Dirac operators
on noncommutative principal torus bundles

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(based on works with A. Sitarz & A. Zucca)

Plan

- Why?
- Classical
- Quantum:

  -Projectability

  -Isometric fibers

  -Real structure

  -Dirac calculus

  -Strong connections

  -Twisted Dirac
- Examples:

  -classical

  -NC torus

  -theta deormations

Why ? NCG marries 'geometric' & 'quantum':

space $\leftrightarrow$ algebra

metric & spin $\leftrightarrow$ spectral triple (S.T.)

Prototype: canonical S.T. $\Big(C^\infty(M), L^2(\Sigma(M), vol_g), D\Big)$

$\exists$ reconstruction thm

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Envisage more:
fibre bundles; start with principal, even with (usual) Lie group, and first $U(1)^n$...

Classics: $U(1)$ [Amman-Bär], many works on G/H await systematic study

Compatible metrics

Let M be a $(m+n)$-dimensional oriented compact smooth manifold which is the total space of a $\boxed{principal\, \mathbb{T}^n - bundle}$ over the m-dimensional oriented manifold $N = M/\mathbb{T}^n$.

Let K_a be the fundamental vector field of the standard basis T_a, $1 \leq a \leq n$ of $\mathbb{R}^n := Lie(\mathbb{T}^n)$).

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Assume that $(M, \widetilde{g})$ & (N, g) are Riemannian, s.t.

- the action of $\mathbb{T}^n$ is isometry of $\widetilde{g}$;
- the bundle projection $\pi : M \to N$ is an oriented Riemannian submersion;
- the fibres are isometric one to each other; moreover, $|K_a|$ is constant along M.

Then $\exists!$ principal $\boxed{connection}$ 1-form $\omega = \sum_{a=1}^{n} \omega_a \otimes T_a$, s.t. $\ker \omega \perp$ fibres.

Compatible classical ST

Assume M is spin and spinor bundle ΣM is $\boxed{projectable}$, ie. admits lift of $\mathbb{T}^n$.
Then the Dirac operators $\widetilde{D}$ on $L^2(\Sigma M, \tilde{g})$ and D on $L^2(\Sigma N, g)$ are <u>compatible</u> [AB],[DZ] (for simplicity m even):

$$\boxed{\widetilde{D} = D_v + D_h + Z,}$$

where:

$$D_v = \sum_{a=1}^{n} \frac{1}{l_a} \gamma^a \partial_{K_a} = \sum_{a=1}^{n} \gamma^a \partial_{e_a}, \quad \text{'vertical'},$$

$$Z = -\frac{1}{4} \sum_{a=1}^{n} \gamma(K_a)\gamma(d\omega_a), \quad \text{'order 0'}$$

and

$$D_h = \cdots = \bigoplus_{k \in \mathbb{Z}^n} Q_k \circ (D_k \otimes \mathsf{id}) \circ Q_k^{-1}, \quad \mathsf{'horizontal'}.$$

Here D_k is D twisted by ω_a, and it acts on $\Sigma N \otimes L_k$,
where $L_k = M \times_{\mathbb{T}^n} \mathbb{C}$ is the line bundle on N associated with weight k of $\mathbb{T}^n$.
Next,

$$Q_k : L^2(\Sigma N \otimes L_k) \otimes \Sigma_n \to V_k$$

are certain isomorphisms, where we decompose into $\mathbb{T}^n$-irreps as

$$L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}^n} V_k.$$

Noncommutative analogue ?   No $\widetilde{g}, g \ldots$

We extended (part of) [AB] to NC circle [DS], [DAZ] and $\mathbb{T}^n$-bundles [DZ]

Will present ("growing") ingredients:

Noncommutative principal bundles

Group $\rightsquigarrow$ Hopf algebra H, with coproduct Δ, invertible antipode S and counit ε.
Quantum principal bundle $\rightsquigarrow$ a right H-comodule algebra $\mathcal{A}$, which is *principal*, i.e. $\exists$ *strong connection* (for the universal calculus $\Omega^1\mathcal{A}$).
Base space $\rightsquigarrow \mathcal{B} := \{a \in \mathcal{A} \mid \Delta_{\mathcal{A}}(a) = a \otimes 1\}$.

