Dirac operators on noncommutative principal torus bundles

Ludwik Dąbrowski

SISSA, Trieste, Italy

(based on works with A. Sitarz & A. Zucca)
Plan

- Why?
- Classical
- Quantum:
  - Projectability
  - Isometric fibers
  - Real structure
  - Dirac calculus
  - Strong connections
  - Twisted Dirac
- Examples:
  - classical
  - NC torus
  - theta deformations
Why?  NCG marries ‘geometric’ & ’quantum’:

space $\leftrightarrow$ algebra
metric & spin $\leftrightarrow$ spectral triple (S.T.)

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\(\exists\) reconstruction thm
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Phys: dim regularization, Standard Model of elementary particles
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Envisage more:
fibre bundles; start with principal, even with (usual) Lie group, and first $U(1)^n$...

Classics: $U(1)$ [Amman-Bär], many works on $G/H$ await systematic study
Compatible metrics
Let $M$ be a $(m + n)$-dimensional oriented compact smooth manifold which is the total space of a principal $\mathbb{T}^n$ - bundle over the $m$-dimensional oriented manifold $N = M/\mathbb{T}^n$.

Let $K_a$ be the fundamental vector field of the standard basis $T_a, 1 \leq a \leq n$ of $\mathbb{R}^n := \text{Lie}(\mathbb{T}^n))$. 
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Assume that $(M, \tilde{g})$ & $(N, g)$ are Riemannian, s.t.
- the action of $\mathbb{T}^n$ is isometry of $\tilde{g}$;
- the bundle projection $\pi : M \to N$ is an oriented Riemannian submersion;
- the fibres are isometric one to each other; moreover, $|K_a|$ is constant along $M$.

Then $\exists!$ principal $\text{connection}$ 1-form $\omega = \sum_{a=1}^{n} \omega_a \otimes T_a$, s.t. $\ker \omega \perp$ fibres.
Compatible classical ST

Assume $M$ is spin and spinor bundle $\Sigma M$ is projectable, i.e. admits lift of $\mathbb{T}^n$. Then the Dirac operators $\widetilde{D}$ on $L^2(\Sigma M, \tilde{g})$ and $D$ on $L^2(\Sigma N, g)$ are compatible [AB],[DZ] (for simplicity $m$ even):

\[
\widetilde{D} = D_v + D_h + Z,
\]

where:

\[
D_v = \sum_{a=1}^{n} \frac{1}{l_a} \gamma^a \partial_{K_a} = \sum_{a=1}^{n} \gamma^a \partial_{e_a}, \quad \text{‘vertical’},
\]

\[
Z = -\frac{1}{4} \sum_{a=1}^{n} \gamma(K_a) \gamma(d\omega_a), \quad \text{‘order 0’}
\]

and
\[ D_h = \cdots = \bigoplus_{k \in \mathbb{Z}^n} Q_k \circ (D_k \otimes \text{id}) \circ Q_k^{-1}, \quad \text{'horizontal'.} \]

Here \( D_k \) is \( D \) twisted by \( \omega_a \), and it acts on \( \Sigma N \otimes L_k \), where \( L_k = M \times_{\mathbb{T}^n} \mathbb{C} \) is the line bundle on \( N \) associated with weight \( k \) of \( \mathbb{T}^n \).

Next,

\[ Q_k : L^2(\Sigma N \otimes L_k) \otimes \Sigma_n \to V_k \]

are certain isomorphisms, where we decompose into \( \mathbb{T}^n \)-irreps as

\[ L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}^n} V_k. \]
Noncommutative analogue? No $\tilde{g}, g \ldots$

We extended (part of) [AB] to NC circle [DS], [DAZ] and $T^n$-bundles [DZ]

Will present ("growing") ingredients:
Noncommutative principal bundles
Group $\rightarrow$ Hopf algebra $H$, with coproduct $\Delta$, invertible antipode $S$ and counit $\varepsilon$.
Quantum principal bundle $\rightarrow$ a right $H$-comodule algebra $\mathcal{A}$, which is principal, i.e. $\exists$ strong connection (for the universal calculus $\Omega^1\mathcal{A}$).
Base space $\rightarrow B := \{a \in \mathcal{A} \mid \Delta_{\mathcal{A}}(a) = a \otimes 1\}$.

