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Fourier Series and Twisted Crossed Products

Villa Mondragone, Frascati, June 2014

JFAA 15, 2009

MJM 5, 2012

arxiv:1303.7381

arxiv:1405.1908

+ work in progress...

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Plan:

- Motivations: classical theory
- Background
- Preparation (lengthy)
- Results
  - for group algebras
  - for crossed products
- Outlook

Classical Fourier theory and harmonic analysis

$f : \mathbb{R} \to \mathbb{C}$, 2π-periodic (i.e., a $\mathbb{C}$-valued function on the unit circle $\mathbb{T}$)

f continuous (in this talk)

$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$ n-th Fourier coeff. of f $(n \in \mathbb{Z})$

$S[f](t) := \sum_{n=-\infty}^{\infty} c_n e^{int}$ (formal) Fourier series of f

$S[f]_N(t) := \sum_{n=-N}^{N} c_n e^{int}$ N-partial sum

Question: when does $S_N[f]$ converge to f? In which sense?

Convergence: pointwise, uniform (in this talk), a.-e., absolute, square mean, mean, ...

Kolmogorov (1923): example of a function in L^1 (but $\notin L^2$) whose Fourier series diverges a.-e., later improved to divergence everywhere (1926).

Lusin problem (1920): does the Fourier series of any continuous function converges a.-e.?

Carleson (1966): the Fourier expansion of any function in L^2 converges a.-e. (later generalized by R. Hunt to L^p for any $p > 1$)

Pointwise convergence

P. du Bois Reymond (1873) showed the existence of a continuous function for which CFS fails at a point

Theorem (Kolmogorov): There is a function in $L^1(\mathbb{T})$ such that $\limsup_{N\to\infty} |S_N[f](t)| = \infty$ for every $t \in [-\pi, \pi)$

Theorem (Carleson): If $f \in L^2(\mathbb{T})$, then $\lim_{N\to\infty} S_N[f](t) = f(t)$ for almost every t in $[-\pi, \pi)$ (In particular, Lusin conjecture is valid: if $f \in C(\mathbb{T})$, then its Fourier series converges to f a.-e. in $[-\pi, \pi)$)

Theorem (Kahane and Katznelson) If $E \subset [-\pi, \pi)$ is a set of (Lebesgue) measure 0, then there exists an $f \in C(\mathbb{T})$ such that $\limsup_{N\to\infty} |S_N[f](t)| = \infty$ for every $t \in E$

Theorem (Hunt): If $f \in L^p(\mathbb{T})$, $1 < p < \infty$, then one has $\lim_{N\to\infty} S_N[f](t) = f(t)$ for almost every t in $[-\pi, \pi)$

Norm convergence: a standard result

f piecewise continuous on $[a, b]$ if it is continuous everywhere except at finitely many points $x_1, \ldots, x_k \in [a, b]$ and the left/right limits of f exist at each x_i

f piecewise smooth if f and f' are piecewise continuous

Theorem: $f : \mathbb{R} \to \mathbb{R}$ 2π-periodic and piecewise smooth. Then the Fourier series of f converges to f uniformly in every interval $[c, d]$ in which f is continuous.

Open Question: characterize the class of continuous f's for which the Fourier series of f converges uniformly to f.

Summability: Abel, Cesáro, Poisson, Fejér, ...

$f \in C(\mathbb{T})$, $e_k(z) = z^k$ $(z \in \mathbb{T}, k \in \mathbb{Z})$, $\widehat{f}(k) = \int_{\mathbb{T}} \overline{e_k} f d\mu$, μ normalized Haar measure on $\mathbb{T}$, $\sum_{k \in \mathbb{Z}} \widehat{f}(k) e_k$ (formal) Fourier series of f.

Let $(\varphi_n)_{n \in \mathbb{N}} \subset \ell^1(\mathbb{Z})$. For each $n \in \mathbb{N}$, define

$$M_n(f) := \sum_{k \in \mathbb{Z}} \varphi_n(k) \widehat{f}(k) e_k$$

(1) the series in the r.h.s. is absolutely convergent w.r.t. the uniform norm $\|\cdot\|_\infty$ on $C(\mathbb{T})$.

(2) $\|M_n(f)\|_\infty \leq \|\varphi_n\|_1 \|f\|_\infty$, thus each M_n is a bounded linear map on $(C(\mathbb{T}), \|\cdot\|_\infty)$ with $\|M_n\| \leq \|\varphi_n\|_1$.

(3) $M_n(f)$ converges uniformly (necessarily to f) iff
(i) $\varphi_n \to 1$ pointwise on $\mathbb{Z}$
(ii) $\sup_n \|M_n\| < \infty$.
In this case, say that $C(\mathbb{T})$ has the summation property w.r.t. (φ_n).

Many obvious candidates for (φ_n) satisfy (i) and the main difficulty is to compute (or estimate) $\|M_n\|$!

Examples:

- $\varphi_n(k) = d_n(k) := 1$ if $|k| \leq n$ and 0 otherwise. Then $\|M_n\| \to \infty$, showing the existence of functions in $C(\mathbb{T})$ whose Fourier series diverges at some point;

- $\varphi_n(k) = f_n(k) := 1 - \frac{|k|}{n}$ if $|k| \leq n-1$ and 0 otherwise. Then $\|M_n\| = 1, \ \forall n$, showing that the Fourier series of any $f \in C(\mathbb{T})$ is uniformly Fejér summable to f;

- $\varphi_n(k) = p_n(k) := {r_n}^{|k|}$, where $r_n \in (0, 1), r_n \to 1$ as $n \to \infty$. (More generally, consider $p_r(k) = r^{|k|}$ for $r \in (0, 1)$, introduce M_r and let $r \to 1$. Use nets instead of sequences to accomodate for such situations!) Then $\|M_n\| = 1 (= \|M_r\|)$, showing that the Fourier series of any $f \in C(\mathbb{T})$ is uniformly Abel-Poisson summable to f

Dirichlet kernel $D_n(x) = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)},\ n \in \mathbb{N}$

Fejér kernel $F_n(x) = \frac{1}{n+1}(\frac{\sin\left((n+1)\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)})^2,\ n \in \mathbb{N}$

Poisson kernel $P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2},\ r \in (0,1)$

TERMINOLOGY

Some analytic invariants

Def. A C^*-algebra $\mathcal{A}$ is nuclear if there exists a unique C^*-norm on the algebraic tensor product of $\mathcal{A}$ with any other C^*-algebra $\mathcal{B}$, namely for any other C^*-algebra $\mathcal{B}$, one has $\|\cdot\|_{min} = \|\cdot\|_{max}$ on $\mathcal{A} \odot \mathcal{B}$.

A Banach space E has the Metric Approximation Property (M.A.P.) if there exists a net of finite rank contractions on E approximating the identity map in the SOT on $B(E)$
(i.e., $(T_\alpha) \subset B(E)$, $\|T_\alpha\| \leq 1$, $T_\alpha(E)$ f.-d. for all α and $\lim_\alpha \|T_\alpha(x) - x\| = 0$, $\forall x \in E$)

nuclear ($\Leftrightarrow$ C.P.A.P.) $\Rightarrow$ M.A.P., but the converse is false

For instance, $B(\mathcal{H})$ does not have the M.A.P. (Thus it is not nuclear, or equivalently it has not the C.P.A.P.).

$C^*_r(\Gamma)$ nuclear iff Γ amenable

$C^*_r(\mathbb{F}_n)$ has the M.A.P.

Group cohomology

- σ normalized 2-cocycle on Γ:

  $\sigma : \Gamma \times \Gamma \to \mathbb{T}$,

  $\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk) \quad (g,h,k \in \Gamma)$,

  $\sigma(g,e) = \sigma(e,g) = 1 \qquad (g \in \Gamma)$

- $Z^2(G,\mathbb{T})$, the set of normalized 2-cocycles, abelian group (under pointwise product, inverse $\sigma^{-1} = \overline{\sigma}$, identity the trivial cocycle 1.)

- $\beta \in Z^2(\Gamma,\mathbb{T})$ *coboundary* if $\beta(g,h) = b(g)b(h)\overline{b(gh)}$ for all $g,h \in \Gamma$, for some $b : \Gamma \to \mathbb{T}$, $b(e) = 1$; then $\beta = \mathrm{d}b$ (b uniquely determined up to multiplication by a character of Γ).

- $B^2(G,\mathbb{T})$, the set of all coboundaries, a subgroup of $Z^2(G,\mathbb{T})$.

- $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ quotient group, with elements $[\sigma]$; also write $\tilde{\sigma} \sim \sigma$ when $[\tilde{\sigma}] = [\sigma]$, $\sigma, \tilde{\sigma} \in Z^2(G,\mathbb{T})$.

Example:

$\Gamma = \mathbb{Z}^N, \quad N \in \mathbb{N}$.

Given an $N \times N$ real matrix Θ, define $\sigma_\Theta \in Z^2(\mathbb{Z}^N, \mathbb{T})$ by

$$\sigma_\Theta(x, y) = e^{ix\cdot(\Theta y)}.$$

Then $\sigma_\Theta \in B^2(\mathbb{Z}^N, \mathbb{T})$ whenever $\Theta \in M_N(\mathbb{R})$ is symmetric: indeed, in this case, $\sigma_\Theta = \mathrm{d}b_\Theta$ where

$$b_\Theta(x) := e^{-i\frac{1}{2}x\cdot(\Theta x)}$$

In general, $[\sigma_\Theta] = [\sigma_{\tilde{\Theta}}]$ where $\tilde{\Theta}$ denotes the skew-symmetric part of Θ.

Every element in $H^2(\mathbb{Z}^N, \mathbb{T})$ may be written as $[\sigma_\Omega]$ for some skew-symmetric Ω.

Projective regular representations

A *σ-projective unitary representation* U of Γ on the Hilbert space $\mathcal{H}$ is a map from Γ into the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ such that

$$U(g)U(h) = \sigma(g,h)U(gh) \qquad (g,h \in \Gamma).$$

Then $U(e) = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) and

$$U(g)^* = \overline{\sigma(g,g^{-1})}U(g^{-1}), \quad g \in \Gamma.$$

If we pick $b : \Gamma \to \mathbb{T}$ satisfying $b(e) = 1$ and set $\tilde{U} = b\,U$, then $\tilde{U}$ becomes a $\tilde{\sigma}$-projective unitary representation of Γ on $\mathcal{H}$ with 2-cocycle $\tilde{\sigma} \sim \sigma$ given by $\tilde{\sigma} = (\mathrm{d}b)\sigma$. Such a $\tilde{U}$ is called a *perturbation* of U (by b).

If $\omega \in Z^2(\Gamma, \mathbb{T})$ and V is some ω-projective unitary representation of Γ on $\mathcal{K}$, can form the tensor product representation $U \otimes V$ acting on $\mathcal{H} \otimes \mathcal{K}$ in the obvious way, which is then $\sigma\omega$-projective. Further, letting $\overline{U}$ denote the conjugate of U, which acts as U on the conjugate Hilbert space $\overline{\mathcal{H}}$ of $\mathcal{H}$, one sees easily that $\overline{U}$ is $\overline{\sigma}$-projective.

To each $\sigma \in Z^2(\Gamma, \mathbb{T})$ one may associate a left (resp. right) regular σ-projective unitary representation λ_σ (resp. ρ_σ) of Γ on $\ell^2(\Gamma)$ defined by

$$(\lambda_\sigma(g)\xi)(h) = \sigma(h^{-1}, g)\xi(g^{-1}h),$$

$$(\rho_\sigma(g)\xi)(h) = \sigma(h, g)\xi(hg),$$

$\xi \in \ell^2(\Gamma),\ g, h \in \Gamma.$

Then $\lambda_\sigma \cong \rho_\sigma$, in fact

$$\rho_\sigma(g) = U\lambda_\sigma(g)U, \quad g \in \Gamma$$

where U is the involutive unitary operator on $\ell^2(\Gamma)$ given by

$$U\xi(g) = \xi(g^{-1}), \quad \xi \in \ell^2(\Gamma), g \in \Gamma.$$

Choosing $\sigma = 1$ gives the usual left and right regular representations of Γ, denoted by λ and ρ.

