Analysis on singular spaces, Lie manifolds, and non-commutative geometry I

Index theory

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Abstract of series

My four lectures are devoted to **Analysis and Index Theory** on singular and non-compact spaces.

We will concentrate more on the **analysis needed** to set up the index problems than on the index problems themselves.

We are also interested in the eventual applications, which, however, will not be restricted to index problems.

From a technical point of view, a central place in my presentation will be occupied by **exact sequences:**

$$0 \to I \to A \to Symb \to 0 .$$

Abstract of series (cont.)

The role of the exact sequence $0 \to I \to A \to Symb \to 0$ in a nut shell, is as follows:

1. A is a suitable **algebra of operators** that describes the analysis on a given (class of) singular space(s). Will be constructed using Lie algebroids and Lie groupoids.
2. the **ideal** $I = A \cap \mathcal{K}$ of compact operators (to be described). (Foliations: $I \not\subset \mathcal{K}$, different!)
3. **the algebra of symbols** $Symb := A/I$ needs to be described and leads to Fredholm conditions.
4. I, A, or $Symb$ will appear in applications.

The contents of the four talks

1. Motivation: Index Theory (a) Exact sequences and index theory (b) The Atiyah-Singer index theorem (c) Foliations (d) The Atiyah-Patodi-Singer index theorem (e) More singular examples. **No new results.**
2. Lie Manifolds: (a) Definition (b) The APS example (c) Lie algebroids (d) Metric and connection (e) Fredholm conditions (f) Examples :Lie manifolds and Fredholm c.
3. Pseudodifferential operators on groupoids: (a) Groupoids, (b) Pseudodifferential operators, (c) Principal symbol, (d) Indicial operators, (e) **Groupoid C^*-algebras** and Fredholm conditions, (f) The index problem and c. homology.
4. Applications: (a) Well posedness on polyhedral domains (L2), (b) Essential spectrum (L3), (c) An index theorem for Callias-type operators (L4).

First lecture: ‘**Motivation: Index Theory**’

Abstract

We begin with a straightforward introduction to differential and pseudodifferential operators.

We then discuss three basic index theorems and their associated analysis (or exact sequences):

- The Atiyah-Singer (AS) index theorem.
- Connes’ index theorem for foliations.
- The Atiyah-Patodi-Singer (APS) index theorem.

We will see that Connes’ and APS frameworks extend the AS framework in complementary directions. **“Higher APS.”**

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Derivations

We fix notation and recall a few basic concepts.

On $\mathbb{R}^n$ we consider the derivations (or *vector fields*)

$$\partial_j = \frac{\partial}{\partial x_j},$$

$j = 1, \ldots, n$ and form the differential monomials

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}, \qquad \alpha \in \mathbb{Z}_+^n .$$

We denote $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \in \mathbb{Z}_+$ the *order* of the differential monomial ∂^α.

Differential operators

A *differential operator* P of order $\leq m$ on $\mathbb{R}^n$ is then an operator

$$P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

of the form

$$Pu = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha u \,.$$

The functions a_α are the *coefficients* of the operator P and will be assumed in this talk to be *smooth functions.* Then

$$P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}_c^\infty(\mathbb{R}^n) \,.$$

Systems ...

It is easy, but important, to consider systems (vector Laplacians, elasticity, signature, Maxwell, ...). Then

$$u = (u_1, u_2, \dots, u_k) \in \mathcal{C}_c^\infty(\mathbb{R}^n)^k = \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^k)$$

is a (smooth, compactly supported) section of the trivial vector bundle $\underline{\mathbb{R}^k} = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ and hence

$$a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n; M_k(\mathbb{R}))$$

is a matrix valued function. It is *an endomorphism* of the trivial bundle $\underline{\mathbb{R}^k}$.