It follows [Haj96] that $\mathcal{A}$ is a Hopf-Galois extension of $B$, and so a quantum principal bundle as in [BrzMaj93], with respect to $\Omega^1\mathcal{A}$.
I'll say what happens for a general differential calculus.

In [Haj96] also strong connection for a general calculus was defined. I'll specify it to Dirac calculi, in equivalent form.
**Our** $H$: unital * Hopf algebra $H(\mathbb{T}^n)$ of the Lie group $\mathbb{T}^n = U(1)^n$; generated by commuting unitaries $z_1, \ldots, z_n$, &

$$\Delta(z_i) = z_i \otimes z_i, \quad S(z_i) = z_i^* = z_i^{-1}, \quad \varepsilon(z_i) = 1.$$ 

Notation: for $k \in \mathbb{Z}^n$ we set $z^k := z_1^{k_1} \ldots z_n^{k_n}$. 
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We call quantum principal $T^n$-bundle a principal $H(\mathbb{T}^n)$-comodule algebra $\mathcal{A}$. It is strongly $\mathbb{Z}^n$-graded

$$\mathcal{A} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{A}^{(k)}, \quad [\mathcal{A}^{(k)} \mathcal{A}^{(k')} = \mathcal{A}^{(k+k')}],$$

(see e.g. [Sch04]), where

$$a \in \mathcal{A}^{(k)} \iff \Delta_{\mathcal{A}}(a) = a \otimes z^k \quad (\iff \delta_j(a) = k_ja).$$
Spectral triples
In NC Riemannian spin geometry = spectral triple \((\mathcal{A}, \mathcal{H}, D)\) ’over’ algebra \(\mathcal{A}\) with a representation \(\pi\) (often omitted) on the Hilbert space \(\mathcal{H}\) & "Dirac operator" \(D = D^*\), s.t. \([D, a] \in B(\mathcal{H})\) for \(a \in \mathcal{A}\), and \(D^{-1} \in K(\mathcal{H})\).

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even if \(\exists\) grading \(\gamma\), s.t. \(\gamma a = a \gamma, D \gamma = -\gamma D.\)
real if \(\exists\) antilinear \(J\) s.t. \([JAJ, A] = 0\) and

\[
J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J \gamma = \epsilon'' \gamma J.\
\]

Here \(\epsilon, \epsilon', \epsilon'' = \pm 1\) depend mod 8 on the so-called \(KR\)-dimension.
We require first order condition

\[
[[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in A.
\] (1)

Then there is right action of \( A \) on \( \mathcal{H} \)

\[
\phi \cdot a = Ja^* J^{-1} \phi, \quad \phi \in \mathcal{H}, a \in A
\]

and of Dirac 1-forms

\[
\Omega^1_D(A) := \pi_D(\Omega^1 A), \quad \pi_D(p \otimes q) := p[D, q], \\
\phi a[D, b] := -\varepsilon' J[D, b^*]a^* J^{-1} \phi, \quad \phi \in \mathcal{H}
\] (2)
$\mathbb{T}^n$-equivariance:
on $\mathcal{H}$ commuting operators $\delta^*_j = \delta_j$, which lift the derivations $\delta_j : \mathcal{A} \to \mathcal{A}$,

$$\delta_j(a\phi) = \delta_j(a)\phi + a\delta_j(\phi),$$

such that

$$[\delta_j, D] = 0, \quad [\delta_j, \gamma] = 0, \quad \delta_j J + J\delta_j = 0.$$