It follows [Haj96] that $\mathcal{A}$ is a *Hopf-Galois extension* of $\mathcal{B}$, and so a quantum principal bundle as in [BrzMaj93], with respect to $\Omega^1\mathcal{A}$.
I'll say what happens for a general differential calculus.

In [Haj96] also strong connection for a general calculus was defined.
I'll specify it to *Dirac* calculi, in equivalent form.

Our H**:** unital * Hopf algebra $H(\mathbb{T}^n)$ of the Lie group $\mathbb{T}^n = U(1)^n$; generated by commuting unitaries $z_1, \dots, z_n$, &

$$\Delta(z_i) = z_i \otimes z_i, \qquad S(z_i) = z_i^* = z_i^{-1}, \qquad \varepsilon(z_i) = 1.$$

Notation: for $k \in \mathbb{Z}^n$ we set $z^k := z_1^{k_1} \dots z_n^{k_n}$.

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We call *quantum principal T^n-bundle* a principal $H(\mathbb{T}^n)$-comodule algebra $\mathcal{A}$. It is strongly $\mathbb{Z}^n$-graded

$$\mathcal{A} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{A}^{(k)}, \quad \boxed{\mathcal{A}^{(k)} \mathcal{A}^{(k')} = \mathcal{A}^{(k+k')}},$$

(see e.g. [Sch04]), where

$$a \in \mathcal{A}^{(k)} \quad \Leftrightarrow \quad \Delta_{\mathcal{A}}(a) = a \otimes z^k \quad (\Leftrightarrow \quad \delta_j(a) = k_j a).$$

Spectral triples

In NC Riemannian spin geometry = spectral triple $(\mathcal{A}, \mathcal{H}, D)$
'over' algebra $\mathcal{A}$ with a representation π (often omitted) on the Hilbert space $\mathcal{H}$
& "Dirac operator" $D = D^*$, s.t. $[D, a] \in B(\mathcal{H})$ for $a \in \mathcal{A}$, and $D^{-1} \in K(\mathcal{H})$.

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even if $\exists$ grading γ, s.t. $\gamma a = a\gamma$, $D\gamma = -\gamma D$.
real if $\exists$ antilinear J s.t. $[JAJ, A] = 0$ and

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J.$$

Here $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ depend mod 8 on the so-called KR-dimension.

We require *first order condition*

$$[[D,a],\ Jb^*J^{-1}] = 0, \quad \forall a,b \in \mathcal{A}. \tag{1}$$

Then there is right action of $\mathcal{A}$ on $\mathcal{H}$

$$\phi \cdot a = Ja^*J^{-1}\phi, \quad \phi \in \mathcal{H}, a \in \mathcal{A}$$

and of Dirac 1-forms

$$\Omega^1_D(\mathcal{A}) := \pi_D(\Omega^1\mathcal{A}), \quad \pi_D(p \otimes q) := p[D,q],$$

$$\phi\, a[D,b] := -\varepsilon' J[D,b^*]a^*J^{-1}\phi, \quad \phi \in \mathcal{H} \tag{2}$$

$\mathbb{T}^n$**-equivariance**:

on $\mathcal{H}$ $\exists$ commuting operators $\delta_j^* = \delta_j$, which lift the derivations $\delta_j : \mathcal{A} \to \mathcal{A}$,

$$\delta_j(a\phi) = \delta_j(a)\phi + a\delta_j(\phi),$$

such that

$$[\delta_j, D] = 0, \qquad [\delta_j, \gamma] = 0, \qquad \delta_j J + J\delta_j = 0.$$

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Accordingly

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{H}_k.$$

We have $D\mathcal{H}_k \subseteq \mathcal{H}_k$ and $\mathcal{A}^{(l)}\mathcal{H}_k \subseteq \mathcal{H}_{k+l}$.

In particular $\mathcal{H}_0$ is stable under D and $\mathcal{B} = \mathcal{A}^{coH} := \mathcal{A}^{(0)}$.

Moreover, since $\mathcal{A}$ is strongly graded, $\boxed{\mathcal{H}_k = \mathcal{H}_0\mathcal{A}^{(k)}}$ as reps of $\mathcal{B}$.