Sobolev spaces

Let $\Delta = -\partial_1^2 - \ldots - \partial_n^2 \geq 0$ and $s \in \mathbb{Z}_+$. We denote as usual

$$H^s(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{C},\ \partial^\alpha u \in L^2(\mathbb{R}^n),\ |\alpha| \leq s\} = \mathcal{D}(\Delta^{s/2}),$$

Both definitions of Sobolev spaces extend to "Lie manifolds."

This definition extends immediately to vector valued functions and, if the coefficients a_α of P are bounded (together with enough derivatives: $P \in W^{m,\infty}$), then we obtain that P

$$P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n).$$

Principal symbol

Let then $a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n; M_k(\mathbb{R}))$ and

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha : \mathcal{C}_c^\infty(\mathbb{R}^n)^k \to \mathcal{C}_c^\infty(\mathbb{R}^n)^k .$$

The function

$$\sigma_m(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)(\imath\xi)^\alpha \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; M_k)$$

is called the **principal symbol** of P (note $\sigma_{m+1}(P) = 0$).

Here $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ is the *dual* variable.

Invariant definition

The fact that ξ is a dual varialbe to $x \in \mathbb{R}^n$ is seen by using transformations of coordinates.

The principal symbol is thus seen to be a function on $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$.

It turns out that the principal symbol $\sigma_m(P)$ of P has a much simpler transformation formula than the *(full) symbol* of P

$$\overline{\sigma}(P)(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)(\imath\xi)^\alpha \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; M_k)\,.$$

Pseudodifferential operators

The full symbol $p(x, \xi)$ of P

$$p(x,\xi) = \overline{\sigma}(P)(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)(\imath\xi)^\alpha \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; M_k)$$

is nevertheless important because $P = p(x, \partial)$, where

$$p(x,D)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\imath x\cdot\xi} p(x,\xi)\hat{u}(\xi) d\xi \, .$$

For more general p, $P := p(x, D)$ will be a **pseudodifferential operator.**

Symbols and pseudodifferential operators

Let $S^m_{1,0}(\mathbb{R}^k \times \mathbb{R}^N)$ to be the space of functions $a : \mathbb{R}^{k+N} \to \mathbb{C}$ that satisfy, for any $i, j \in \mathbb{Z}_+$, the estimate

$$|\partial_x^i \partial_\xi^j a(x, \xi)| \leq C_{i,j}(1 + |\xi|)^{m-j} .$$

If $a \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) = S^m_{1,0}(\mathbb{R}^{2n})$, we define

$$a(x, D)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\imath x \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi .$$

The operator $a(x, D) : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is called a **pseudodifferential operator.**

Classical pseudodifferential operators

The pseudodifferential operator $a(x, D)$ is **classical** if $a \sim \sum_{j=0}^{\infty} a_j \in S^m_{cl}(\mathbb{R}^{2n})$, with a_j **eventually homogeneous** of order $m - j$, in which case we define the principal symbol

$$\sigma_m(a(x, D)) = a_0 \in \mathcal{C}^\infty(T^*\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^{2n}),$$

Summarizing, we see that if $P = \sum_{|\alpha| \leq m} p_\alpha \partial^\alpha$ and $p(x, \xi) := \sum_{|\alpha| \leq m} p_\alpha (\imath\xi)\alpha$ is its full symbol, then $P = p(x, D)$ and P is a **classical pseudodifferential operator** of order m.

The two definitions of principal symbol clearly coincide.

Manifolds

The definition of a (pseudo)differential operator P (of order $\leq m$) and of its principal symbol $\sigma_m(P)$ then extend to manifolds (using vector bundles). We replace

$$\mathbb{R}^n \leftrightarrow M = \text{ a smooth manifold}$$

$$\mathcal{C}_c^\infty(\mathbb{R}^n)^k \leftrightarrow \mathcal{C}_c^\infty(M;E) = \text{ sections of } E \text{ (or } F\text{), so}$$

$$P : \mathcal{C}_c^\infty(M;E) \to \mathcal{C}_c^\infty(M;F)$$

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M; \mathrm{Hom}(E,F))$$

For Sobolev spaces, however, we need also a **metric** on M (a Lipschitz equivalence classes of such metrics).