Require also that the spec($\delta_j$) = $\mathbb{Z}$, $\forall j$, so it exponentiates to action of $\mathbb{T}^n$. 
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Require also that the \text{spec}(\delta_j) = \mathbb{Z}, \forall j, \text{ so it exponentiates to action of } \mathbb{T}^n.
Accordingly
\[ \mathcal{H} = \bigoplus_{k \in \mathbb{Z}^n} \mathcal{H}_k. \]

We have \( D\mathcal{H}_k \subseteq \mathcal{H}_k \) and \( \mathcal{A}^{(l)}\mathcal{H}_k \subseteq \mathcal{H}_{k+l} \).
In particular \( \mathcal{H}_0 \) is stable under \( D \) and \( \mathcal{B} = \mathcal{A}^{coH} := \mathcal{A}^{(0)}. \)

Moreover, since \( \mathcal{A} \) is strongly graded, \( \mathcal{H}_k = \mathcal{H}_0 \mathcal{A}^{(k)} \) as reps of \( \mathcal{B}. \)
DEF: Projectable spectral triples
A $\mathbb{T}^m$-equivariant real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \{\delta_j\})$, of $KR$-dimension $n+m$, is said to be projectable if $\exists$ a $\mathbb{Z}_2$ grading $\Gamma$ on $\mathcal{H}$, s.t.

$$\left[ \Gamma, \pi(a) \right] = 0 \ \forall a \in \mathcal{A}, \quad \left[ \Gamma, \delta_j \right] = 0 \ \forall 1 \leq j \leq n,$$

$$J\Gamma = \begin{cases} \frac{(-1)^{n(n-1)/2}}{2} \Gamma J & \text{if } n + m = 1 \pmod{4}, \\ \frac{(-1)^{n(n+1)/2}}{2} \Gamma J & \text{otherwise}, \end{cases}$$

$$\Gamma \gamma = (-1)^n \gamma \Gamma \quad \text{if } n + m \text{ is even.}$$
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We define horizontal Dirac operator

$$D_h := \frac{1}{2} \Gamma [\Gamma, D]_{\pm} \quad \text{for } n \text{ even/odd.} \quad (3)$$

(This is weaker than iterated or n-fold $U(1)$-projectability).
DEF: Isometric fibres: \( \exists \) a selfadjoint operator \( D_v \) on \( \mathcal{H} \) and a bounded \( Z \), s.t. 

\[
D = D_v + D_h + Z
\]

and:

(a) \( D_v|_{\mathcal{H}_k} \) is bounded with finite (uniformly in \( k \)) spectrum and \( D_v|_{\mathcal{H}_0} = 0 \)

(b) \([D_v, b] = 0\) for \( b \in \mathcal{B} \), and \( D_vJ = \varepsilon'JD_v \):

(c) \([D_v, \delta_i] = 0\) and \( \delta_i \) are relatively bounded w.r.t \( D_v \), \( \forall i = 1, \ldots, n \);

(d) \( Z \in A' \).
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(d) \( Z \in A' \).

This generalizes 'constant length of fibres' of [DS] in the case \( n = 1, m = 2 \) but we don't require it for each \( U(1) \)-action; neither that \( D_v \sim \Gamma \delta \).

Moreover instead of \( Z \in A'' \cap B' \) we require now (d), e.g. \( Z \in JAJ \).

This (and (a)) assures that \( D_h \) and \( D \) determine the same calculus on \( \mathcal{B} \):

\[
[D_h, b] = [D, b], \quad b \in \mathcal{B}.
\]
First main result
Denote $D_k := D_h|\mathcal{H}_k$, $J_0 := J|\mathcal{H}_0$ and $\gamma_0 = \Gamma|\mathcal{H}_0$.

PROP: Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma, \{\delta_j\}, \Gamma)$ be a projectable S.T. with isometric fibres. Then, $(\mathcal{B}, \mathcal{H}_k, D_k)$ is a spectral triple (in general reducible).