It is also useful to introduce their unitarily equivalent versions Λ_σ and R_σ, still acting on $\ell^2(\Gamma)$, given by

$$\Lambda_\sigma(g) = S_\sigma \lambda_\sigma(g) S_\sigma^*, \quad R_\sigma(g) = S_\sigma \rho_\sigma(g) S_\sigma^*, \quad g \in \Gamma,$$

with S_σ the unitary multiplication operator on $\ell^2(\Gamma)$

$$(S_\sigma \xi)(g) = \sigma(g, g^{-1})\xi(g), \quad \xi \in \ell^2(\Gamma), g \in \Gamma.$$

In fact, one could just assume that $\sigma(g, g^{-1}) = 1$ for all $g \in \Gamma$, which would not be a real loss of generality as this may be achieved by pertubing with a coboundary. But in some cases it seems convenient not to "regularize" the given cocycle in this way.

Letting $\{\delta_h\}_{h\in\Gamma}$ denote the canonical basis of $\ell^2(\Gamma)$, one has

$$\Lambda_\sigma(g)\delta_h = \sigma(g, h)\delta_{gh}, \quad g, h \in G$$

and, in particular, $\Lambda_\sigma(g)\delta_e = \delta_g$. We also record that

$$(\Lambda_\sigma(g)\xi)(h) = \sigma(g, g^{-1}h)\xi(g^{-1}h), \ \ \xi \in \ell^2(\Gamma), \ g, h \in \Gamma$$

and the following commutation relations

$$\Lambda_\sigma(g)\rho_{\overline{\sigma}}(h) = \rho_{\overline{\sigma}}(h)\Lambda_\sigma(g),$$

$$\lambda_\sigma(g)R_{\overline{\sigma}}(h) = R_{\overline{\sigma}}(g)\lambda_\sigma(h),$$

hold for all $g, h \in \Gamma$. Hence the "right" companion of Λ_σ is $\rho_{\overline{\sigma}}$ (while $R_{\overline{\sigma}}$ is the one for λ_σ).

Twisted group operator algebras

Def.: the *reduced twisted group C^*-algebra* $C^*_r(\Gamma, \sigma)$ (resp. the *twisted group von Neumann algebra* $\mathcal{L}(\Gamma, \sigma)$) is the C^*-subalgebra (resp. von Neumann subalgebra) of $B(\ell^2(\Gamma))$ generated by the set $\Lambda_\sigma(\Gamma)$, that is, as the closure in the operator norm (resp. weak operator) topology of the *-algebra $\mathbb{C}(\Gamma, \sigma) :=$Span$(\Lambda_\sigma(\Gamma))$.

Set $\delta = \delta_e$, a cyclic ($=$ generating) vector for all these algebras.

The (normal) state τ on these algebras given by restricting the vector state ω_δ associated to δ is easily seen to be tracial. Further, τ is faithful as δ is separating for $\mathcal{L}(\Gamma, \sigma)$. Hence $\mathcal{L}(\Gamma, \sigma)$ is finite as a von Neuman algebra.

Remark:

- $\mathcal{L}(\Gamma, \sigma)$ is a factor iff the conjugacy class of each non-trivial σ-regular element in Γ is infinite (by definition, $g \in \Gamma$ is σ-regular whenever $\sigma(g, h) = \sigma(h, g)$ for all $h \in \Gamma$ commuting with g).

- the commutant of $\mathcal{L}(\Gamma, \sigma)$ is the von Neumann algebra generated by $\rho_{\overline{\sigma}}(\Gamma)$, that is, we have

$$\mathcal{L}(\Gamma, \sigma)' = \rho_{\overline{\sigma}}(\Gamma)'',$$

  or equivalently

$$\mathcal{L}(\Gamma, \sigma) = \rho_{\overline{\sigma}}(\Gamma)'.$$

  One inclusion follows readily from the commutation relations, while the converse inclusion can also be shown by going through some elementary, but somewhat more involved considerations. A cheap way to deduce equality directly is to apply (pre-)Tomita-Takesaki theory to the pair $(\mathcal{L}(\Gamma, \sigma), \delta)$: the J-operator is easily seen to be given by $(J_\sigma \xi)(g) = \overline{\sigma}(g, g^{-1})\overline{\xi(g^{-1})}$ and one computes that $J_\sigma \Lambda_\sigma(g) J_\sigma = \rho_{\overline{\sigma}}(g), g \in \Gamma$. Thus

  $\mathcal{L}(\Gamma, \sigma)' = J_\sigma \mathcal{L}(\Gamma, \sigma) J_\sigma = (J_\sigma \Lambda_\sigma(\Gamma) J_\sigma)'' = \rho_{\overline{\sigma}}(\Gamma)''.$

Also, we may consider $\mathcal{L}(\Gamma, \sigma)$ as a Hilbert algebra w.r.t. the inner product $< x, y >:= \tau(y^* x) = (x\delta, y\delta)$. Denoting by $\| \cdot \|_2$ the associated norm, the linear map

$$x \to \widehat{x} := x\delta$$

is then an isometry from $(\mathcal{L}(\Gamma, \sigma), \| \cdot \|_2)$ to $(\ell^2(\Gamma), \| \cdot \|_2)$, which sends $\Lambda_\sigma(g)$ to δ_g for each $g \in \Gamma$. (This map is the analogue of the Fourier transform when Γ is abelian, $\sigma = 1$, and one identifies $\mathcal{L}(\Gamma)$ with $L^\infty(\widehat{\Gamma})$).

The value $\widehat{x}(g)$ is called the *Fourier coefficient* of $x \in \mathcal{L}(\Gamma, \sigma)$ at $g \in \Gamma$. Considering τ as the normalized "Haar functional" on $\mathcal{L}(\Gamma, \sigma)$, we have indeed

$$\widehat{x}(g) = (x\delta, \delta_g) = (x\delta, \Lambda_\sigma(g)\delta) = \tau(x\Lambda_\sigma(g)^*).$$

Further, we have $\|\widehat{x}\|_\infty \leq \|\widehat{x}\|_2 = \|x\|_2 \leq \|x\|$.

Fourier Series

The (formal) *Fourier series* of $x \in \mathcal{L}(\Gamma, \sigma)$ is defined as $\sum_{g \in \Gamma} \widehat{x}(g) \Lambda_\sigma(g)$. This series does not necessarily converge in the weak operator topology. However, we have

$$x = \sum_{g \in \Gamma} \widehat{x}(g) \Lambda_\sigma(g)$$

(convergence w.r.t. $\| \cdot \|_2$.)

The Fourier series representation of $x \in \mathcal{L}(\Gamma, \sigma)$ is unique.

Let $f \in \ell^1(\Gamma)$. The series $\sum_{g\in\Gamma} f(g)\Lambda_\sigma(g)$ is clearly absolutely convergent in operator norm and we shall denote its sum by $\pi_\sigma(f)$. Then we have $\|\pi_\sigma(f)\| \leq \|f\|_1$ and

$$\widehat{\pi_\sigma(f)} = (\sum_{g\in\Gamma} f(g)\Lambda_\sigma(g))\delta = \sum_{g\in\Gamma} f(g)\delta_g = f.$$

Let now $x \in \mathcal{L}(\Gamma, \sigma)$ and assume that $\widehat{x} \in \ell^1(\Gamma)$. Then we get $\widehat{\pi_\sigma(\widehat{x})} = \widehat{x}$, hence $\pi_\sigma(\widehat{x}) = x$. Therefore, *in this case*, we have $\|x\| = \|\pi_\sigma(\widehat{x})\| \leq \|\widehat{x}\|_1$ and

$$x = \sum_{g\in\Gamma} \widehat{x}(g)\Lambda_\sigma(g) \ \ \text{(convergence w.r.t. } \|\cdot\|\text{)},$$

which especially shows that $x \in C_r^*(\Gamma, \sigma)$. Hence, setting $CF(\Gamma, \sigma) := \{x \in C_r^*(\Gamma, \sigma) \mid \sum_{g\in\Gamma} \widehat{x}(g)\Lambda_\sigma(g)$ is convergent in operator norm $\}$ we have $\pi_\sigma(\ell^1(\Gamma)) \subseteq CF(\Gamma, \sigma)$.

As in classical Fourier analysis, one may consider other kinds of decay properties to ensure convergence of Fourier series in operator norm!

The subspace of $\ell^2(\Gamma)$ defined by

$$\mathcal{U}(\Gamma, \sigma) := \{\widehat{x} \mid x \in \mathcal{L}(\Gamma, \sigma)\}$$

becomes a Hilbert algebra when equipped with the involution $\widehat{x}^* := \widehat{x^*}$ and the product $\widehat{x} * \widehat{y} := \widehat{xy}$. We have $\widehat{x}^*(g) = \overline{\sigma}(g, g^{-1})\overline{\widehat{x}(g^{-1})}$. Further, as our notation indicates, the product $\widehat{x} * \widehat{y}$ may be expressed as a twisted convolution product.

To see this, let $\xi, \eta \in \ell^2(\Gamma)$. The *$\sigma$-convolution product* $\xi * \eta$ is defined as the complex function on Γ given by

$$(\xi * \eta)(h) = \sum_{g \in \Gamma} \xi(g)\sigma(g, g^{-1}h)\eta(g^{-1}h), \ h \in \Gamma.$$

As $|(\xi * \eta)(h)| \leq (|\xi| * |\eta|)(h), \ h \in \Gamma$, it is straightforward to check that $\xi * \eta$ is a well defined bounded function on Γ satisfying

$$\|\xi * \eta\|_\infty \leq \||\xi| * |\eta|\|_\infty \leq \|\xi\|_2 \|\eta\|_2.$$

We notice that $\delta_a * \delta_b = \sigma(a, b)\delta_{ab}$, $a, b \in \Gamma$.

Now, if $x \in \mathcal{L}(\Gamma, \sigma)$ and $\eta \in \ell^2(\Gamma)$, one has $x\eta = \widehat{x} * \eta$. This implies that $\widehat{xy} = xy\delta = x\widehat{y} = \widehat{x} * \widehat{y}$ for all $x, y \in \mathcal{L}(\Gamma, \sigma)$, where the last expression is defined through the σ-convolution product, thus justifying our comment above.

BTW, $\mathcal{U}(\Gamma, \sigma)$ may be described as the space of all $\xi \in \ell^2(\Gamma)$ such that $\xi * \eta \in \ell^2(\Gamma)$ for all $\eta \in \ell^2(\Gamma)$ and the resulting linear map $\eta \to \xi * \eta$ from $\ell^2(\Gamma)$ into itself is bounded.

Since $\widehat{\pi_\sigma(f)} = f$ for all $f \in \ell^1(\Gamma)$, we have $\ell^1(\Gamma) \subseteq \mathcal{U}(\Gamma, \sigma)$. Further, $\ell^1(\Gamma)$ is a *-subalgebra of $\mathcal{U}(\Gamma, \sigma)$ which becomes a unital Banach *-algebra with respect to the ℓ^1-norm $\|\cdot\|_1$, the unit being given by δ. This Banach *-algebra is usually denoted by $\ell^1(\Gamma, \sigma)$. Its involution is explicitely given by $f^*(g) = \overline{\sigma}(g, g^{-1})\overline{f(g^{-1})}, g \in \Gamma$.