Principal symbol multiplicativity

The main property of the principal symbol, more important than its diffeomorphism invariance, is the *multiplicativity property*

$$\sigma_{m+m'}(PP') = \sigma_m(P)\sigma_{m'}(P'),$$

a property that is enjoyed by its extension to *pseudodifferential operators* (which are allowed to also have negative orders).

Definition. A (classical, pseudo)differential operator P is called **elliptic** if its principal symbol is invertible away from the zero section of T^*M.

Fredholm operators

Let now M be a **compact**, smooth manifold, so the Sobolev spaces $H^s(M)$ are uniquelly defined.

Let also P be a (classical, pseudo) differential operator of order $\leq m$ acting between sections of vector bundles E, F.

Recall that $T : X \to Y$ is Fredholm, if and only if, $\ker(P) := \{u \in X, Tu = 0\}$ and $\mathrm{coker}(P) := Y/TX$ are finite dimensional. Our **model result** is then:

Theorem. *The operator $P : H^s(M; E) \to H^{s-m}(M; F)$ is Fredholm if, and only if, P is elliptic.*

The index

Fredholm operators appear all the time in applications (because elliptic operators are so fundamental).

Since the invertibility of an operator P, which is equivalent to $\dim \ker(D) = \dim \operatorname{coker}(D) = 0$, it is important then to calculate the **index** $\operatorname{ind}(P)$:

$$\operatorname{ind}(P) = \dim \ker(P) - \dim \operatorname{coker}(P) .$$

The index has better stability properties than simply $\dim \ker(P)$ or $\dim \operatorname{coker}(P)$. The index is, for instance, *homotopy invariant and depends only on the principal symbol of* P.

◇ Atiyah-Singer index formula

Closer to our goals: INDEX THEORY. Switch gears. ◇.

Theorem. (Atiyah-Singer) *Let M be compact and P be elliptic, then* $\operatorname{ind}(P) = \langle\, ch[\sigma_m(P)]\mathcal{T}(M)\,,\,[T^*M]\,\rangle$.

- $ch[\sigma_m(P)] \in H_c^{even}(T^*M)$ the Chern character of the principal symbol $\sigma_m(P)$ of P.
- $\mathcal{T}(M)$ is the Todd class of M.
- $[T^*M]$ is the fundamental class of T^*M.

The A.-S. index formula was much studied and has found many applications. We want to extend it to the noncompact and singular cases. **Main focus of my presentation: Analysis**.

◇ Abstract index theorems

As explained in the Abstract, the main focus of this presentation is the analysis of index theorems. So, in order to motivate our results, we now look at an abstract index theorem.

Let

$$0 \to I \to A \to Symb \to 0$$

be an exact sequence of algebras.

Then the boundary map in K-theory gives rise to the connecting (or boundary) morphism

$$\partial : K_1(Symb) \to K_0(I).$$

◇ Abstract index theorems

If the ideal I of the exact sequence $0 \to I \to A \to Symb \to 0$ consists of compact operators (i.e. $I \subset \mathcal{K}$), we have a natural map

$$Tr_* : K_0(I) \to \mathbb{Z} = K_0(\mathcal{K})\,.$$

Recall the boundary map $\partial : K_1(Symb) \to K_0(I)$. Then $Tr_* \circ \partial$ **computes the usual (Fredholm) index:**

$$\text{ind} \;=\; Tr_* \circ \partial \;:\; K_1(Symb) \xrightarrow{\partial} K_0(I) \xrightarrow{Tr_*} \mathbb{C}\,.$$

Indeed, if $P \in A$ is invertible in $Symb$, then P defines a class $[P] \in K_1(Symb)$. Moreover, P is **Fredholm** and its index is

$$\text{ind}(P) = Tr_* \circ \partial[P].$$

◇ Abstract index theorems

The map Tr_* of the basic equation $\text{ind}(P) = Tr_* \circ \partial[P]$ when $I \subset \mathcal{K}$ is particular instance of the pairing between **cyclic cohomology** and K-theory. More important when $I \not\subset \mathcal{K}$.