\textit{Proof.} Selfadjointness and $[D_k, b] \in B(\mathcal{H}_k)$ follow since $D_k = D_h|\mathcal{H}_k$, and compact resolvent since $D_k$ is a bounded perturbation of $D|\mathcal{H}_k$. \qed
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Proof. Selfadjointness and \([D_k, b] \in \mathcal{B}(\mathcal{H}_k)\) follow since \( D_k = D_h|_{\mathcal{H}_k} \), and compact resolvent since \( D_k \) is a bounded perturbation of \( D|_{\mathcal{H}_k} \). \hfill \Box

REM: \( J|_{\mathcal{H}_0} \) and \( \Gamma|_{\mathcal{H}_0} \) yield real even structure on \( \mathcal{B} \) of KR-dimension \( m + n \); however by a tidious and scrupulous analysis we constructed \( j_0, \gamma_0 \), s.t. \( (\mathcal{B}, \mathcal{H}_0, D_0, j_0, \gamma_0) \) is a real even spectral triple of KR-dimension \( m \). Moreover, the first order condition holds and so have right action of \( \mathcal{B} \) on \( \mathcal{H}_0 \).
Principal $\mathbb{T}^n$-bundles with Dirac calculus

PROP: Let $(\mathcal{A}, \mathcal{H}, D, J)$ as above and $\mathcal{A}$ a principal $H(\mathbb{T}^n)$-comodule algebra. Let $N = \ker \pi_D$ and $Q = (\ker \varepsilon)^2$.

Then $(\mathcal{A}, H(\mathbb{T}^n), \Delta_{\mathcal{A}}, N, Q)$ is a quantum principal bundle of [BrzMaj93] iff

(a)

$$\sum p[D, q] = 0 \Rightarrow \sum p \delta_i(q) = 0, \quad \forall i \quad (4)$$

(b)

$$\sum pq = 0 \text{ and } \sum p \delta_i(q) = 0, \forall i \Rightarrow \sum p[D, q] \in \mathcal{A}[D, \mathcal{B}]\mathcal{A}. \quad (5)$$

REM: Valid for more general right covariant calculi $\Omega^1(\mathcal{A})$.

The compatibility condition (a) appeared in [DS10] for circle bundles; we add (b).
Strong connections

DEF: A family of $n$ 1-forms $\{\omega_i\} \subset \Omega^1_D(\mathcal{A})$ is *strong $\mathbb{T}^n$-connection* on $\mathcal{A}$ iff

(i) $[\delta_j, \omega_i] = 0$, $\forall i, j = 1, \ldots, n$;

(ii) if $\omega_i = \sum_j p_j[D, q_j]$, with $p_j, q_j \in \mathcal{A}$,
    then $\sum_j p_j \delta_i(q_j) = 1$ and $\sum_j p_j \delta_l(q_j) = 0$ for $l \neq i$;

(iii) $\forall a \in \mathcal{A}$, $([D, a] - \sum_i \delta_i(a) \omega_i) \in \Omega^1_D(B)\mathcal{A}$.

& MS: (ii), (iii) meaningful (depend only on $\omega_i$ and not $p_j, q_j$) due to (a), (b).

This Def. (with (a), (b)) is equivalent to strong connection [Haj96], specified to quantum principal $\mathbb{T}^n$-bundle with calculi $\Omega^1_D(\mathcal{A})$ and $\Omega^1_{dR}(\mathbb{T}^n)$:
Twisted Dirac operators
We want to interpret $D_h|\mathcal{H}_k$ as $D_0$ coupled (twisted) with connection.

Let $\{\omega_i\} \subset \Omega^1_D(\mathcal{A})$ be a strong $\mathbb{T}^n$-connection with selfadjoint (operators) $\omega_i$. Using the identification $\mathcal{H}_k = \mathcal{H}_0 \mathcal{A}^{(k)}$ (as reps of $\mathcal{B}$) and right actions, define

$$D^k_\omega : (\text{Dom}(D_0))\mathcal{A}^{(k)} \to \mathcal{H}_k,$$

$$D^k_\omega(\phi a) \equiv (D_0 \phi)a + \phi\left([D, a] - \sum_{j=1}^n k_j a \omega_j\right), \quad \phi \in \mathcal{H}_0, a \in \mathcal{A}^{(k)}. \quad (6)$$
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\]