Consider the map $\pi_\sigma : \ell^1(\Gamma) \to C_r^*(\Gamma, \sigma) \subseteq B(\ell^2(\Gamma))$ defined by $f \to \pi_\sigma(f)$. Clearly we have

$$\pi_\sigma(f)\eta = f * \eta,\ f \in \ell^1(\Gamma),\ \eta \in \ell^2(\Gamma).$$

Further, π_σ is easily seen to be a faithful *-representation of $\ell^1(\Gamma, \sigma)$ on $\ell^2(\Gamma)$. Hence, the enveloping C*-algebra of $\ell^1(\Gamma, \sigma)$ is just the completion of $\ell^1(\Gamma, \sigma)$ w.r.t. the norm

$$\|f\|_{max} := \sup_\pi \{\|\pi(f)\|\}$$

where the supremum is taken over all non-degenerate *-representations of $\ell^1(\Gamma, \sigma)$ on Hilbert spaces. This C*-algebra is denoted by $C^*(\Gamma, \sigma)$ and called the *full twisted group C^*-algebra associated to* (Γ, σ). We will identify $\ell^1(\Gamma, \sigma)$ with its canonical image in $C^*(\Gamma, \sigma)$, which is then generated as a C*-algebra by its canonical unitaries δ_g.

The twisted group C^*-algebras of the form $C^*(\mathbb{Z}^N, \sigma_\Theta)$ are often called noncommutative N-tori (since $C^*(\mathbb{Z}^N, \sigma_\Theta)$ is $*$-isomorphic to $C(\mathbb{Z}^N)$ in the case where Θ is symmetric).

Any non-degenerate $*$-representation of $\ell^1(\Gamma, \sigma)$ extends uniquely to a non-degenerate $*$-representation of $C^*(\Gamma, \sigma)$, and we will always use the same symbol to denote the extension. There is a bijective correspondence $U \longrightarrow \pi_U$ between σ-projective unitary representations of Γ and non-degenerate $*$-representations of $C^*(\Gamma, \sigma)$ determined by

$$\pi_U(f) = \sum_{g \in G} f(g)U(g), \ f \in \ell^1(\Gamma),$$

(the series above being obviously absolutely convergent in operator norm), the inverse correspondence being simply given by $U_\pi(g) = \pi(\delta_g), g \in \Gamma$. As $\pi_{\Lambda_\sigma} = \pi_\sigma$ we have

$$C_r^*(\Gamma, \sigma) = \overline{\pi_\sigma(\ell^1(\Gamma, \sigma))}^{\|\cdot\|} = \pi_\sigma(C^*(\Gamma, \sigma)).$$

When G is amenable, then π_σ is faithful.

Summary (case $\sigma = 1$):

$$C^*(\Gamma)$$

$$\downarrow \lambda$$

$$\mathbb{C}\Gamma (= \mathcal{K}\Gamma) \hookrightarrow \ell^1(\Gamma) \subset C^*_r(\Gamma) \subset \mathcal{L}(\Gamma) \hookrightarrow \ell^2(\Gamma)$$

and

$$\|f\|_2 \leq \|\lambda(f)\| \leq \|f\|_1$$

The dual space of $C^*(\Gamma, \sigma)$ may be identified as a subspace $B(\Gamma, \sigma)$ of $\ell^\infty(\Gamma)$ through the linear injection $\Phi : \phi \to \tilde{\phi}$ where $\tilde{\phi}(g) := \phi(\delta_g), g \in \Gamma$. Equip $B(\Gamma, \sigma)$ with the transported norm $\|\Phi(\phi)\| := \|\phi\|$. Now, if ϕ is a positive linear functional on $C^*(\Gamma, \sigma)$, then $\tilde{\phi}$ is σ-positive definite according to the following definition : a complex function φ on Γ is *σ-positive definite* (σ-p.d.) whenever we have

$$\sum_{i,j=1}^{n} \overline{c_i} c_j \varphi(g_i^{-1} g_j) \overline{\sigma}(g_i, g_i^{-1} g_j) \geq 0$$

for all $n \in \mathbb{N}, c_1, \ldots c_n \in \mathbb{C}, g_1, \ldots g_n \in \Gamma$.
One checks readily that φ is σ-p.d. if and only if there exists a σ-projective unitary representation U of Γ on a Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ (which may be chosen to be cyclic for U) s.t.

$$\varphi(g) = (U(g)\xi, \xi),\ g \in \Gamma,$$

which implies that φ is then bounded with $\|\varphi\|_\infty = \|\xi\|^2 = \varphi(e)$. Further, as we then have $(\pi_U(f)\xi, \xi) = \sum_{g \in G} f(g)\varphi(g)$ for all $f \in \ell^1(\Gamma)$, we also get an unambiguously defined positive linear functional L_φ on $C^*(\Gamma, \sigma)$ via $L_\varphi(x) := (\pi_U(x)\xi, \xi)$, which satisfies that $\Phi(L_\varphi) = \varphi$. Denoting by $P(\Gamma, \sigma)$ the cone of all σ-p.d. functions on Γ, then

$$B(\Gamma, \sigma) = \mathrm{Span}(P(\Gamma, \sigma)).$$

By considering the universal *-representation of $C^*(\Gamma, \sigma)$, one deduces further that $B(\Gamma, \sigma)$ consists precisely of all coefficient functions associated to σ-projective unitary representations of Γ.

Remark: if φ is σ-p.d. and ψ is ω-p.d. for some $\omega \in Z^2(\Gamma, \mathbb{T})$ then $\varphi\psi$ is $\sigma\omega$-p.d. Hence $B(\Gamma, \sigma)B(\Gamma, \omega) \subseteq B(\Gamma, \sigma\omega)$.

Especially, $B(\Gamma, \sigma)$ is not a priori an algebra w.r.t. to pointwise multiplication (unless we have $\sigma = 1$, in which case it is usually called the Fourier-Stieltjes algebra of Γ). It is not a priori closed under complex conjugation either : if $\varphi \in P(\Gamma, \sigma)$, then $\overline{\varphi} \in P(\Gamma, \overline{\sigma})$. Similarly, if $\tilde{\varphi}(g) := \sigma(g, g^{-1})\varphi(g^{-1})$, then $\tilde{\varphi} \in P(\Gamma, \overline{\sigma})$. Hence $\varphi^* \in P(\Gamma, \sigma)$, where $\varphi^*(g) := \overline{\sigma}(g, g^{-1})\overline{\varphi}(g^{-1})$. (This just corresponds to the fact that $L_{\varphi^*} = (L_\varphi)^*$ is then also positive linear functional on $C^*(\Gamma, \sigma)$).

As $C_r^*(\Gamma, \sigma)$ is a quotient of $C^*(\Gamma, \sigma)$, we may identify its dual space as a closed subspace $B_r(\Gamma, \sigma)$ of $B(\Gamma, \sigma)$ consisting of the span of all σ-p.d. functions on Γ associated to unitary representations of Γ which are weakly contained in Λ_σ (that is, such that the associated representation of $C^*(\Gamma, \sigma)$ is weakly contained in π_σ). Further, the predual of $\mathcal{L}(\Gamma, \sigma)$ can be regarded as a closed subspace of the dual of $C_r^*(\Gamma, \sigma)$, hence as a closed subspace $A(\Gamma, \sigma)$ of $B_r(\Gamma, \sigma)$, and of $B(\Gamma, \sigma)$, which may be described as the set of all coefficient functions of Λ_σ.

Dual Spaces: a summary (untwisted case)

$P(\Gamma) =$ cone of all pos.def. functions on Γ

$A(\Gamma)(\simeq \mathcal{L}(\Gamma)_*) =$ set of all matrix coefficients of λ, the Fourier algebra of Γ

$B_r(\Gamma)(\simeq C_r^*(\Gamma)^*)$ set of all matrix coefficients of unitary rep's of Γ weakly contained in λ

$B(\Gamma)(\simeq C^*(\Gamma)^*) =$ set of all matrix coefficients of unitary reps of Γ, the Fourier-Stieltjes algebra of Γ

$$\ell^2(\Gamma) \subseteq A(\Gamma) \subseteq B_r(\Gamma) \subseteq B(\Gamma) = \mathrm{span}\, P(\Gamma)$$

Amenable groups

Γ is *amenable* if there exists a (left or/and right) translation invariant state on $\ell^\infty(\Gamma)$. Amenability of Γ can be formulated in a huge number of equivalent ways. In particular, TFAE:

1) Γ has a *Følner net* $\{F_\alpha\}$, that is, each F_α is a finite non-empty subset of Γ and we have

$$\lim_\alpha \frac{|gF_\alpha \triangle F_\alpha|}{|F_\alpha|} = 0, \quad g \in \Gamma\ . \tag{1}$$

2) there exists a net (φ_α) of normalized positive definite functions on Γ with finite support such that $\varphi_\alpha \to 1$ pointwise on Γ.

   (As usual, a complex function on Γ is called *normalized* when it takes the value 1 at e).

3) there exists a net $\{\psi_\alpha\}$ of normalized positive definite functions in $\ell^2(\Gamma)$ such that $\psi_\alpha \to 1$ pointwise on Γ.

4) $|\sum_{g\in\Gamma} f(g)\,| \leq \|\sum_{g\in G} f(g)\lambda(g)\|$ $(= \|\pi_1(f)\|)$ for all $f \in \ell^1(\Gamma)$.

Here, take 1) as the running definition of the amenability of Γ, and regard 2), 3) and 4) as properties.

Indeed, assume 1) holds and set $\xi_\alpha := |F_\alpha|^{-1/2}\chi_{F_\alpha}$, which is a unit vector in $\ell^2(\Gamma)$. Then 2) is satisfied with $\varphi_\alpha(g) := (\lambda(g)\xi_\alpha, \xi_\alpha) = \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|}$: each φ_α is clearly p.d., has finite support given by $\mathrm{supp}(\varphi_\alpha) = F_\alpha \cdot F_\alpha^{-1}$ and the Følner condition (1) is equivalent to $\varphi_\alpha \to 1$ pointwise. Condition 3) is then trivially satisfied with $\psi_\alpha = \varphi_\alpha$. Further, letting ϵ being the state on $B(\ell^2(\Gamma))$ obtained by picking any weak*-limit point of the net of vector states $\{\omega_{\xi_\alpha}\}$, we get $\epsilon(\lambda(g)) = 1$ for all $g \in \Gamma$, hence

$$|\sum_{g\in\Gamma} f(g)\,| = |\,\epsilon(\sum_{g\in\Gamma} f(g)\lambda(g))\,| \le \|\sum_{g\in\Gamma} f(g)\lambda(g)\|$$

for all $f \in \ell^1(\Gamma)$, which shows that 4) holds.

Haagerup property

Γ has the *Haagerup property* if there exists a net $\{\varphi_\alpha\}$ of normalized positive definite functions on Γ, vanishing at infinity on Γ (that is, $\varphi_\alpha \in c_0(\Gamma)$ for all α), and converging pointwise to 1. When Γ is countable, this property is equivalent to the fact that there exists a negative definite function $h : \Gamma \to [0, \infty)$ which is proper, that is, $\lim_{g\to\infty} h(g) = \infty$, or, equivalently, $(1+h)^{-1} \in c_0(\Gamma)$. We will call such a function h a *Haagerup function* on Γ.

This class of groups includes all amenable groups (by 3) and also the nonabelian free groups (Haagerup).

Negative definite functions (case $\sigma = 1$)

Recall that a function $\psi : \Gamma \to \mathbb{C}$ is called *negative definite* (or conditionally negative definite) whenever ψ is Hermitian, that is $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in \Gamma$, and

$$\sum_{i,j=1}^{n} \overline{c_i} c_j \psi(g_i^{-1} g_j) \leq 0$$

$\forall\, n \in \mathbb{N}, g_1, \ldots, g_n \in \Gamma$, $c_1, \ldots, c_n \in \mathbb{C} : \sum_{i=1}^{n} c_i = 0$.

By Schoenberg theorem, a function $\psi : \Gamma \to \mathbb{C}$ is negative definite iff $e^{-t\psi}$ is p.d. for all $t > 0$ (equivalently, r^{ψ} is p.d for all $0 < r < 1$).

$(t + \psi)^{-1}$ is p.d. for all $t > 0$ whenever $\psi : \Gamma \to \{z \in \mathbb{C}, \Re(z) \geq 0\}$ is negative definite.

If $\psi : \Gamma \to \{z \in \mathbb{C}, \Re(z) \geq 0\}$ is negative definite and satisfies $\psi(e) \geq 0$, then $\psi^{1/2}$ is negative definite.