HP^* =**periodic cyclic cohomology** groups (Connes).

Let $\phi \in HP^0(I)$. A more **general (higher) index theorem** is then to compute

$$\phi_* \circ \partial : K_1(Symb) \to \mathbb{C}\,.$$

(Also given by a cyclic cocycle V.N.) Useful for **foliations**, since $I \not\subset \mathcal{K}$. **Typically, however, $I \subset \mathcal{K}$ in my talks.**

◇ A.-S. and exact sequences (of Ψdos)

Let $\Psi^m(M)$ be the space of classical pseudodifferential operators of order $\le m$ on M. We then have an exact sequence of *algebras* **(analysis!)**

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \xrightarrow{\tilde{\sigma}_0} \mathcal{C}^\infty(S^*M) \to 0$$

where the first map is the inclusion, S^*M is the set of rays (directions) in T^*M, and the second map $(\tilde{\sigma}_0)$ is defined by

$$\tilde{\sigma}_0(P)(x,\xi) := \lim_{t\to\infty} \sigma_0(P)(x,t\xi)\,, \quad |\xi| = 1\,,$$

since the principal symbol $\sigma_0(P)$ of an operator $P \in \Psi^0(M)$ is (asymptotically) homogeneous of order zero.

◇ Boundary map

The exact sequence of *algebras* (we drop the $\sim$ from σ_0)

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \xrightarrow{\sigma_0} \mathcal{C}^\infty(S^*M) \to 0$$

gives rise to a natural (boundary) map in (algebraic or topological) K-theory

$$\partial : K_1(\mathcal{C}^\infty(S^*M)) = K^1(S^*M) \to K_0(\Psi^{-1}(M)) \simeq \mathbb{Z},$$

(last isomorphism defined by the trace) such that

$$Tr_* \circ \partial[\sigma_0(P)] = \mathrm{ind}(P) \in \mathbb{Z}.$$

◇ Connes' approach

It is therefore enough to compute $Tr_* \circ \partial : K_1(\mathcal{C}^\infty(S^*M)) \to \mathbb{Z}$, which was shown by Connes to be given by a **cyclic cocycle** ψ_{ind}.

The **Atiyah-Singer index formula is thus equivalent to the computation of** $[\psi_{\text{ind}}] \in HP^*(\mathcal{C}^\infty(S^*M)) \simeq H_*(S^*M)$.

It is therefore possible to express the A.-S. Index Formula purely **in classical terms** (vector bundles and cohomology) because the quotient $\Psi^0(S^*M)/\Psi^{-1}(S^*M) \simeq \mathcal{C}^\infty(S^*M)$ **is commutative.** (NC case: Connes.)

◇ Useful example: Foliation algebras

Cyclic homology is especially useful for **foliations.**

We regard a foliation $(M, \mathcal{F})$ of a smooth, compact manifold M as a sub-bundle $\mathcal{F} \subset TM$ that is *integrable* ($\Gamma(\mathcal{F})$ Lie algebra).

Connes' construction of pseudodifferential operators along the leaves of a foliation then yields the exact sequence *algebras*

$$0 \to \Psi_{\mathcal{F}}^{-1}(M) \to \Psi_{\mathcal{F}}^{0}(M) \xrightarrow{\sigma_0} \mathcal{C}^{\infty}(S^*\mathcal{F}) \to 0\,,$$

For $\mathcal{F} = TM$, it reduces to the earlier exact sequence

$$0 \to \Psi^{-1}(M) \to \Psi^{0}(M) \xrightarrow{\sigma_0} \mathcal{C}^{\infty}(S^*M) \to 0\,.$$