DEF: Projectable spectral triples

A $\mathbb{T}^n$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \{\delta_j\})$, of KR-dimension $n+m$, is said to be *projectable* if $\exists$ a $\mathbb{Z}_2$ grading Γ on $\mathcal{H}$, s.t.

$$[\Gamma, \pi(a)] = 0 \quad \forall a \in \mathcal{A}, \quad [\Gamma, \delta_j] = 0 \quad \forall 1 \leq j \leq n,$$

$$J\Gamma = \begin{cases} (-)^{n(n-1)/2}\,\Gamma J & \text{if } n+m = 1 \,(\mathrm{mod}\, 4), \\ (-)^{n(n+1)/2}\,\Gamma J & \text{otherwise}, \end{cases}$$

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We define *horizontal* Dirac operator

$$D_h := \frac{1}{2}\Gamma[\Gamma, D]_\pm \quad \text{for } n \text{ even/odd.} \tag{3}$$

(This is <u>weaker</u> than iterated or n-fold $U(1)$-projectability).

DEF: Isometric fibres: $\exists$ a selfadjoint operator D_v on $\mathcal{H}$ and a bounded Z, s.t.

$$D = D_v + D_h + Z$$

and:

(a) $D_v|_{\mathcal{H}_k}$ is bounded with finite (uniformly in k) spectrum and $D_v|_{\mathcal{H}_0} = 0$
(b) $[D_v, b] = 0$ for $b \in \mathcal{B}$, and $D_v J = \varepsilon' J D_v$:
(c) $[D_v, \delta_i] = 0$ and δ_i are relatively bounded w.r.t D_v, $\forall i = 1, \ldots, n$;
(d) $Z \in A'$.

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This generalizes 'constant length of fibres' of [DS] in the case $n = 1, m = 2$ but we don't require it for each $U(1)$-action; neither that $D_v \sim \Gamma\delta$.

Moreover instead of $Z \in \mathcal{A}'' \cap B'$ we require now (d), e.g. $Z \in J\mathcal{A}J$.
This (and (a)) assures that D_h and D determine the same calculus on $\mathcal{B}$:

$$[D_h, b] = [D, b], \; b \in \mathcal{B}.$$

First main result

Denote $D_k := D_h|_{\mathcal{H}_k}$, $J_0 := J|_{\mathcal{H}_0}$ and $\gamma_0 = \Gamma|_{\mathcal{H}_0}$.

PROP: Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\}, \Gamma)$ be a projectable S.T. with isometric fibres. Then, $(\mathcal{B}, \mathcal{H}_k, D_k)$ is a spectral triple (in general reducible).

Proof. Selfadjointness and $[D_k, b] \in B(\mathcal{H}_k)$ follow since $D_k = D_h|_{\mathcal{H}_k}$, and compact resolvent since D_k is a bounded perturbation of $D|_{\mathcal{H}_k}$. $\square$

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REM: $J|_{\mathcal{H}_0}$ and $\Gamma|_{\mathcal{H}_0}$ yield real even structure on B of KR-dimension $m+n$; however by a tidiuos and scrupulous analysis we constructed j_0, γ_0, s.t. $(\mathcal{B}, \mathcal{H}_0, D_0, j_0, \gamma_0)$ is a real even spectral triple of KR-dimension m. Moreover, the first order condition holds and so have right action of $\mathcal{B}$ on $\mathcal{H}_0$.

Principal $\mathbb{T}^n$-bundles with Dirac calculus

PROP: Let $(\mathcal{A}, \mathcal{H}, D, J)$ as above and $\mathcal{A}$ a principal $H(\mathbb{T}^n)$-comodule algebra.
Let $N = \ker \pi_D$ and $Q = (\ker \varepsilon)^2$.
Then $(\mathcal{A}, H(\mathbb{T}^n), \Delta_{\mathcal{A}}, N, Q)$ is a quantum principal bundle of [BrzMaj93] iff

(a)

$$\sum p[D, q] = 0 \qquad \Rightarrow \qquad \sum p\, \delta_i(q) = 0, \quad \forall i \tag{4}$$

(b)

$$\sum pq = 0 \text{ and } \sum p\, \delta_i(q) = 0, \forall i \quad \Rightarrow \quad \sum p[D, q] \in \mathcal{A}[D, \mathcal{B}]\mathcal{A}. \tag{5}$$

REM: Valid for more general right covariant calculi $\Omega^1(A)$.
The compatibility condition (a) appeared in [DS10] for circle bundles; we <u>add</u> (b).