Example: consider a homomorphism $b : \Gamma \to \mathcal{H}$ (Hilbert space $\mathcal{H}$ regarded as a group w.r.t. addition). Then $\psi(g) := \|b(g)\|^2$ is negative definite on Γ. Especially, $|\cdot|_2$ denoting the Euclidean norm-function on $\mathbb{Z}^N, N \in \mathbb{N}$, it follows that $|\cdot|_2^2$, and therefore also $|\cdot|_2$ (taking the square root), are negative definite on $\mathbb{Z}^N$. The $|\cdot|_1$-norm function on $\mathbb{Z}^N$ is also negative definite. Last claim proved by induction : the inductive step being straightforward, it suffices to show this when $N = 1$. Then appeal to Schoenberg's theorem : it suffices to show that $\varphi(m) := r^{|m|}$ is p.d. on $\mathbb{Z}$ for all $0 < r < 1$. Let U denote the unitary representation of $\mathbb{Z}$ on $L^2(\mathbb{T})$ associated to the unitary operator on $L^2(\mathbb{T})$ given by multiplication with the function $z \to z^{-1}, z \in \mathbb{T}$. With $\xi_r := \sum_{k=-\infty}^{\infty} r^{|k|} e_k \in L^2(\mathbb{T})$ for $r \in (0, 1)$, one has $\varphi(k) = r^{|k|} = (U(k)\xi_r, \xi_r)$ for all $k \in \mathbb{Z}$, and the assertion is then clear.

Length

An interesting class of functions on Γ are the so-called length functions (which are basically left Γ-invariant metrics on Γ).

Definition: A function $L : \Gamma \to [0, \infty)$ is a *length* function if

$L(e) = 0$,

$L(g^{-1}) = L(g)$

$L(gh) \leq L(g) + L(h)$

for all $g, h \in \Gamma$.

Examples:
(1) If Γ acts isometrically on a metric space (X, d) and $x_0 \in X$, then

$$L(g) := d(g \cdot x_0, x_0)$$

gives a *geometric* length function on Γ.
(2) If Γ is finitely generated and S is a finite generator set for Γ, then the obvious word-length function $g \to |g|_S$ (w.r.t. to the letters from $S \cup S^{-1}$) is an *algebraic* length function on Γ. All such algebraic length functions are equivalent in a natural way. Any algebraic length function is clearly proper.

Remark: for any $t > 0$ and any algebraic length function L on Γ, the "Gaussian" function e^{-tL^2} is summable (this corresponds to the fact that the naturally associated unbounded Fredholm module $(\ell^2(\Gamma), D_L)$ is θ-summable in Connes' terminology).

Length functions may be used to define growth conditions. Let L be a length function on Γ; look at the ball of radius r

$$B_{r,L} := \{g \in \Gamma | L(g) \leq r\}, r \in \mathbb{R}, r \geq 0.$$

Then Γ is said to be

(i) of *polynomial growth* (w.r.t. L) if there exist some constants $K, p > 0$ such that, for all $r \geq 0$,

$$|B_{r,L}| \leq K(1 + r)^p$$

(ii) *exponentially bounded* (w.r.t. L) if for any $b > 1$, there is some $r_0 \in \mathbb{R}, r_0 \geq 0$, such that, for all $r \geq r_0$,

$$|B_{r,L}| < b^r$$

Clearly, exponential boundedness is weaker than polynomial growth.

If Γ is finitely generated, one just says that Γ has polynomial growth (resp. is exponentially bounded) if the property holds w.r.t. some or, equivalently, any algebraic length on Γ. Any exponentially bounded group is necessarily amenable.

A famous result of M. Gromov says that Γ is of polynomial growth if (and only if) Γ is almost nilpotent (the only if part being due to W. Woess). Further, R. I. Grigorchuk has produced examples of exponentially bounded groups which are not of polynomial growth. Finally, if Γ is finitely generated and has polynomial growth (resp. is exponentially bounded) w.r.t. to some length function L on Γ, then Γ has polynomial growth (resp. is exponentially bounded).

Remark: Algebraic length functions on finitely generated groups have been used to define (formal) growth series of the type $\sum_{g \in G} z^{L_S(g)}$; We consider summability aspects of this kind of series (for real z between 0 and 1) in the case where the length function is not necessarily algebraic.

Γ fin. gen., S generator set

Theorem:
1) If Γ has polynomial growth then $\{B_{k,L_S}\}_k$ is a Følner sequence for Γ
2) If Γ has subexponential growth then there is a subsequence of $\{B_{k,L_S}\}_k$ which is a Følner sequence for Γ
3) Γ has polynomial growth iff it is almost nilpotent
4) Γ may have subexponential growth without having polynomial growth

Remark (length functions vs. Haagerup property): assume that h is a Haagerup function for some (countable) Γ s.t. WLOG $h(e) = 0$ and $h(g) > 0$ for $g \neq e$. Then $L := h^{1/2}$ is negative definite, and it is also a length function on Γ. Hence L is a Haagerup length function on Γ. This means that a countable group has the Haagerup property if and only if it has a Haagerup length function.

In some cases, a Haagerup length function is naturally geometrically given: this is for example the case when Γ acts isometrically and metrically properly on a tree, or on a $\mathbb{R}$-tree, X (equipped with its natural metric). In general, one can show that a countable group Γ has the Haagerup property if and only if there exists an isometric and metrically proper action of Γ on some metric space (X, d), a unitary representation U of Γ on some Hilbert space $\mathcal{H}$ and a map $c : X \times X \to \mathcal{H}$ satisfying the following conditions :

$$c(x,z) = c(x,y) + c(y,z), \quad c(g \cdot x, g \cdot y) = U(g)\, c(x,y)$$

$$\|c(x,y)\| \to \infty \quad \text{as} \quad d(x,y) \to \infty, \text{ for all } x, y, z \in X, g \in G.$$

In this case, picking any $x_0 \in X$, $h(g) := d(g \cdot x_0, x_0)^2$ is then a Haagerup function for Γ, while $L(g) := d(g \cdot x_0, x_0)$ is a Haagerup length function for Γ.

In the case of finitely generated groups, a Haagerup length function is sometimes algebraically given : this is at least true for finitely generated free groups and Coxeter groups.

Remark: let Γ be finitely generated and assume that it has an algebraic length function L such that L^2 is negative definite (this implies that L itself is negative definite). Then Γ is amenable: indeed, the "Gaussian" net of functions on Γ defined by $\psi_t := e^{-tL^2}, t > 0$ consists then of summable functions which are all normalized and p.d., and it converges pointwise to 1 on Γ as $t \to 0^+$.

PREPARATION

Fourier series and multipliers

Setup: $\mathcal{A} = C^*_r(\Gamma, \sigma) \subset \mathcal{B} = \mathcal{L}(\Gamma, \sigma) \subset B(\ell^2(\Gamma))$

τ canonical tracial state on $\mathcal{B}$

To each $x \in \mathcal{B}$, attach its (formal) Fourier series

$$\sum_{g \in \Gamma} \widehat{x}(g) \Lambda_\sigma(g) \ ,$$

where $\Lambda_\sigma(g)$ is the (left) σ-projective regular representation of Γ on $\ell^2(\Gamma)$ and $\widehat{x}(g) = \tau(x \Lambda_\sigma(g)^*)$ is the Fourier coefficient of x at g

This series is trivially convergent in the $\| \cdot \|_2$ norm, but it is not necessarily convergent in the WOT on $\mathcal{B}$ (even if $\sigma = 1$).

Main Goal: set up a general framework for discussing norm convergence of Fourier series in twisted group C^*-algebras of discrete groups

However, in general, for $x \in \mathcal{A}$, the Fourier series will not always be convergent to x in norm: for abelian Γ (say $\mathbb{Z}$) and σ trivial one has $C^*_r(\Gamma, 1) \simeq C(\widehat{\Gamma})$ and recover the classical situation!

Way out: summation properties of Fejér, resp. Abel-Poisson type!

Tool: multipliers (Haagerup, 1982)

Let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Then there exists a unique completely positive map $M_\varphi \in B(C_r^*(\Gamma))$ s.t., for all $g \in \Gamma$,

$$M_\varphi(\lambda(g)) = \varphi(g)\lambda(g)$$

Also, $\|M_\varphi\| = \varphi(e)$.

In particular, such a φ is a "multiplier" on Γ.

Haagerup's results (1982): although $\mathbb{F}_n$ is not amenable, $C_r^*(\mathbb{F}_n)$ has the M.A.P. ($n \leq \infty$).

Let $\Gamma = \mathbb{F}_2$

$|\cdot|$ the word length function w.r.t. $S = \{a, b, a^{-1}, b^{-1}\}$

- The function $\mathbb{F}_2 \ni g \mapsto e^{-\lambda|g|}$ is (vanishing at infinity and) positive definite, for every $\lambda > 0$

  By Schoenberg theorem, $|\cdot|$ is (proper) negative definite

- $\|\lambda(f)\| \leq 2(\sum_{g\in\Gamma} |f(g)|^2(1+|g|)^4)^{1/2}$, $\forall f \in \mathbb{C}\Gamma$

- Let $\varphi : \Gamma \to \mathbb{C}$ be s.t.

  $$K := \sup_{g\in\Gamma} |\varphi(g)|(1+|g|)^2 < \infty .$$

  Then φ is a multiplier with $\|\varphi\| \leq 2K$.

a-T-menable groups

A discrete group Γ has the Haagerup property (or is a-T-menable) if there exists a proper conditionally negative type function d on Γ (in that case, one can choose d to be a length function)

Bekka-Cherix-Jolissaint-Valette:

For a second countable, l.c. group G, TFAE:

(1) there exists a continuous function $d : G \to \mathbb{R}_+$ which is of conditionally negative type and proper, that is, $\lim_{g\to\infty} d(g) = \infty$

(2) G has the Haagerup approximation property, in the sense of C.A. Akemann and M. Walter or M. Choda, or property C_0 in the sense of V. Bergelson and J. Rosenblatt: there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous, normalized (i.e., $\varphi_n(e) = 1$) positive definite functions on G, vanishing at infinity on G, and converging to 1 uniformly on compact subsets of G. (In other words, $C_0(G)$ has an approximate unit of continuous normalized positive definite functions).

(3) G is a-T-menable, as Gromov meant it in 1986: there exists a (strongly continuous, unitary) representation of G, weakly containing the trivial representation, whose matrix coefficients vanish at infinity on G (a representation with matrix coefficients vanishing at infinity will be called a C_0-representation)

(4) G is a-T-menable, as Gromov meant it in 1992: there exists a continuous, isometric action α of G on some affine Hilbert space $\mathcal{H}$, which is metrically proper (that is, for all bounded subsets B of $\mathcal{H}$, the set $\{g \in G : \alpha_g(B) \cap B \neq \emptyset\}$ is relatively compact in G).

Moreover, if these conditions hold, one can choose in (1) a proper, continuous, conditionally negative definite function d such that $d(g) > 0$ for all $g \neq e$, and similarly the representation π in (3) may be chosen such that, for all $g \neq e$, there exists a unit vector $\xi \in \mathcal{H}$ with $|(\xi, \pi(g)\xi)| < 1$. In particular, π is faithful.

Jolissaint's Property RD

For any $s \geq 0$, define the s-Sobolev space $H_\ell^s(\Gamma) := (\mathbb{C}\Gamma)^{\|\cdot\|_{\ell,s}}$, where

$$\|f\|_{\ell,s} = \sqrt{\sum_{g\in\Gamma} |f(g)|^2(1+\ell(g))^{2s}} = \|f(1+\ell)^s\|_2, f \in \mathbb{C}\Gamma$$

is the weighted ℓ^2-norm associated with the length ℓ.

A discrete group Γ has property RD (rapid decay) w.r.t. some length function ℓ if there exists positive reals C, s such that, for all $f \in \mathbb{C}\Gamma$,

$$\|\lambda(f)\| \leq C\|f\|_{\ell,s} .$$

A group Γ has property RD if it satisfies property RD w.r.t. some length function ℓ.