◇ Different type of example: Foliation indices

Connes' exact sequence *algebras*

$$0 \to \Psi_{\mathcal{F}}^{-1}(M) \to \Psi_{\mathcal{F}}^{0}(M) \xrightarrow{\sigma_0} \mathcal{C}^\infty(S^*\mathcal{F}) \to 0\,,$$

then yields a boundary (or index) map

$$\partial\,:\, K_1(\mathcal{C}^\infty(S^*\mathcal{F})) = K^1(S^*\mathcal{F}) \,\to\, K_0(\Psi_{\mathcal{F}}^{-1}(M)) \simeq K_0(\mathcal{C}_c^\infty(\mathcal{F}))\,.$$

Difficult: There are few determinations of $K_0(\Psi_{\mathcal{F}}^{-1}(M))$.

Moreover, unlike our other examples, $\Psi_{\mathcal{F}}^{-1}(M))$ has no canonical proper ideals, so there are no other index map.

◇ Cyclic homology

The determination of **periodic cyclic cohomology** of foliations in terms of the twisted cohomology of the classifying space (Connes, Brylinsky-V.N., Crainic), yields a large set of maps

$$\phi_* : K_0(\mathcal{C}_c^\infty(\mathcal{F})) \to \mathbb{C}\,,$$

each of which defines an index map

$$\phi_* \circ \partial : K_0(\mathcal{C}^\infty(S^*\mathcal{F})) \to \mathbb{C}\,.$$

We will not pursue this, but we note Connes' results. (Benameur-Heitch for Haeffliger homology, V. N. for foliated bundles.)

Manifolds with cylindrical ends

Warning: New ideas. Switch gears.

Let $\overline{M}$ be a manifold with **smooth boundary** $\partial\overline{M}$.

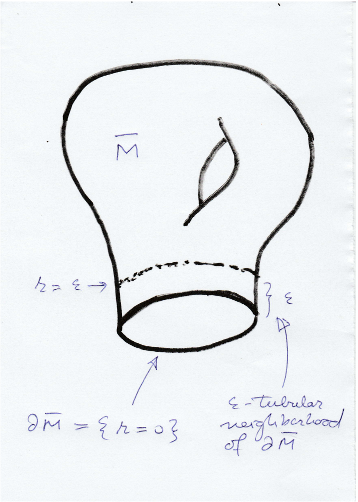
To $\overline{M}$ we attach the semi-infinite cylinder

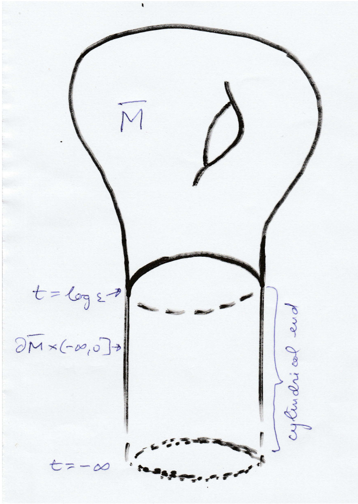
$$\partial\overline{M} \times (-\infty, 0]\,,$$

yielding a **manifold with cylindrical ends.** The metric is taken to be a product metric $g = g_{\partial\overline{M}} + dt^2$ far on the end.

Kondratiev's transform $r = e^t$ maps the cylindrical end to a tubular neighborhood of the boundary $g = g_{\partial\overline{M}} + (r^{-1}dr)^2$.

Kondratiev transform $t = \log r$





Differential operators for APS

We want differential operators with coefficients that extend to smooth functions even at infinity.

Because of this, it is more convenient to work on $\overline{M}$ than on $\overline{M} \cup \partial\overline{M} \times (-\infty, 0]$. We take the coefficients to be smooth functions up to the boundary. However, ∂_t **becomes** $r\partial_r$.

In local coordinates (r, x') on the distinguished tubular neighborhood of $\partial\overline{M}$, we obtain ($n = \dim(\overline{M})$)

$$P = \sum_{|\alpha| \leq m} a_\alpha(r, x')(r\partial_r)^{\alpha_1} \partial_{x'_2}^{\alpha_2} \ldots \partial_{x'_n}^{\alpha_n} .$$

totally characteristic differential operators.

