Monthubert, Mazzeo-Melrose, Piazza, Melrose-V.N., ...)

$\diamond$ Exact sequence

Let the fibered product $Symb := \mathcal{C}^\infty(S^*M) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}$ consists of pairs (f, Q) such that the principal symbol of the $\mathbb{R}$ invariant pseudodifferential operator Q matches the restriction of $f \in \mathcal{C}^\infty(S^*\overline{M})$ at the boundary.

Let $I(P) = \widehat{P} \in \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}$. **Since,** $r\Psi^{-1}(\overline{M}) = \Psi^0(\overline{M}) \cap \mathcal{K}$,

$$0 \to r\Psi^{-1}(\overline{M}) \to \Psi^0(\overline{M}) \xrightarrow{\sigma_0 \oplus I} \mathcal{C}^\infty(S^*\overline{M}) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}} \to 0$$

$0 \to \Psi^{-1}(\overline{M}) \to \Psi^0(\overline{M}) \xrightarrow{\sigma_0} \mathcal{C}^\infty(S^*\overline{M}) \to 0$ is not interesting.

◇ Cyclic homology

The pair $(\sigma_0(P), I(P)) \in Symb := \mathcal{C}^\infty(S^*M) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}$ is invertible if, and only if, P is Fredholm.

Combining $\partial : K_1(Symb) \to K_0(r\Psi^{-1}(\overline{M}))$ with the boundary map

$$\mathrm{ind} = Tr_* \circ \partial : K_1(Symb) \to \mathbb{C}$$

we see that the APS index formula is also equivalent to the calculation of the class of the cyclic cocycle $\partial Tr \in HP^1(Symb)$.

Important: The **noncommutativity of the algebra of symbols** $Symb$ explains the fact that the **APS formula is non-local.**

Summary: Exact sequences and index

$$0 \to \Psi^{-1}(M) \to \Psi^{0}(M) \to \mathcal{C}^\infty(S^*M) \to 0\,, \quad \text{(AS)}$$
$$0 \to \Psi^{-1}_{\mathcal{F}}(M) \to \Psi^{0}_{\mathcal{F}}(M) \to \mathcal{C}^\infty(S^*\mathcal{F}) \to 0\,, \quad \text{(Connes)}$$
$$0 \to r\Psi^{-1}(\overline{M}) \to \Psi^{0}(\overline{M}) \to \mathcal{C}^\infty(S^*\overline{M}) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}} \to 0\,.$$

The **index** is given by ($Symb$ =the quotient)

$$\text{ind} = \phi_* \circ \partial = \psi_* : K_1(Symb) \to \mathbb{C}\,,$$

where $\phi = Tr$ in the AS and APS cases and ϕ is a foliation cyclic cocycle in Connes' exact sequence.

Important: $r\Psi^{-1}(\overline{M}) \subset \mathcal{K}$, whereas $\Psi^{-1}_{\mathcal{F}}(M) \not\subset \mathcal{K}$, in general.

Motivating examples

Laplacian in **polar** coordinates (ρ, θ) in **2D** is a totally characteristic differential operator (ignoring ρ^{-2}) generated by the Lie algebra of vector fields spanned by $\rho\partial_\rho$ and ∂_θ,

$$\Delta u = \rho^{-2}\big(\rho^2\partial_\rho^2 u + \partial_\theta^2 u\big).$$

The Laplacian in **cylindrical** coordinates (ρ, θ, z) in **3D** is

$$\Delta u = \rho^{-2}\big((\rho\partial_\rho)^2 u + \partial_\theta^2 u + (\rho\partial_z)^2\big).$$

It is not totally characteristic, but generated by the Lie algebra of vector fields $\rho\partial_\rho$, ∂_θ, and $\rho\partial_z$. Nonabelian: $[\rho\partial_\rho, \rho\partial_z] = \rho\partial_z$.

Lie algebras of vector fields

In general, motivated by our examples, we shall consider: **differential operators generated by a Lie algebra of vector fields on a manifold $\overline{M}$.** This deals with the **degeneracies.**

Moreover, we see that the manifold $\overline{M}$ is:

- $(\rho, \theta) \in \overline{M} = [0, \infty) \times S^1$ or $\overline{M} = [0, \infty) \times [0, \alpha]$, for Δ in **2D.**
- For Δ in dihedral angle in **3D**:

$$(\rho, \theta, z) \in [0, \infty) \times [0, \alpha] \times \mathbb{R}\,.$$

We are thus lead to consider manifolds $\overline{M}$ **locally** of the form $[0, 1]^k$: **Manifolds with corners.** (Kondratiev, Mazya, Melrose).

Lie manifolds: Notation (manifolds with corners)

Switch gears: more details.

In what follows $\overline{M}$ will denote a compact manifold **with corners** (locally like $[0,1]^n$).