[Roughly, RD w.r.t. ℓ means that $\frac{1}{(1+\ell)^s} : \ell^2(\Gamma) \hookrightarrow C_r^*(\Gamma)$]

Rem. if Γ is amenable, then it has RD (w.r.t. ℓ) iff Γ has polynomial growth (w.r.t. ℓ).

The functions in the intersection of all Sobolev spaces

$$H_\ell^\infty(\Gamma) = \bigcap_{s \geq 0} H_\ell^s(\Gamma)$$

are called *rapidly decaying* functions, as their decay at infinity is faster than any inverse of a polynomial in ℓ. Property RD w.r.t. ℓ is equivalent to having $H_\ell^\infty(\Gamma) \subseteq C_r^*(\Gamma)$, which somehow explains the terminology.

Example: $\Gamma = \mathbb{Z}$, under Fourier transform $C_r^*(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$, and $H_\ell^\infty(\Gamma)$ corresponds to smooth functions.

Decay properties

Let $\mathcal{L}$ be a linear space s.t. $\mathcal{K}\Gamma \subset \mathcal{L} \subset \ell^2\Gamma$.

Say that (G, σ) has the $\mathcal{L}$-decay property if there exists a norm $\|\cdot\|'$ on $\mathcal{L}$ such that

i) $\forall \epsilon > 0$ there exists a finite $F_0 \subset \Gamma$ such that $\|\xi\chi_F\|' < \epsilon$ for all finite $F \subset \Gamma$ disjoint from F_0

ii) the map $f \mapsto \pi_\sigma(f)$ from $(\mathcal{K}\Gamma, \|\cdot\|')$ to $(C_r^*(\Gamma, \sigma), \|\cdot\|)$ is bounded.

Under very mild conditions, if $(G, 1)$ has $\mathcal{L}$-decay then (G, σ) has $\mathcal{L}$-decay, too.

Theorem: Suppose that (G, σ) has $\mathcal{L}$-decay. Then

(1) Given $\xi \in \mathcal{L}$, the series $\sum_{g \in \Gamma} \xi(g)\Lambda_\sigma(g)$ converges in operator norm to some $a \in C_r^*(\Gamma, \sigma)$ such that $\widehat{a} = \xi$.

Set $a =: \tilde{\pi}_\sigma(\xi)$.

(2) $\tilde{\pi}_\sigma(\mathcal{L}) = \{x \in \mathcal{L}(\Gamma, \sigma) \mid \widehat{x} \in \mathcal{L})\} \subset CF(\Gamma, \sigma)$.

Clearly $\mathcal{L} = \ell^1(\Gamma)$ always works

For other examples, look at the weighted spaces

$$\mathcal{L}^p_\kappa := \{\xi : \Gamma \to \mathbb{C} \mid \xi\kappa \in \ell^p(\Gamma)\} \subseteq \ell^p(\Gamma) \ ,$$

$1 \leq p \leq \infty$, equipped with the norm $\|\xi\|_{p,\kappa} = \|\xi\kappa\|_p$.
Here, $\kappa :\in \Gamma \to [1, +\infty)$.
Note that $\mathcal{L}^p_\kappa \subset \mathcal{L}^q_\kappa$, $1 \leq p \leq q \leq +\infty$.

Def. Say that (G, σ) is κ-decaying if it has the $\mathcal{L}^2_\kappa$-decay property (w.r.t. $\|\cdot\|_{2,\kappa}$).

Examples:

(i) Γ fin.gen., L algebraic length function. For $t > 0$, set $\kappa_t = e^{tL^2}$, then $(\kappa_t)^{-1} \in \ell^2\Gamma$ and Γ is κ_t-decaying

(ii) any Γ with subexponential growth is a^L-decaying, for all $a > 1$.

(iii) Γ has RD-property (w.r.t. length L) iff there exists $s_0 > 0$ s.t. Γ is $(1 + L)^{s_0}$-decaying.

Haagerup content and H-growth

Let $\emptyset \neq E \subset \Gamma$ be finite. Set

$c(E) := \sup\{\|\pi_\lambda(f)\| \mid f \in \mathcal{K}\Gamma,\ \mathrm{supp}(f) \subseteq E, \|f\|_2 = 1\}$

Then $1 \leq c(E) \leq |E|^{1/2}$.
If G is amenable, $c(E) = |E|^{1/2}$ for all E.

Def. For Γ countable and $L : \Gamma \to [0, +\infty)$ a proper function, set $B_{r,L} = \{g \in \Gamma \mid L(g) \leq r\}$. Then

Γ has polynomial H-growth (w.r.t. L) if there exist $K, p > 0$ such that

$$c(B_{r,L}) \leq K(1 + r)^p, \quad r \in \mathbb{R}_+ .$$

Γ has subexponential H-growth if, for any $b > 1$, there exists $r_0 \in \mathbb{R}_+$ such that

$$c(B_{r,L}) < b^r, \quad r \geq r_0 .$$

(For Γ amenable with length function L, these definitions reduce to the usual ones)

Examples:

(i) $\mathbb{F}_n$ has polynomial H-growth w.r.t. word-length.

(ii) More generally, the same holds for any Gromov hyperbolic group.

(iii) Any Coxeter group has polynomial H-growth.

(iv) Under mild assumptions, polynomial H-growth is stable under amalgamated free products $\Gamma_1 *_A \Gamma_2$ with finite A.

(v) Γ fin. gen., with subexponential but not polynomial growth, then $\Gamma \times \mathbb{F}_2$ has subexponential (but not polynomial) H-growth w.r.t. $L(g_1, g_2) = L_1(g_1) + L_2(g_2)$

Fundamental Lemma: any countably infinite Γ is κ-decaying, for a suitable $\kappa : \Gamma \to [1, +\infty)$.

Theorem: Γ countably infinite, $L : \Gamma \to [0, +\infty)$ proper.

1) Suppose that Γ has polynomial H-growth (w.r.t. L). Then there exists $s_0 > 0$ such that (Γ, σ) is $(1 + L)^{s_0}$-decaying. In particular, if L is a length function, then Γ has the σ-twisted RD-property.

(2) Suppose that Γ has subexponential H-growth. Then (Γ, σ) is a^L-decaying for any $a > 1$.

Corollary: Let $L : \Gamma \to [0, +\infty)$ be a proper function.

(1) If Γ has polynomial H-growth (w.r.t. L), then there exists some $s > 0$ such that the Fourier series of $x \in C_r^*(\Gamma, \sigma)$ converges to x in operator norm, whenever

$$\sum_{g \in \Gamma} |\widehat{x}(g)|^2 (1 + L(g))^s < +\infty \ .$$

(2) If Γ has subexponential H-growth, then the Fourier series of $x \in C_r^*(\Gamma, \sigma)$ converges to x in operator norm, whenever there exists some $t > 0$ such that

$$\sum_{g \in \Gamma} |\widehat{x}(g)|^2 e^{tL(g)} < +\infty \ .$$

Intermezzo: Twisted Haagerup's Lemma

$\sigma \in Z^2(\Gamma, \mathbb{T})$, V proj. unitary repr. of Γ with 2-cocycle ω

Twisted Fell Absorbing Property: $\Lambda_\sigma \otimes V \cong \Lambda_{\sigma\omega} \otimes I_{\mathcal{H}}$

Twisted Haagerup Lemma: $\omega \in Z^2(\Gamma, \mathbb{T})$, $\varphi \in P(\Gamma, \omega)$, V ω-projective repr. on $\mathcal{H}$, $\eta \in \mathcal{H}$ s.t. $\varphi(g) = (V(g)\eta, \eta)$. Then there exists a c.p. normal map $\tilde{M}_\varphi : \mathcal{L}(\Gamma, \sigma\omega) \to \mathcal{L}(\Gamma, \sigma)$ s.t.

$$\tilde{M}_\varphi(\Lambda_{\sigma\omega}(g)) = \varphi(g)\Lambda_\sigma(g), g \in \Gamma \ .$$

By restriction, get a c.p. map $M_\varphi : C_r^*(\Gamma, \sigma\omega) \to C_r^*(\Gamma, \sigma)$. Moreover,

$$\|\tilde{M}_\varphi\| = \|M_\varphi\| = \varphi(e) = \|\eta\|_{\mathcal{H}}^2$$

In particular, if φ is p.-d. (i.e., $\omega = 1$) then get a c.p. map $M_\varphi \in B(C_r^*(\Gamma, \sigma))$.

Byproduct: elementary proof of

Theorem (Zeller-Meier, 1968): Γ amenable, $\omega \in Z^2(\Gamma, \mathbb{T})$. Then $C^*(\Gamma, \omega) \simeq C_r^*(\Gamma, \omega)$, canonically (it also holds for certain twisted crossed products)

Question: is the converse true ?

Remark (about twisted Haagerup Lemma): Likewise, get twisted analogues of results about ω-projective uniformly bounded representation of Γ on a Hilbert space by invertible operators

Twisted Multipliers

Consider $\varphi : \Gamma \to \mathbb{C}$, $\sigma, \omega \in Z^2(\Gamma, \mathbb{T})$

Let $M_\varphi : \mathbb{C}(\Gamma, \omega) \to \mathbb{C}(\Gamma, \sigma)$ be the linear map given by

$$M_\varphi(\pi_\omega(f)) = \pi_\sigma(\varphi f), \ f \in \mathbb{C}\Gamma.$$

Definition: (1) φ is a (σ, ω)-*multiplier* if M_φ is bounded w.r.t. the operator norms on $\mathbb{C}(\Gamma, \omega)$ and $\mathbb{C}(\Gamma, \sigma)$.
In that case, denote by M_φ the (unique) extension of M_φ to an element in $B(C_r^*(\Gamma, \omega), C_r^*(\Gamma, \sigma))$. Note that M_φ is then the unique element in this space satisfying

$$M_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), \quad g \in \Gamma.$$

(2) $MA(\Gamma, \sigma, \omega) :=$ the set of all (σ, ω)-multipliers φ on Γ

(a subspace of $\ell^\infty(G)$ containing $\mathcal{K}\Gamma$ and a Banach space equipped with the norm $\|\varphi\|_{MA} = \|M_\varphi\|$)

(3) $MA(\Gamma, \sigma) := MA(\Gamma, \sigma, \sigma)$, $MA(\Gamma) := MA(\Gamma, 1)$.

Then $B(\Gamma, \omega) \subseteq MA(\Gamma, \sigma, \sigma\omega)$ and $\|\varphi\|_{MA} \le \|\varphi\|$, $\forall \varphi \in B(\Gamma, \omega)$; if $\omega = 1$ then $B(\Gamma) \subset MA(\Gamma, \sigma)$; if $\varphi \in P(\Gamma)$, then $\|\varphi\|_{MA} = \|\varphi\| = \varphi(e)$.

Remark: Γ amenable $\Rightarrow B(\Gamma, \omega) = MA(\Gamma, 1, \omega)$

(but $B(\Gamma) = MA(\Gamma, \sigma)$? True in the case $\sigma = 1$)

$\ell^2(\Gamma) \subset MA(\Gamma, \sigma, \omega)$; for $\varphi \in \ell^2(\Gamma)$, $\|\varphi\|_{MA} \leq \|\varphi\|_2$.
Moreover, for every $x \in C_r^*(\Gamma, \omega)$,

$$M_\varphi(x) = \sum_{g \in \Gamma} \varphi(g) \widehat{x}(g) \Lambda_\sigma(g)$$

(sum convergent in operator norm)

Thm (twisted Haagerup-de Cannière, case $\sigma = \omega$): a function $\varphi : \Gamma \to \mathbb{C}$ is in $MA(\Gamma, \sigma)$ iff there exists a (unique) normal operator $\tilde{M}_\varphi : \mathcal{L}(\Gamma, \sigma) \to \mathcal{L}(\Gamma, \sigma)$ s.t.

$$\tilde{M}_\varphi(\Lambda_\sigma(g) = \varphi(g)\Lambda_\sigma(g), \; g \in \Gamma$$

In this case, $\|M_\varphi\| = \|\tilde{M}_\varphi\|$ and $(MA(\Gamma, \sigma), |||\cdot|||)$ is a Banach space w.r.t. the norm $|||\varphi||| := \|M_\varphi\|$.