Principal symbol

We shall write the resulting differential operators simply as

$$P = \sum_{|\alpha| \leq m} a_\alpha(r, x')(r\partial_r)^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} = \sum_{|\alpha| \leq m} a_\alpha (r\partial_r)^{\alpha_1} \partial^{\alpha'} .$$

The right notion of principal symbol (near $\partial \overline{M}$) is then simply

$$\sigma_m(P) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha .$$

(**Not** $\sum_{|\alpha|=m} a_\alpha r^{\alpha_1} \xi^\alpha$ as one might think first!)

Indicial family

The **indicial family** of $P = \sum_{|\alpha|\le m} a_\alpha(r, x')(r\partial_r)^{\alpha_1}\partial^{\alpha'}$ is

$$\widehat{P}(\tau) := \sum_{|\alpha|\le m} a_\alpha(0, x')(\imath\tau)^{\alpha_1}\partial^{\alpha'}.$$

Note that $\widehat{P}(\tau)$ is a family of differential operators on $\partial\overline{M}$.

Last ingredient for Fredholmness: Let $M := \overline{M} \smallsetminus \partial\overline{M}$ with the cylindrical end metric. Since the cylindrical end metric is complete, the Laplacian Δ is self-adjoint, and hence we cand **define Sobolev space** $H^s(M) := \mathcal{D}(\Delta^{s/2})$, the domain of $\Delta^{s/2}$.

Fredholm operators

We have a characterization of Fredholm *totally characteristic differential operators* similar to the compact case (difference in **red**).

Theorem. *Assume M has cylindrical ends and P is totally characteristic. We have that $P : H^s(M; E) \to H^{s-m}(M; F)$ is Fredholm if, and only if, P is elliptic and* $\widehat{P}(\tau)$ **is invertible for all $\tau \in \mathbb{R}$.**

Lockhart-Owen for differential operators, Melrose-Mendoza for totally characteristic pseudodifferential operators. (**Kondratiev '67**, Mazya, Schröhe, Schultze).

Degeneration and singularities: Our setting

We shall work more in the setting inspired by the APS framework. ($r\Psi^{-1}(\overline{M}) = \Psi^0(\overline{M}) \cap \mathcal{K}$, whereas $\Psi_{\mathcal{F}}^{-1}(M) \not\subset \mathcal{K}$, in general.)

Our extensions go in the direction of more general APS analysis and index formulas since they model a different type of degeneracies than foliations.

The foliation formalism nevertheless provides **many useful insights.**

(Simplest-APS type) motivating examples

- Laplacian in polar coordinates (ρ, θ) in **2D**

$$\Delta u = \rho^{-2}\big(\rho^2\partial_\rho^2 u + \rho\partial_\rho u + \partial_\theta^2 u\big).$$

- Schrödinger operator (**3D**)

$$-(\Delta + \frac{Z}{\rho})u = -\rho^{-2}\big(\rho^2\partial_\rho^2 u + 2\rho\partial_\rho u + \Delta_{S^2} u + Z\rho u\big).$$

- Black-Scholes equation (parabolic, backward Kolmogorov)

$$\partial_t u - \left(\frac{\sigma^2}{2}x^2\partial_x^2 u + rx\partial_x u - ru\right) = 0.$$

A 'higher rank' example

The Laplacian in cylindrical coordinates (ρ, θ, z) in **3D** is

$$\Delta u = \rho^{-2}\big((\rho\partial_\rho)^2 u + \partial_\theta^2 u + (\rho\partial_z)^2\big).$$

Ignoring the factor ρ^{-2}, we see that our differential operator is generated by the **vector fields**

$$\rho\partial_\rho,\ \partial_\theta,\ \rho\partial_z\ ,$$

and that these vector fields form a **Lie algebra**.

Lie algebras of vector fields = main ingredient.