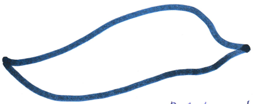
A face $H \subset \overline{M}$ of maximal dimension is called a **hyperface**.

Recall that a **defining function** of a hyperface H of $\overline{M}$ is a function x such that $H = \{x = 0\}$ and $dx \neq 0$ on H.

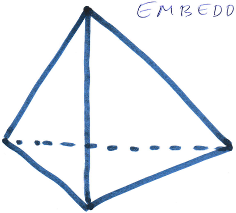
The hyperface $H \subset \overline{M}$ is called **embedded** if it has a defining function.

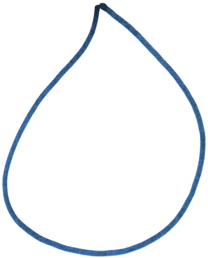
Embedded and non-embedded faces

“TEAR – DROP DOMAIN”



ALL H EMBEDDED





EXACTLY ONE HYPERFACE, NON EMBEDDED.

Lie manifolds: more notation

- $M =$ the *interior* of $\overline{M}$:

$$M = \overline{M} \smallsetminus \cup \text{ faces } .$$

- $\Gamma(E) =$ space of *smooth sections* of $E \to \overline{M}$, so
- $\Gamma(T\overline{M}) =$ the space of *smooth vector fields* on $\overline{M}$.
- $\mathcal{V}_b \subset \Gamma(T\overline{M})$ is the set of vector fields *tangent to all faces*.
- $\mathcal{V} \subset \mathcal{V}_b$ is a **Lie algebra of vector fields:** $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

Definition of Lie manifolds

Definition. [Ammann-Lauter-V.N.] A **Lie manifold** is pair $(\overline{M}, \mathcal{V})$ consisting of a compact manifold with corners $\overline{M}$ and a subspace $\mathcal{V} \subset \mathcal{V}_b$ of vector fields that satisfy:

- $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$;
- $\mathcal{V}$ is a finitely-generated, projective $\mathcal{C}^\infty(\overline{M})$–module;
- the vector fields $X_1, \ldots, X_n$ that locally generate $\mathcal{V}$ around an interior point p also give a local basis of T_pM.

Particular cases: Cordes, Melrose's, Parenti, Schulze.

We observe that $\Gamma_c(T\overline{M}) \subset \mathcal{V}$ (equivalent to the last condition).

First example: cylindrical ends

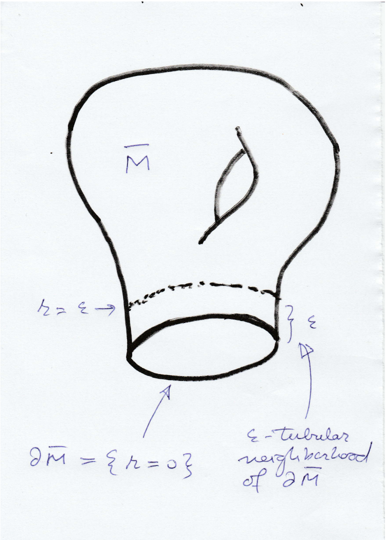
- $\overline{M}$ = a manifold with **smooth boundary** with defining function x (so $\partial\overline{M} = \{x = 0\}$).
- $\mathcal{V} = \mathcal{V}_b$ the space of vector fields on $\overline{M}$ that are **tangent** to the boundary $\partial\overline{M}$.
- At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis of $\mathcal{V}$ is given by $x\partial_x$, ∂_{y_2}, ... , ∂_{y_n}.

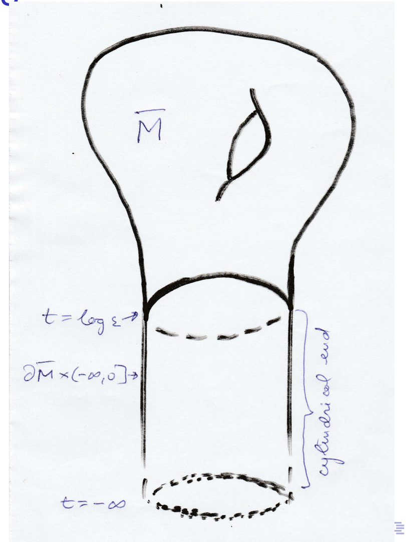
  ($y_2, \ldots, y_n$ are local coordinates on $\partial\overline{M}$.)
- There is **no condition on these vector fields in the interior** (valid for all Lie manifolds).

The Riemannian metric of a **manifold with cylindrical ends**.

APS: APS, Debord-Lescure, Kondratiev, Melrose, Schulze.