Rem. the predual $\mathcal{L}(\Gamma, \sigma)_*$ identifies with a certain space $A(\Gamma, \sigma)$ of functions on Γ (corresponding to the Fourier algebra in the untwisted setting). $MA(\Gamma, \sigma)$ multiplies $A(\Gamma, \sigma)$ into itself.

Completely bounded multipliers

Def. $M_0A(\Gamma,\sigma) = \{\varphi \in MA(\Gamma,\sigma) \mid M_\varphi$ c.b. map$\}$, equipped with the norm $\|\varphi\|_{cb} = \|M_\varphi\|_{cb}$.

$M_0A(\Gamma) := M_0A(\Gamma, 1)$

The existence of cb-multipliers is well-known in the untwisted setting:

$$\ell^2(\Gamma) \subset B(\Gamma) = \mathrm{span}P(\Gamma) \subset M_0A(\Gamma) \subset MA(\Gamma)$$

Also, for $\varphi \in B(\Gamma)$, $|||\varphi||| \le \|\varphi\|_{cb} \le \|\varphi\|$ (the latter is the norm of φ as an element $C^*(\Gamma)^*$)

For $\varphi \in P(\Gamma)$, $|||\varphi||| = \|\varphi\|_{cb} = \|\varphi\| = \varphi(e)$.
For $\varphi \in \ell^2(\Gamma)$, $\|\varphi\|_{cb} \le \|\varphi\|_2$.

Rem. in case $\sigma = 1$, Γ is amenable iff $B(\Gamma) = MA(\Gamma)$, iff $B(\Gamma) = M_0A(\Gamma)$ (Bozejko, Nebbia)

Rem. c.b. multipliers closely related to (Herz-)Schur multipliers.

Prop. $M_0A(\Gamma,\sigma) = M_0A(\Gamma)$
(and the cb-norm of $\varphi \in M_0A(\Gamma,\sigma)$ is indep. of σ)

Question: $MA(\Gamma,\sigma) = MA(\Gamma)$?

For $\varphi \in MA(\Gamma, \sigma)$, $x \in C_r^*(\Gamma, \sigma)$ it holds $\widehat{M_\varphi(x)} = \varphi \widehat{x}$. That is, the Fourier series of $M_\varphi(x)$ is

$$\sum_{g \in \Gamma} \varphi(g) \widehat{x}(g) \Lambda_\sigma(g)$$

(not necessarily convergent in operator norm; but it does, if $\varphi \in \ell^2(\Gamma)$, since then $\varphi \widehat{x} \in \ell^1(\Gamma)$).

Define $MCF(\Gamma, \sigma) =$

$$\{\varphi : \Gamma \to \mathbb{C} \mid \sum_{g \in \Gamma} \varphi(g) \widehat{x}(g) \Lambda_\sigma(g) \text{ norm-convergent, } x \in C_r^*(\Gamma, \sigma$$

Prop. $\ell^2(\Gamma) \subset MCF(\Gamma, \sigma) \subset MA(\Gamma, \sigma)$.
Moreover,

$$MCF(\Gamma, \sigma) = \{\varphi \in MA(\Gamma, \sigma) \mid M_\varphi(C_r^*(\Gamma, \sigma)) \subset CF(\Gamma, \sigma)\}$$

If $\varphi \in MCF(\Gamma, \sigma)$ then, for all $x \in C_r^*(\Gamma, \sigma)$,

$$\sum_g \varphi(g) \widehat{x}(g) \Lambda_\sigma(g) = M_\varphi(x)$$

(norm convergent sums)

Rem. Other elements in $MCF(\Gamma, \sigma)$ can be obtained by considering suitable κ-decaying subspaces (building over RD property).

Summation Processes

Def: A net (φ_α) in MA(Γ,σ) is an approximate multiplier unit whenever $M_{\varphi_\alpha} \to$ id in the SOT on $B(C_r^*(\Gamma,\sigma))$. Such a net (φ_α) is bounded if (M_{φ_α}) is uniformly bounded (that is, $\sup_\alpha \|M_{\varphi_\alpha}\| < \infty$)

Remark: a net (φ_α) in MA(Γ,σ) is a bounded approximate multiplier unit iff $\varphi \to 1$ pointwise on Γ and (φ_α) is bounded.

Example: a net of normalized p.-d. functions on Γ converging pointwise to 1 is a bounded approximate multiplier unit.
(Such nets always exist if Γ has the Haagerup property)

Definition: Let (φ_α) be a net of complex functions on Γ. Say that $C_r^*(\Gamma,\sigma)$ has the *Summation Property (S.P.) w.r.t.* (φ_α), or, equivalently, that (φ_α) is a *Fourier summing net for* (Γ,σ), if (φ_α) is an approximate multiplier unit s.t. $\varphi_\alpha \in MCF(\Gamma,\sigma)$ for all α.

In this case, the series $\sum_{g\in G} \varphi_\alpha(g)\widehat{x}(g)\Lambda_\sigma(g)$ is convergent in operator norm for all α, and we have

$$\sum_{g\in G} \varphi_\alpha(g)\widehat{x}(g)\Lambda_\sigma(g) \to_\alpha x$$

for all $x \in C_r^*(\Gamma,\sigma)$ (convergence in operator norm).

Question: given (Γ,σ), is it always possible to find a Fourier summing net?

Classical Examples:

1) Fejér summation theorem can be restated by saying that $C_r^*(\mathbb{Z}, 1)$ has the S.P. w.r.t. $(f_n) \subset \mathcal{K}\mathbb{Z}$.

2) For each $0 < r < 1$, let $\psi_r(k) = r^{|k|},\ k \in \mathbb{Z}$. Then the Abel-Poisson summation theorem corresponds to the fact that $C_r^*(\mathbb{Z}, 1)$ has the S.P. w.r.t. $(\psi_r)_{0<r<1} \subset \ell^2(\mathbb{Z})$ (letting $r \to 1$).

In order to produce Fourier summing nets, look for $(\varphi_\alpha) \subset \ell^2(\Gamma)$ or satisfying a suitable decay property.

Definition: Say that (Γ, σ) has
(1) the Fejér property (resp. the Abel-Poisson property) if there exists a net (φ_α) in $\mathbb{C}\Gamma$ (resp. in $\ell^2(\Gamma)$) such that $C_r^*(\Gamma, \sigma)$ has the S.P. w.r.t. (φ_α);
(2) the bounded Fejér property (resp. the bounded Abel-Poisson property) if the net (φ_α) can be chosen to be bounded;
(3) metric Fejér property (resp. the metric Abel-Poisson property), if this net can be chosen to satisfy $\sup_\alpha \|M_{\varphi_\alpha}\| = 1$.

If (Γ, σ) metric Fejér then $C_r^*(\Gamma, \sigma)$ has the M.A.P.

Haagerup actually showed that $\mathbb{F}_n$ has the metric Fejér property

Examples of groups with the metric Fejér property include $\mathbb{Z}$ and, more generally all amenable groups (see below), but also $\mathbb{F}_n, 0 < n < \infty$ (Haagerup).

Problem: when does (Γ, σ) have the metric Fejér/Abel-Poisson property?

In particular, if Γ has the Haagerup property does (Γ, σ) have the metric Fejér property?

(So far, all known examples of groups with the metric Fejér property have the Haagerup property)

Corollary (cf. Zeller-Meier, 1968) Let Γ be amenable. Then (Γ, σ) has the metric Fejér property.

Indeed, if (φ_α) is any net of normalized positive definite functions in $\mathbb{C}\Gamma$ converging to 1 pointwise on Γ, $C_r^*(\Gamma, \sigma)$ has the S.P. w.r.t. (φ_α) and $\|M_{\varphi_\alpha}\| = 1$ for all α.

Any net (φ_α) as in the last Corollary gives a net (M_{φ_α}) of finite rank completely positive maps on $C_r^*(\Gamma, \sigma)$ converging to the identity in the SOT. Hence we recover: if Γ is amenable, then $C_r^*(\Gamma, \sigma)$ has the so-called C.P.A.P., a property which is known to be equivalent to nuclearity. Actually,

Proposition: TFAE:

1) Γ is amenable.

2) $C^*(\Gamma, \sigma)$ is nuclear.

3) $C_r^*(\Gamma, \sigma)$ is nuclear.

4) $\mathcal{L}(\Gamma, \sigma)$ is injective.

Example (About Følner and Fejér): suppose that Γ is amenable, and pick a a Følner net (F_α) for Γ. Set

$$\varphi_\alpha(g) = \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|}, \quad g \in \Gamma.$$

(E.g., When $\Gamma = \mathbb{Z}$, one may choose $F_n = \{0, 1, \ldots, n-1\}$, which gives $\varphi_n(g) = 1 - \frac{|g|}{n}$ if $|g| \leq n-1$ and 0 otherwise, that is, we get the Fejér functions on $\mathbb{Z}$ used in the classical Fejér summation theorem.)

We have $\varphi_\alpha(g) = (\xi_\alpha, \lambda(g)\xi_\alpha)$, with $\xi_\alpha = |F_\alpha|^{-1/2}\chi_{F_\alpha}$ and $\mathrm{supp}(\varphi_\alpha) = F_\alpha \cdot F_\alpha^{-1}$. Hence the following analogue of Fejér's summation theorem holds : for all $x \in C_r^*(G, \sigma)$,

$$\sum_{g \in F_\alpha \cdot F_\alpha^{-1}} \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|} \, \widehat{x}(g)\Lambda_\sigma(g) \to_\alpha x$$

(in operator norm).

Example: the following analogue of the Abel-Poisson summation theorem holds: for all $x \in C_r^*(\mathbb{Z}^N, \sigma_\Theta)$ we have

$$\sum_{m \in \mathbb{Z}^N} r^{|m|_j^k} \, \widehat{x}(m)\Lambda_\sigma(m) \to_{r \to 1^-} x$$

(in operator norm), $j = 1, 2$, $1 \leq k \leq j$.

Other analogues of the Abel Poisson summation theorem hold for finitely generated free groups and for Coxeter groups (replacing the ℓ^2-condition with suitable decaying conditions).

Indeed, in both cases, the natural word-length L_S is a Haagerup function and the group has polynomial H-growth w.r.t. L_S so point (1) of the result below applies:

Theorem: Γ countable group with Haagerup function L.
(1) Assume that Γ has polynomial H-growth (w.r.t. L). Then there exists $q \in \mathbb{N}$ s.t. $((1+tL)^{-q})_{t\to 0+}$ is a bounded Fourier summing net for (Γ, σ).
(2) Assume that Γ has subexponential H-growth (w.r.t. L). Then $\{r^L\}_{r\to 1-}$ is a bounded Fourier summing net for (Γ, σ).

A generalized Haagerup theorem

Theorem: Suppose that the following three conditions hold:

(1) There exists an approximate multiplier unit (φ_α) in $MA(\Gamma,\sigma)$ satisfying $\|M_{\varphi_\alpha}\| = 1$ for all α.

(2) For each α there exists a function $\kappa_\alpha : \Gamma \to [1,+\infty)$ such that (Γ,σ) is κ_α-decaying.

(3) We have $\varphi_\alpha\kappa_\alpha \in c_0(\Gamma)$ for all α.

Then (Γ,σ) has the metric Fejér property.

Corollary: Γ countable with subexponential H-growth w.r.t. a Haagerup function, then (Γ,σ) has the metric Fejér property.

Corollary: If there exists a Haagerup length function L on Γ s.t. Γ has the R.D. property w.r.t. L, then $C_r^*(\Gamma,\sigma)$ has the M.A.P.
(cf. Jolissaint-Valette, 1991; Brodzki-Niblo 2004, case $\sigma = 1$)

Def. A group Γ is weakly amenable if there exists a net $\{\varphi_i\}$ of finitely supported functions converging pointwise to 1, s.t. $\sup_i \|M_{\varphi_i}\| < +\infty$.