REM: \( D^k_\omega \) is "\( D_0 \) minimally coupled with \( \omega_j \)"; \ or with \( D_0 \)-connection [DS10,DSZ13]

\[
\nabla_\omega : \mathcal{A}^{(k)} \to \Omega^1_D(\mathcal{B}) \mathcal{A}^{(k)}, \quad \nabla_\omega(a) := [D, a] - \sum_j k_j a \omega_j \in \Omega^1_D(\mathcal{B}) \mathcal{A}^{(k)}
\]

(by (iii) of DEF [DZ]).

\( \diamond \)
Second main result

PROP: $D^k_\omega$ is a well-defined selfadjoint operator on $\mathcal{H}_k$, with compact resolvent. Moreover, it has bounded commutators with $B$ (acting on the left).

Proof. Used (d) ($Z \in \mathcal{A}'$) and (a) ($D_v|_{\mathcal{H}_0} = 0$) of DEF (isometric fibers) we rewrite (6) in the form

$$D^k_\omega(\phi a) = (D_h \phi)a + \phi \left( [D_h + D_v, a] - \sum_{i=1}^n k_i \phi a \omega_i \right)$$

$$= D_h(\phi a) + D_v(\phi a) - \sum_{i=1}^n k_i \phi a \omega_i = \left( D - Z + \varepsilon' \sum_{i=1}^n J \omega^*_i J^{-1} k_i \right)(\phi a),$$

(7)

Thus $D^k_\omega$ depends only on the product $\phi a$.

The rest follows since $D^k_\omega - D$ is selfadjoint & bounded.
This result gives us a family of spectral triples \((B, \mathcal{H}_k, D^k_\omega), k \in \mathbb{Z}^n\).

PROP: \(D_\omega := \bigoplus_k D^k_\omega\) is selfadjoint operator on \(\mathcal{H}\) with bounded commutators with \(a \in \mathcal{A}\).

*Proof.* built on (7) + Kato-Rellich theorem
Compatibility
DEF: We say that $\omega$ is compatible with $D$ if $D_\omega$ and $D_h$ coincide on a dense subset of $\mathcal{H}$.

Can such $D_\omega$ be 'projected' from some spectral triple on $\mathcal{A}$?

PROP: Let $D_v$ be as in definition and let

$$D_\omega = D_v + D_\omega.$$ 

If $D_v$ anticommutes with $D_\omega$, then $(\mathcal{A}, \mathcal{H}, D_\omega)$ is a projectable spectral triple with isometric fibres, and the horizontal part of the operator $D_\omega$ coincides with $D_\omega$.

Proof. $\{D_\omega, D_v\}_+ = 0$ permits to analyse the spectrum of $D_\omega$. \qed
EXAMPLES

Classical II: principal $\mathbb{T}^n$-bundles

The classical principal $\mathbb{T}^n$-bundle $M$ over the $m$-dimensional oriented manifold $N = M/\mathbb{T}^n$, with compatible metrics, constant length, projectable spin bundles, fits our NC reformulation:

It is clear that with $\Gamma = i^{[n/2]}\gamma^1\gamma^2\ldots\gamma^n$, where $\gamma^a, a = 1, \ldots, n$ are vertical $(C^\infty(M), L^2(\Sigma), \widehat{D})$ is a projectable spectral triple in our sense, and $D_h$ comes as before.
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Next, with $D_v$ and $Z$ as before the fibres are isometric. In particular, $D_v$ anticommutes with $D_h$ since a principal $\mathbb{T}^n$ connection $\omega_i$ is Ad-invariant (=invariant). Moreover, identifying the de Rham calculus with the Dirac calculus $\Omega^1_D(C^\infty(M))$, $\omega_i$ is indeed a strong connection in our sense.
In addition $\omega$ is also compatible with $\mathcal{D}_\omega = D_v + D_\omega$, which corresponds to a metric connection (but maybe nonzero torsion).
Flat noncommutative tori:

\[ D = \sum_{j} \gamma_j \delta_j, \]

where \( \delta_j \) are the standard derivations implemented on \( \mathcal{H} = L^2(\mathbb{T}_\theta^k, \tau) \otimes \mathbb{C}^{2^{[k/2]}} \) and \( \tau \) is the usual trace.