Strong connections

DEF: A family of n 1-forms $\{\omega_i\} \subset \Omega^1_D(\mathcal{A})$ is *strong* $\mathbb{T}^n$*-connection* on $\mathcal{A}$ iff

(i) $[\delta_j, \omega_i] = 0, \ \forall\, i, j = 1, \ldots, n$;
(ii) if $\omega_i = \sum_j p_j[D, q_j]$, with $p_j, q_j \in \mathcal{A}$,
then $\sum_j p_j \delta_i(q_j) = 1$ and $\sum_j p_j \delta_l(q_j) = 0$ for $l \neq i$;
(iii) $\forall a \in \mathcal{A}, \ \ ([D, a] - \sum_i \delta_i(a)\omega_i) \in \Omega^1_D(\mathcal{B})\mathcal{A}$.

& **MS**: (ii), (iii) meaningful (depend only on ω_i and not p_j, q_j) due to (a), (b).

This Def. (with (a), (b)) is equivalent to strong connection [Haj96], specified to quantum principal $\mathbb{T}^n$-bundle with calculi $\Omega^1_D(\mathcal{A})$ and $\Omega^1_{dR}(\mathbb{T}^n)$:

Twisted Dirac operators

We want to interpret $D_h|_{\mathcal{H}_k}$ as D_0 coupled (twisted) with connection.

Let $\{\omega_i\} \subset \Omega^1_D(\mathcal{A})$ be a strong $\mathbb{T}^n$-connectionn with selfadjoint (operators) ω_i. Using the identification $\mathcal{H}_k = \mathcal{H}_0 \mathcal{A}^{(k)}$ (as reps of $\mathcal{B}$) and right actions, define

$$D^k_\omega : (\mathrm{Dom}(D_0))\mathcal{A}^{(k)} \to \mathcal{H}_k,$$

$$D^k_\omega(\phi a) \equiv (D_0 \phi)a + \phi\left([D, a] - \sum_{j=1}^{n} k_j a \omega_j\right), \quad \phi \in \mathcal{H}_0, a \in \mathcal{A}^{(k)}. \quad (6)$$

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REM: D^k_ω is "D_0 *minimally coupled* with ω_j"; or with D_0-connection [DS10,DSZ13]

$$\nabla^\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{B})\mathcal{A}^{(k)}, \quad \nabla^\omega(a) := [D,a] - \sum_j k_j a\omega_j \in \Omega^1_D(\mathcal{B})\mathcal{A}^{(k)}$$

(by (iii) of DEF [DZ]). $\diamond$

Second main result

PROP: D^k_ω is a well-defined selfadjoint operator on $\mathcal{H}_k$, with compact resolvent. Moreover, it has bounded commutators with $\mathcal{B}$ (acting on the left).

Proof. Uused (d) ($Z \in \mathcal{A}'$) and (a) ($D_v|_{\mathcal{H}_0} = 0$) of DEF (isometric fibers) we rewrite (6) in the form

$$\begin{aligned} D^k_\omega(\phi a) &= (D_h\phi)a + \phi\left([D_h + D_v, a] - \sum_{i=1}^n k_i \phi a \omega_i\right) \\ &= D_h(\phi a) + D_v(\phi a) - \sum_{i=1}^n k_i \phi a \omega_i = \left(D - Z + \varepsilon' \sum_{i=1}^n J\omega_i^* J^{-1} k_i\right)(\phi a), \end{aligned} \tag{7}$$

Thus D^k_ω depends only on the product ϕa.
The rest follows since $D^k_\omega - D$ is selfadjoint & bounded.

$\square$

This result gives us a family of spectral triples $(\mathcal{B}, \mathcal{H}_k, D_\omega^k)$, $k \in \mathbb{Z}^n$.

PROP: $D_\omega := \overline{\bigoplus_k D_\omega^k}$ is selfadjoint operator on $\mathcal{H}$ with bounded commutators with $a \in \mathcal{A}$.