Cowling Conjecture: Any countable group Γ with the Haagerup property is weakly amenable with CH constant 1, i.e. there exists a net $\{\varphi_\alpha\} \subset \mathcal{K}\Gamma$, converging pointwise to 1, s.t. $\sup_\alpha \|\varphi_\alpha\|_{cb} = 1$. (True in a number of cases)

The latter groups are said to have the complete metric approximation property (CMAP)

Converse fails: de Cornulier, Stalder and Valette (2008) construct certain wreath products which are a-T-menable but do not have the CMAP

Ozawa (2007): all Gromov hyperbolic groups are weakly amenable, hence they have the bounded Fejér property.

However, not all groups have the bounded Fejér property:

Haagerup: $H := \mathbb{Z}^2 \rtimes \mathsf{SL}(2,\mathbb{Z})$ (does not have the bounded Fejér property and thus) is not weakly amenable.

Still, H has the Fejér property, as it has property AP (Haagerup and Kraus), stronger than Fejér.

Γ weakly amenable $\Rightarrow$ Γ has AP $\Rightarrow$ Γ exact

(opposite implications false)

Lafforgue, de la Salle (2011): $SL(3,\mathbb{Z})$ (linear, thus exact but) fails to have AP. Not known if it has Fejér property.

C*-dynamical systems and covariant representations

We consider a unital, discrete, twisted C^*-dynamical system

$$\Sigma = (A, G, \alpha, \sigma)\,.$$

So A is a C*-algebra with 1, G is a discrete group, and the maps

$\alpha : G \to \mathsf{Aut}(A)$ (= the group of $*$-automorphisms of A)

$\sigma : G \times G \to \mathcal{U}(A)$ (= the unitary group of A)

satisfy

$$\alpha_g\,\alpha_h = \mathsf{Ad}(\sigma(g,h))\,\alpha_{gh}$$

$$\sigma(g,h)\,\sigma(gh,k) = \alpha_g(\sigma(h,k))\,\sigma(g,hk)$$

$$\sigma(g,e) = \sigma(e,g) = 1$$

where e denotes the unit of G.

(sometimes also called a cocycle G-action)

All the C-algebras we consider are assumed to be unital, and homomorphisms between these are assumed to be unit- and *-preserving.*

A *covariant homomorphism* of Σ is a pair (ϕ, u), where ϕ is a homomorphism from A into a C^*-algebra C and u is a map from G into $\mathcal{U}(C)$, satisfying

$$u(g)\, u(h) = \phi(\sigma(g,h))\, u(gh)$$

and the covariance relation

$$\phi(\alpha_g(a)) = u(g)\, \phi(a)\, u(g)^*\,.$$

If X is a (right) Hilbert C*-module (e.g. a Hilbert space) and $C = \mathcal{L}(X)$ (the adjointable operators on X), then (ϕ, u) is called a *covariant representation* of Σ on X.

The vector space $C_c(\Sigma)$ of functions from G into A with finite support becomes a (unital) $*$-algebra when equipped with the operations

$$(f_1 * f_2)\,(h) = \sum_{g\in G} f_1(g)\,\alpha_g(f_2(g^{-1}h))\,\sigma(g,g^{-1}h),$$

$$f^*(h) = \sigma(h,h^{-1})^*\,\alpha_h(f(h^{-1}))^*\ .$$

The *full* C^**-algebra* $C^*(\Sigma)$ is generated by (a copy of) $C_c(\Sigma)$

and has the universal property that whenever $(\phi,u) : A \to C$ is a covariant homomorphism of Σ, then there exists a unique homomorphism $\phi \times u : C^*(\Sigma) \to C$ such that

$$(\phi \times u)(f) = \sum_{g\in G} \phi(f(g))\,u(g), \quad f \in C_c(\Sigma)\,.$$

As is well known, any representation π of A on some Hilbert B-module Y induces a covariant representation $(\tilde{\pi}, \tilde{\lambda}_\pi)$ of Σ on the B-module

$$Y^G = \{\xi : G \to Y \mid \sum_{g \in G} \langle \xi(g), \xi(g) \rangle \quad \text{is norm-convergent in } B\}$$

Considering A itself as a (right) Hilbert A-module in the obvious way and letting $\ell : A \to \mathcal{L}(A)$ denote left multiplication, we may form *the regular covariant representation of* Σ

$$\Lambda = \tilde{\ell} \times \tilde{\lambda}_\ell : C^*(\Sigma) \to \mathcal{L}(A^G)$$

The *reduced C^*-algebra of* Σ is then be defined as the C^*-subalgebra of $\mathcal{L}(A^G)$ given by

$$C_r^*(\Sigma) = \Lambda(C^*(\Sigma)) \,.$$

It is convenient to consider also the Hilbert A-module $A^{\Sigma} =$

$\{\xi : G \to A \mid \sum_{g \in G} \alpha_g^{-1}(\xi(g)^* \, \xi(g))$ is norm-convergent in $A\}$

where

$$\langle \xi, \eta \rangle_\alpha = \sum_{g \in G} \alpha_g^{-1}(\xi(g)^* \eta(g)) \,,$$

$$(\xi \cdot a)(g) = \xi(g) \, \alpha_g(a) \,.$$

A covariant representation $(\ell_\Sigma \,, \lambda_\Sigma)$ of Σ on A^Σ is given by

$[\ell_\Sigma(a) \, \xi](h) = a \, \xi(h)$

$[\lambda_\Sigma(g) \, \xi \,](h) = \alpha_g(\xi(g^{-1}h)) \, \sigma(g, g^{-1}h) \,.$

Identifying A with $\ell_\Sigma(A)$ (acting on A^Σ) gives

$$\Lambda_\Sigma(f) = \sum_{g \in G} f(g) \, \lambda_\Sigma(g), \quad f \in C_c(\Sigma) \,.$$

As $\Lambda_\Sigma = \ell_\Sigma \times \lambda_\Sigma$ is unitarily equivalent to Λ, we have

$$C_r^*(\Sigma) \simeq \Lambda_\Sigma(C^*(\Sigma)) \,.$$

Let $\xi_0 \in A^\Sigma$ be defined as $1 \odot \delta_e$, i.e.

$$\xi_0(e) = 1\,, \quad \xi_0(g) = 0 \;\; g \neq e\,.$$

Then

$$\Lambda_\Sigma(f)\,\xi_0 = f\,, \quad f \in C_c(\Sigma)\,.$$

Hence, setting $\widehat{x} = x\,\xi_0 \in A^\Sigma$ for $x \in C_r^*(\Sigma)$, we have

$$\widehat{\Lambda_\Sigma(f)} = f\,, \quad f \in C_c(\Sigma)\,.$$

The (injective) linear map $x \to \widehat{x}$ from $C_r^*(\Sigma)$ into A^Σ is called the *Fourier transform*. The canonical conditional expectation E from $C_r^*(\Sigma)$ onto A is simply given by

$$E(x) = \widehat{x}(e)\,, \quad x \in C_r^*(\Sigma)\,,$$

and we have

$$\widehat{x}(g) = E(x\,\lambda_\Sigma(g)^*)\,.$$

Note: $\qquad \Lambda_\Sigma(f)\,\xi = f * \xi\,, \quad f \in C_c(G, A),\, \xi \in A^\Sigma\,.$

(where $*$ = twisted convolution).

Especially: if $\xi_0 = 1_A \odot \delta_e \in A^\Sigma$, then $\Lambda_\Sigma(f)\,\xi_0 = f$.

Some useful properties of Fourier coefficients:

$\widehat{\Lambda_\Sigma(f)} = f$, $f \in C_c(\Sigma)$.; in particular,
$\widehat{\ell_\Sigma(a)} = a \odot \delta_e$, $\widehat{\lambda_\Sigma(g)} = 1 \odot \delta_g$.

$x\xi = \widehat{x} * \xi$, $x \in C_r^*(\Sigma)$, $\xi \in C_c(G, A) \subset A^\Sigma$

For all $x \in C_r^*(\Sigma)$,

$$\|\widehat{x}\|_\infty \leq \|\widehat{x}\|_\alpha \leq \|x\|$$

where
$\|\widehat{x}\|_\infty := \sup_g \|\widehat{x}(g)\|$
$\|\widehat{x}\|_\alpha = \| \sum_g \alpha_g^{-1}(\widehat{x}(g)^* \widehat{x}(g))\|^{1/2}$

(cf. the Riemann-Lebesgue Lemma)

$\widehat{xy} = \widehat{x} * \widehat{y}$, for all $x \in C_r^*(\Sigma)$, $y \in \Lambda(C_c(\Sigma))$

$\widehat{x^*} = \widehat{x}^*$, i.e.

$$\widehat{x^*}(g) = \sigma(g, g^{-1})\alpha_g(\widehat{x}(g^{-1}))^*$$

Moreover,

$E(\Lambda_\Sigma(f)) = f(e)$, $f \in C_c(\Sigma)$; in particular, $E(\ell_\Sigma(a)) = a$ and $E(\lambda_\Sigma(g)) = 0$ for $g \neq e$

$E(x\lambda_\Sigma(g)^*) = \widehat{x}(g)$, $g \in G$

$E(x^*x) = \langle \widehat{x}, \widehat{x} \rangle_\alpha$, for any $x \in C_r^*(\Sigma)$

$E(\lambda_\Sigma(g)x\lambda_\Sigma(g)^*) = \alpha_g(E(x))$ (equivariance), $g \in G, x \in C_r^*(\Sigma)$

Lemma (Rørdam-Sierakowski, 2010) A C^*-algebra, G a countable discrete group acting on A by automorphisms. For each $g \in G$ set $x_g = E(xu_g^*)$. Then, for all $x \in A \rtimes_r G$,

$$E(xx^*) = \sum_g x_g (x_g)^* \ , \quad E(x^* x) = \sum_g \alpha_g (x^*_{g^{-1}} x_{g^{-1}})$$

and the sums are norm-convergent.

(an application of Dini's theorem to obtain norm-convergence from convergence on states)

Given $x \in C_r^*(\Sigma)$, its (formal) Fourier series is defined as

$$\sum_{g \in G} \widehat{x}(g) \Lambda_\Sigma(g)$$

Remark: there are left/right Fourier series

$CF(\Sigma) =$

$$\{x \in C_r^*(\Sigma) \mid \sum_{g \in G} \widehat{x}(g) \Lambda_\Sigma(g) \text{ convergent w.r.t.} \| \cdot \| \}$$

Look for some nice decay subspaces of A^Σ, e.g. $\ell^1(G, A)$...

Theorem: Let $L : G \to [0, +\infty)$ be a proper function.

If G has polynomial H-growth (w.r.t. L) then there exists some $s > 0$ s.t. the Fourier series of $x \in C_r^*(\Sigma)$ converges to x in operator norm whenever

$$\sum_{g \in G} \|\widehat{x}(g)\|^2 (1 + L(g))^s < +\infty$$

If G has subexponential H-growth (w.r.t. L) then there exists some $s > 0$ s.t. the Fourier series of $x \in C_r^*(\Sigma)$ converges to x in operator norm whenever there exists some $t > 0$ s..t.

$$\sum_{g \in G} \|\widehat{x}(g)\|^2 e^{tL(g)} < +\infty$$

Remark: the proof requires $\ell^2_\kappa(G,A)$-decay, where

$$\ell^2_\kappa(G,A) = \{\xi : G \to A \mid \sum_g \|\xi(g)\|^2 \kappa^2(g) < \infty\} \subset A^\Sigma$$

is the weighted version of $\ell^2(G,A)$ and κ is scalar-valued)

However, in general, it is not clear that $\widehat{x} \in \ell^2_\kappa(G,A)$. It would be better to deal with the weaker A^Σ_κ-decay, where $A^\Sigma_\kappa =$

$$\{\xi : G \to A \mid \sum_g \alpha_g^{-1}(\xi(g)^* \xi(g))\kappa^2(g) \text{ norm-convergent in } A\}$$

Problem: find conditions on Σ implying A_κ^Σ-decay, i.e.

$$\| \sum_{g \in G} f(g)\lambda_\Sigma(g)\| \leq C\|f\kappa\|_\alpha \,, \quad f \in C_c(G, A)$$

for some $C > 0$ and $\kappa : G \to [1, +\infty)$.