Using (part of) this action \( \mathcal{A} = \mathcal{A}(\mathbb{T}_\theta^{n+m}) \) becomes a \( \mathbb{T}^n \)-bundle. The base \( \mathcal{B} = \mathcal{A}(\mathbb{T}_\vartheta^m) \) is generated by the last \( m \) generators \( U_i \), and \( \vartheta \) is the \( m \times m \) submatrix of \( \theta \).
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Next, \( \Omega^1_{D} (\mathcal{A}) \) and \( \Omega^1_{dR} (\mathcal{A}(\mathbb{T}^n)) \) are 'derivation based", so "(a), (b)"-compatible.

Explicitly, arbitrary strong connection is

\[ \omega_i = \sum_{j=m+1}^{m+n} b_{ij} \otimes \gamma^j + 1 \otimes \gamma^i, \quad \text{with} \quad b_{ij} \in \mathcal{B}. \] (8)

Moreover \( (\mathcal{A}, D, \mathcal{H}) \) is projectable with \( \Gamma = i^{[n/2]} \gamma^1 \gamma^2 \ldots \gamma^n \), and \( D_0 \) is just the standard Dirac operator on \( \mathcal{B} \).
\( \mathbb{T}_\theta^3 \) as a quantum principal \( U(1) \)-bundle over \( \mathbb{T}_\theta^2 \)

Consider \( \mathbb{T}^1 \)-action along \( U_3 \). Four (of eight) spin structures are projectable and \( \Gamma = \sigma^3 \).

Strong connections are

\[
\omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1,
\]

where \( \omega_1, \omega_2 \in B \). Finally, the compatible Dirac operator is

\[
D_\omega = D - (\sigma^2 J_2 J^{-1} + \sigma^1 J_1 J^{-1}) \delta_3.
\]

The only connection, compatible with the flat \( D \) (\( \mathbb{T}^3 \)-equivariant) is \( \omega = \sigma^3 \).
$\mathbb{T}_3^3$ as a quantum principal $\mathbb{T}^2$-bundle over $\mathbb{T}^1$

Consider the $\mathbb{T}^2$-action along $U_1, U_2$. The flat S.T. is projectable, has isometric fibres with

$$\Gamma = \sigma^3,$$

and

$$D_h = \frac{1}{2} \Gamma [D, \Gamma]_+ = \sigma^3 \delta_3, \quad D_v = \sigma^1 \delta_1 + \sigma^2 \delta_2; \quad Z = 0.$$

Next

$$D'_0 = \Gamma D_0, \quad j_0 = \Gamma J|_{\mathcal{H}_0} = (J_0 \otimes (\sigma^1 \circ c.c.))|_{\mathcal{H}_0}$$

and $(\mathcal{B}, \mathcal{H}_0, D'_0, j_0)$ is a real spectral triple of $KR$-dimension 1.
Twisted Dirac operators
Any selfadjoint strong $\mathbb{T}^2$-connection over $\mathcal{A}$ is

$$\omega_i = \sigma^i + \sigma^3 b_i, \text{ where } b_i = b_i^* \in \mathcal{B}.$$  

Then twisted $D_h$ on $\mathcal{H}$ (direct sum of $D^k_\omega$ on $\mathcal{H}(k)$) is

$$D_\omega = \sigma^3 \delta_3 - \sigma^3 Jb_1 J^{-1} \delta_1 - \sigma^3 Jb_2 J^{-1} \delta_2.$$  

Thus $D$ is compatible with (only) $\omega$ iff if $b_i = 0$, i.e. $\omega_i = \sigma^i$.