Proof. built on (7) + Kato-Rellich theorem $\square$

Compatibility

DEF: We say that ω is $\boxed{\textit{compatible}}$ with D if D_ω and D_h coincide on a dense subset of $\mathcal{H}$.

Can such D_ω be 'projected' from some spectral triple on $\mathcal{A}$?

PROP: Let D_v be as in definition and let

$$\mathcal{D}_\omega = D_v + D_\omega .$$

If D_v anticommutes with D_ω, then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_\omega)$ is a projectable spectral triple with isometric fibres, and the horizontal part of the operator $\mathcal{D}_\omega$ coincides with D_ω.

Proof. $\{D_\omega, D_v\}_+ = 0$ permits to analyse the spectrum of $\mathcal{D}_\omega$ □

EXAMPLES

Classical II: principal $\mathbb{T}^n$-bundles

The classical principal $\mathbb{T}^n$-bundle M over the m-dimensional oriented manifold $N = M/\mathbb{T}^n$, with compatible metrics, constant length, projectable spin bundles, fits our NC refomulation:

It is clear that with $\Gamma = i^{[n/2]}\gamma^1\gamma^2 \ldots \gamma^n$, where $\gamma^a, a = 1, \ldots, n$ are vertical $(C^\infty(M), L^2(\Sigma), \widetilde{D})$ is a projectable spectral triple in our sense, and D_h comes as before.

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Next, with D_v and Z as before the fibres are isometric. In particular, D_v anticommutes with D_h since a principal $\mathbb{T}^n$ connection ω_i is Ad-invariant (=invariant).
Moreover, identifying the de Rham calculus with the Dirac calculus $\Omega^1_{\widetilde{D}}(\mathbb{C}^\infty(M)$, ω_i is indeed a strong connection in our sense.
In addition ω is also compatible with $\mathcal{D}_\omega = D_v + D_\omega$, which corresponds to a metric connection (but maybe nonzero torsion).

Flat noncommutative tori:

$$D = \sum_{j}^{m+n} \gamma_j \delta_j ,$$

where δ_j are the standard derivations implemented on $\mathcal{H} = L^2(\mathbb{T}^k_\theta, \tau) \otimes \mathbb{C}^{2^{[k/2]}}$ and τ is the usual trace.

Using (part of) this action $\mathcal{A} = \mathcal{A}(\mathbb{T}^{n+m}_\theta)$ becomes a $\mathbb{T}^n$-bundle.
The base $\mathcal{B} = \mathcal{A}(\mathbb{T}^m_\vartheta)$ is generated by the last m generators U_i, and ϑ is the $m \times m$ submatrix of θ.

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Next, $\Omega_D^1(\mathcal{A})$ and $\Omega_{dR}^1(\mathcal{A}(\mathbb{T}^n))$ are 'derivation based", so "(a), (b)"-compatible.
Explicitly, arbitrary strong connection is

$$\omega_i = \sum_{j=m+1}^{m+n} b_{ij} \otimes \gamma^j + 1 \otimes \gamma^i, \quad \text{with} \quad b_{ij} \in \mathcal{B}. \tag{8}$$

Moreover $(\mathcal{A}, D, \mathcal{H})$ is projectable with $\Gamma = i^{[n/2]}\gamma^1\gamma^2 \ldots \gamma^n$, and D_0 is just the standard Dirac operator on $\mathcal{B}$.

$\mathbb{T}^3_\theta$ as a quantum principal $U(1)$-bundle over $\mathbb{T}^2_\vartheta$

Consider $\mathbb{T}^1$-action along U_3. Four (of eight) spin structures are projectable and $\Gamma = \sigma^3$.

Strong connections are

$$\omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1, \tag{9}$$

where $\omega_1, \omega_2 \in \mathcal{B}$. Finally, the compatible Dirac operator is

$$\mathcal{D}_\omega = D - (\sigma^2 J \omega_2 J^{-1} + \sigma^1 J \omega_1 J^{-1}) \delta_3. \tag{10}$$

The only connection, compatible with the flat D ($\mathbb{T}^3$-equivariant) is $\omega = \sigma^3$.