Kondratiev transform $t = \log r$





The Lie algebroid $A_{\mathcal{V}}$ associated to a Lie manifold

To discuss metrics on Lie manifolds, we need **algebroids.**

Recall that a **Lie algebroid** $A \to \overline{M}$ is a vector bundle over $\overline{M}$ together with a *Lie algebra* structure on $\Gamma(A)$ and a bundle map (anchor) $\varrho : A \to T\overline{M}$ such that

- $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$ and
- $[X, fY] = f[X, Y] + (\varrho(X)f)Y$, where $X, Y \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(\overline{M})$.

(We extend the anchor map ρ to a map $\varrho : \Gamma(A) \to \Gamma(T\overline{M})$.)

The Lie algebroid $A_{\mathcal{V}}$ associated to a Lie manifold

Let $(\overline{M}, \mathcal{V})$ be a Lie manifold. Recall then that $\mathcal{V}$ is a **finitely generated, projective** $\mathcal{C}^\infty(\overline{M})$–module.

The Serre–Swan Theorem implies then that there exists a finite dimensional vector bundle $A_{\mathcal{V}} \to \overline{M}$, uniquely defined up to isomorphism, such that

$$\mathcal{V} \simeq \Gamma(A_{\mathcal{V}}).$$

We call $A_{\mathcal{V}} \to \overline{M}$ the **the Lie algebroid associated to** $(\overline{M}, \mathcal{V})$.

This leads to the following new definition of a *Lie manifold*.

Definition of Lie manifolds using Lie algebroids

The pair $(\overline{M}, \mathcal{V})$ is a **Lie manifold** if, and only if, there is a Lie algebroid $A_{\mathcal{V}} \to T\overline{M}$ such that:

- the anchor map $\varrho : A_{\mathcal{V}} \to T\overline{M}$ is an **isomorphism** over $M := \overline{M} \smallsetminus \partial M$ and
- the Lie algebra of vector fields

$$\mathcal{V} := \Gamma(A_{\mathcal{V}}) = \varrho(\Gamma(A_{\mathcal{V}}))$$

consists of vector fields **tangent** to all faces of $\overline{M}$.

Metric and geometry

Let $(\overline{M}, \mathcal{V})$ be a **Lie manifold** and $A = A_{\mathcal{V}} \to \overline{M}$ be its associated Lie algebroid, that is

$$\mathcal{V} \simeq \Gamma(A).$$

Then A extends TM to $\overline{M}$, namely

$$A|_M \simeq TM.$$

In particular, **a metric on A will induce a Riemannian metric on TM**, i.e. a metric on M. (Cylindrical ends for our first example.)

Connections

The **Levi-Civita connection** $\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$, extends to an **$A^*$-valued connection**

$$\nabla : \Gamma(A) \to \Gamma(A \otimes A^*),$$

satisfying for all $X, Y, Z \in \mathcal{V} = \Gamma(A)$:

$$\nabla_X(fY) = X(f)Y + f\nabla_X(Y) \quad \text{and} \quad X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

The covariant derivatives $\nabla^k R$ of the curvature R extend to $\overline{M}$, and hence they are **bounded**: **Bounded geometry**.

Differential operators

Define $\text{Diff}(\mathcal{V})$ =**the algebra of differential operators on M generated by $\mathcal{C}^\infty(\overline{M})$ and vector fields $X \in \mathcal{V}$.**

We can extend the definition of $\text{Diff}(\mathcal{V})$ to include operators $\text{Diff}(\mathcal{V}; E, F)$ acting between vector bundles $E, F \to \overline{M}$. Then

$$d \in \text{Diff}(\mathcal{V}; \Lambda^q A^*, \Lambda^{q+1} A^*) \quad \text{and}$$
$$\nabla \in \text{Diff}(\mathcal{V}; A, A \otimes A^*).$$

Theorem. (Ammann–Lauter–N.)

$$\Delta \in \text{Diff}(\mathcal{V}).$$

Similarly, all **geometric** differential operators on $\overline{M}$ are generated by $\mathcal{V}$. (Done "by hand" for the first examples.)

The three basic examples

We have already seen that our first example of a Lie manifold (when $\mathcal{V} = \mathcal{V}_b$) recovers the framework of the APS Index Theorem and $\mathrm{Diff}(\mathcal{V}_b)$ consists of the totally characteristic operators considered in the first lecture.

The “example zero” can be $\mathcal{V} = \Gamma(TM)$, for M compact smooth (no corners). This example of Lie manifold recovers the framework of the AS Index Theorem.

For foliations, the choice $\mathcal{V} = \Gamma(\mathcal{F})$ does not satisfy the assumption that $\Gamma_c(TM) \subset \mathcal{V}$, **unless** we are actually in the AS framework.

Second example: asymptotically hyperbolic manifolds

- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = x\Gamma(T\overline{M})$ = the space of vector fields on $\overline{M}$ that vanish on the boundary.
- At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_n}$.
- No condition in the interior (all Lie manifolds).

The metric is *asymptotically hyperbolic*.

Pseudodifferential calculus: Lauter, Mazzeo, Schulze.