Another “3-dim” Dirac operator (with torsion) is

$$D_\omega = D_v + D_\omega.$$  

REM: $(\mathcal{B}, \mathcal{H}_k, D^L_\omega)$ are reducible since $\sigma^3$ commutes with $\mathcal{A}$ & $D$ (also classically).
Theta deformations
We can combine the two examples to construct a theta deformation of S.T. on principal $\mathbb{T}^n$-bundles.
Any classical $M$ with isomery group $\mathbb{T}^n + \ell = \mathbb{T}^n \times \mathbb{T}^\ell$, $\ell \geq 2$, has a Rieffel quantization $M_\theta$ and this extends to equivariant S.T. over $M$ [CoLa01].
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There is an equivalent but more ‘functorial’ realization [ConD-V]:

$$C^\infty(M_\theta) \approx \left( C^\infty(M) \hat{\otimes} C^\infty(\mathbb{T}^n + \ell) \right)^{\alpha \otimes \beta^{-1}},$$

(11)

$$\mathcal{H}_\theta := \left( L^2(M, \Sigma) \hat{\otimes} L^2(\mathbb{T}^n + \ell) \right)^{\alpha \otimes \beta^{-1}}$$

$$D_\theta := D \otimes I,$$

where on the r.h.s. we have the invariant subalgebra or submodule. Similarly

$$J_\theta = J \otimes \ast.$$  

(This spectral triple satisfies all additional 7 axioms of Connes).
Turns out that this behaves also 'functorially' for maps (eg. bundle projection) and respects the properties of principal $\mathbb{T}^n$-bundles. Then $M_\theta$ constructed with the $(n + \ell) \times (n + \ell)$ matrix $\theta$ is a principal $\mathbb{T}^n$-bundle over $N_\vartheta$ constructed with the $\ell \times \ell$ submatrix $\vartheta$ of $\theta$. The right action of $\mathbb{T}^n$ on $M_\theta$ is just

$$\alpha \otimes \text{id}$$

and the bundle inclusion $N_\theta \hookrightarrow M_\theta$ reads

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where $p^*$ is the pullback bundle projection $p : M \to N$. 
Turns out that this behaves also ’functorially’ for maps (eg. bundle projection) and respects the properties of principal $\mathbb{T}^n$-bundles. Then $M_\theta$ constructed with the $(n + \ell) \times (n + \ell)$ matrix $\theta$ is a principal $\mathbb{T}^n$-bundle over $N_\theta$ constructed with the $\ell \times \ell$ submatrix $\vartheta$ of $\theta$. The right action of $\mathbb{T}^n$ on $M_\theta$ is just
\[
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and the bundle inclusion $N_\theta \hookrightarrow M_\theta$ reads
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\]
where $p^*$ is the pullback bundle projection $p : M \to N$. Next $(M_\theta, \mathcal{H}_\theta, D_\theta)$ is projectable, with
\[
\Gamma_\theta = \Gamma \otimes \text{id}
\]
and $(D_\theta)_h$ is just $(D_h)_\theta$. Moreover $\Omega^1_{D_\theta}(M_\theta)$ is compatible with the de Rham calculus on $\mathbb{T}^n$ and the conditions (a), (b) hold. Finally, the connection compatible with $D$ is
\[
(\omega_\theta)_i := \omega_i \otimes 1.
\]
APPLICATIONS (?):

- dynamical systems
- Hilbert modules and Kasparov product
- T-duality
- asymmetric quantum tori
B. Ammann, C. Bär
The Dirac Operator on Nilmanifolds and Collapsing Circle Bundles

L. Dabrowski, A. Sitarz
Noncommutative circle bundles and new Dirac operators

L. Dabrowski, A. Sitarz, A. Zucca
Dirac operator on noncommutative principal circle bundles
arXiv:1305.6185

A. Zucca, L. Dabrowski
Dirac operators on noncommutative principal torus bundles
arXiv:1308.4738