$\mathbb{T}^3_\theta$ as a quantum principal $\mathbb{T}^2$-bundle over $\mathbb{T}^1$

Consider the $\mathbb{T}^2$-action along U_1, U_2. The flat S.T. is projectable, has isometric fibres with

$$\Gamma = \sigma^3,$$

and

$$D_h = \frac{1}{2}\Gamma[D,\Gamma]_+ = \sigma^3\delta_3, \quad D_v = \sigma^1\delta_1 + \sigma^2\delta_2; \quad Z = 0.$$

Next

$$D_0' = \Gamma D_0, \quad j_0 = \Gamma J|_{\mathcal{H}_0} = (J_0 \otimes (\sigma^1 \circ c.c.))|_{\mathcal{H}_0}$$

and $(\mathcal{B}, \mathcal{H}_0, D_0', j_0)$ is a real spectral triple of KR-dimension 1.

Twisted Dirac operators

Any selfadjoint strong $\mathbb{T}^2$-connection over $\mathcal{A}$ is

$$\omega_i = \sigma^i + \sigma^3 b_i, \text{ where } b_i = b_i^* \in \mathcal{B}.$$

Then twisted D_h on $\mathcal{H}$ (direct sum of D^k_ω on $\mathcal{H}^{(k)}$) is

$$D_\omega = \sigma^3 \delta_3 - \sigma^3 J b_1 J^{-1} \delta_1 - \sigma^3 J b_2 J^{-1} \delta_2.$$

Thus D is compatible with (only) ω iff if $b_i = 0$, i.e. $\omega_i = \sigma^i$.

Another “3-dim" Dirac operator (with torsion) is

$$\mathcal{D}_\omega = D_v + D_\omega.$$

REM: $(\mathcal{B}, \mathcal{H}_k, D^{(k)}_\omega)$ are reducible since σ^3 commutes with $\mathcal{A}$ & D (also classically).

Theta deformations

We can combine the two examples to construct a theta deformation of S.T. on principal $\mathbb{T}^n$-bundles.

Any classical M with isomery group $\mathbb{T}^{n+\ell} = \mathbb{T}^n \times \mathbb{T}^\ell$, $\ell \geq 2$, has a Rieffel quantization M_θ and this extends to equivariant S.T. over M [CoLa01].

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There is an equivalent but more 'functorial' realization [ConD-V]:

$$C^\infty(M_\theta) \approx \left(C^\infty(M)\widehat{\otimes}C^\infty(\mathbb{T}_\theta^{n+\ell})\right)^{\alpha\otimes\beta^{-1}}, \tag{11}$$

$$\mathcal{H}_\theta := \left(L^2(M,\Sigma)\widehat{\otimes}L^2(\mathbb{T}_\theta^{n+\ell})\right)^{\alpha\otimes\beta^{-1}}$$

$$D_\theta := D \otimes I,$$

where on the r.h.s. we have the invariant subalgebra or submodule. Similarly

$$J_\theta = J \otimes *.$$

(This spectral triple satisfies all additional 7 axioms of Connes).

Turns out that this behaves also 'functorially' for maps (eg. bundle projection) and respects the properties of principal $\mathbb{T}^n$-bundles.
Then M_θ constructed with the $(n+\ell) \times (n+\ell)$ matrix θ is a principal $\mathbb{T}^n$-bundle over N_ϑ constructed with the $\ell \times \ell$ submatrix ϑ of θ.
The right action of $\mathbb{T}^n$ on M_θ is just

$$\alpha \otimes \mathrm{id}$$

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where p^* is the pullback bundle projection $p : M \to N$.
Next $(M_\theta, \mathcal{H}_\theta, D_\theta)$ is projectable, with

$$\Gamma_\theta = \Gamma \otimes \mathrm{id}$$

and $(D_\theta)_h$ is just $(D_h)_\theta$. Moreover $\Omega^1_{D_\theta}(M_\theta)$ is compatible with the de Rham calculus on $\mathbb{T}^n$ and the conditions (a), (b) hold.
Finally, the connection compatible with D is

$$(\omega_\theta)_i := \omega_i \otimes 1.$$

APPLICATIONS (?):

- dynamical systems
- Hilbert modules and Kasparov product
- T-duality
- asymmetric quantum tori

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