Remark: we can do this when A is commutative and α is trivial. In this case, $C_r^*(\Sigma)$ is a reduced central twisted transformation group algebra.

Equivariant representations of $\Sigma = (A, G, \alpha, \sigma)$

An *equivariant representation* of Σ on a Hilbert A-module X is a pair (ρ, v) where

- $\rho : A \to \mathcal{L}(X)$ is a representation of A on X,
- $v : G \to \mathcal{I}(\mathcal{X})$ (= the group of all $\mathbb{C}$-linear, invertible, bounded maps from X into itself),

satisfying

(i) $\rho(\alpha_g(a)) = v(g)\,\rho(a)\,v(g)^{-1}\,, \quad g \in G\,,\, a \in A$

(ii) $v(g)\,v(h) = \mathrm{ad}_\rho(\sigma(g,h))\,v(gh)\,, \quad g, h \in G$

(iii) $\alpha_g(\langle x\,,\, x'\rangle) = \langle v(g)x\,,\, v(g)x'\rangle\,, \quad g \in G\,, x, x' \in X$

(iv) $v(g)(x\cdot a) = (v(g)x)\cdot\alpha_g(a)\,,\, g \in G,\, x \in X,\, a \in A.$

In (ii) above, $\mathrm{ad}_\rho(\sigma(g,h)) \in \mathcal{I}(\mathcal{X})$ is defined by

$$\mathrm{ad}_\rho(\sigma(g,h))\,x = (\rho(\sigma(g,h))\,x)\cdot\sigma(g,h)^*\,.$$

Some examples

- $\ell : A \to \mathcal{L}(A)$ and $\alpha : G \to \mathrm{Aut}(A) \subset \mathcal{I}(\mathcal{A})$ give the *trivial equivariant representation* (ℓ, α) of Σ.

- Let (ρ, v) be an equivariant representation of Σ on X. The *induced* equivariant representation $(\check{\rho}, \check{v})$ on X^G is given by

  $$(\check{\rho}(a)\,\xi)(h) = \rho(a)\,\xi(h), \quad (\check{v}(g)\xi)(h) = v(g)\,\xi(g^{-1}h)\,.$$

- More generally, if w is a unitary representation of G on some Hilbert space $\mathcal{H}$, then $(\rho\otimes\iota, v\otimes w)$ is an equivariant representation of Σ on $X \otimes \mathcal{H}$.

- $(\check{\ell}, \check{\alpha})$ is called the *regular equivariant representation* of Σ. It acts on A^G via

  $$[\check{\ell}(a)\,\xi](h) = a\,\xi(h)$$

  $$[\check{\alpha}(g)\,\xi](h) = \alpha_g(\xi(g^{-1}h))\,.$$

Tensoring an equivariant rep. with a covariant rep.

Consider

- an equivariant rep. (ρ, v) of Σ on a Hilbert A-module X ,

- a covariant rep. (π, u) of Σ on a Hilbert B-module Y .

One may then form the covariant representation $(\rho\dot{\otimes}\pi\, ,\ v\dot{\otimes}u)$ of Σ on the internal tensor product Hilbert B-module $X \otimes_\pi Y$.

It acts on simple tensors in $X \otimes_\pi Y$ as follows:

$$[(\rho\dot{\otimes}\pi)(a)]\,(x\dot{\otimes}\, y) = \rho(a)x \,\dot{\otimes}\, y$$

$$[(v\dot{\otimes}u)(g)]\,(x\dot{\otimes}\, y) = v(g)x \,\dot{\otimes}\, u(g)y\,.$$

Some properties

Let (ρ, v) and (π, u) be as before.

- $(\ell \dot{\otimes} \pi) \times (\alpha \dot{\otimes} u) \simeq \pi \times u$

- Fell absorption principle (I):

$$(\rho \dot{\otimes} \ell_{\Sigma}) \times (v \dot{\otimes} \lambda_{\Sigma}) \simeq \tilde{\rho} \times \tilde{\lambda}_{\rho} \,.$$

- Fell absorption principle (II):

  Let $\pi' : \mathcal{L}(X^G) \to \mathcal{L}(X^G \otimes_\pi Y)$ denote the amplification map, so

$$\breve{\rho} \dot{\otimes} \pi = \pi' \circ \breve{\rho} : A \to \mathcal{L}(X^G \otimes_\pi Y) \,.$$

  Then

$$(\breve{\rho} \dot{\otimes} \pi) \times (\breve{v} \dot{\otimes} u) \simeq \pi' \circ (\tilde{\rho} \times \tilde{\lambda}_{\rho}) \,.$$

Equivariant representations and multipliers

Let $T : G \times A \to A$ be a map that is linear in the second variable.

For each $g \in G$, let $T_g : A \to A$ be the linear map given by

$$T_g(a) = T(g, a)\,, \quad a \in A\,.$$

For each $f \in C_c(\Sigma)$, define $T \cdot f \in C_c(\Sigma)$ by

$$(T \cdot f)(g) = T_g(f(g))\,, \quad g \in G\,.$$

We say that T is a (reduced) *multiplier* of Σ whenever there exists a bounded linear map $M_T : C_r^*(\Sigma) \to C_r^*(\Sigma)$ such that

$$M_T(\Lambda_\Sigma(f)) = \Lambda_\Sigma(T \cdot f)\,, \quad \text{that is,}$$

$$M_T\Big(\sum_{g \in G} f(g)\, \lambda_\Sigma(g)\Big) = \sum_{g \in G} T_g(f(g))\, \lambda_\Sigma(g)$$

for all $f \in C_c(\Sigma)$. We then set $|||T||| = \|M_T\|$.

For any $x \in C_r^*(\Sigma)$, $\widehat{M_T(x)}(g) = T_g(\widehat{x}(g))$, $g \in G$.

Set $MA(\Sigma) =$ all (reduced) multipliers of Σ and let $M_0A(\Sigma)$ denote the subspace of $MA(\Sigma)$ consisting of completely bounded multipliers.

Example: consider $\varphi : G \to \mathbb{C}$ and set $T^\varphi(g,a) = \varphi(g)a$. If $T^\varphi \in MA(\Sigma)$ then $\varphi \in MA(G)$. Also, $T^\varphi \in M_0A(\Sigma)$ iff $\varphi \in M_0A(G)$ and, in this case, $|||T^\varphi||| \leq \|M_{T^\varphi}\|_{cb} \leq \|M_\varphi\|_{cb}$.

Theorem 1 *Let (ρ, v) be an equivariant representation of Σ on a Hilbert A-module X and let $x, y \in X$. Define $T : G \times A \to A$ by*

$$T(g,a) = \langle x\,,\, \rho(a)\, v(g)\, y\rangle\,.$$

Then $T \in M_0A(\Sigma)$, with $|||T||| \leq \|M_T\|_{\mathrm{cb}} \leq \|x\|\,\|y\|$.

Moreover, if $x = y$, then M_T is completely positive and

$$|||T||| = \|M_T\|_{\mathrm{cb}} = \|x\|^2\,.$$

The proof relies on the Fell absorption principle (I). With the help of this result one may construct Fejér-like summation processes for Fourier series of elements in $C_r^*(\Sigma)$ in many cases.

Remarks

Let T be as in the previous theorem.

- Set $Z = \{z \in X \mid \rho(a)\, z = z \cdot a,\ \ a \in A\}$. Then we have

  $$T(g,a) = \langle x, v(g)y\rangle\, a \quad \text{if } y \in Z,$$

  while

  $$T(g,a) = a\,\langle x, v(g)y\rangle \quad \text{if } x \in Z.$$

- Let w be a unitary representation of G on a Hilbert space $\mathcal{H}$ and $\xi, \eta \in \mathcal{H}$.

  Considering $(\rho, v) = (\ell \otimes \iota, \alpha \otimes w)$ on $X = A \otimes \mathcal{H}$ and $x = 1 \otimes \xi$, $y = 1 \otimes \eta$ gives

  $$T(g,a) = \langle 1\,,\, a\,\alpha_g(1)\rangle\, \langle \xi, w(g)\eta\rangle = \langle \xi, w(g)\eta\rangle\, a\,.$$

  and we recover a result of U. Haagerup.

Coefficients functions of equivariant representations of Σ may also be shown to give (completely bounded) **full** multipliers of Σ. The sets of all these functions may be organized as an algebra, analogous to the Fourier-Stieltjes algebra of a group, which we are presently studying.

Using the Fell absorption principle (II), we can prove:

Theorem 2 *Let* (ρ, v) *be an equivariant representation of* Σ *on a Hilbert* A*-module* X *and let* $\xi, \eta \in X^G$*. Define* $\check{T} : G \times A \to A$ *by*

$$\check{T}(g, a) = \langle\, \xi,\, \check{\rho}(a)\, \check{v}(g)\, \eta \,\rangle\,.$$

Then $\check{T}$ *is a completely bounded* **rf***-multiplier of* Σ*, that is, there exists a completely bounded map* $\Phi_T : C_r^*(\Sigma) \to C^*(\Sigma)$ *such that*

$$\Phi_T(\Lambda_\Sigma(f)) = T \cdot f$$

for all $f \in C_c(\Sigma)$*, satisfying* $\|\Phi_T\|_{cb} \leq \|\xi\|\,\|\eta\|\,.$

The weak approximation property

Σ is said to have the *weak approximation property* if there exist an equivariant representation (ρ, v) of Σ on some A-module X and nets $\{\xi_i\}$, $\{\eta_i\}$ in X^G, both having finite support, satisfying

- there exists some $M > 0$ s.t. $\|\xi_i\| \cdot \|\eta_i\| \leq M$ for all i;

- for all $g \in G$ and $a \in A$ we have

$$\lim_i \|\langle \xi_i \, , \, \check{\rho}(a)\check{v}(g)\eta_i \rangle - a\| = 0 \, .$$

Note that if (ρ, v) can be chosen as (ℓ, α), one recovers *Exel's approximation property* for Σ. This property is known to imply that Σ is regular, that is, $\Lambda : C^*(\Sigma) \to C^*_r(\Sigma)$ is an isomorphism.

From our previous theorem, one can deduce that

Theorem 3 *If Σ has the weak approximation property, then Σ is regular (i.e., $\Lambda : C^*(\Sigma) \to C^*_r(\Sigma)$ is an isomorphism). Moreover, $C^*(\Sigma) \simeq C^*_r(\Sigma)$ is nuclear iff A is nuclear.*

Theorem: Assume that A is abelian. TFAE:

(a) Σ has the approximation property

(b) α is amenable in the sense of Delaroche

(c) Σ has the central weak approximation property

If σ is scalar-valued, they are also equivalent to

(d) Σ has the weak approximation property

Rem. Exel-Ng (2002) showed equivalence of (a) and (b) in the untwisted case.

A permanence result

Assume

- Σ *has the weak approximation property*

- B *is a C*-subalgebra of* A *containing the unit of* A

- B *is invariant under each* α_g, $g \in G$

- σ *takes values in* $\mathcal{U}(\mathcal{B})$

- *there exists an equivariant conditinal expectation* $E : A \to B$.

Then $(B, G, \alpha_{|B}, \sigma)$ *has the weak approximation property.*

Example. *Let* G *be an exact group,* H *be an amenable subgroup of* G, $\sigma \in Z^2(G, \mathbb{T})$. *Let* α *denote the action of* G *on* $A = \ell^\infty(G)$ *by left translations. Then it is well-known that* α *is amenable, so that* Σ *has the approximation property. Let* β *denote the natural action of* G *on* $B = \ell^\infty(G/H)$. *Then* (B, β, G, σ) *has the weak approximation property.*

Summation processes for Fourier series in crossed products

$$MCF(\Sigma) = \{T \in MA(\Sigma) \mid M_T(x) \in CF(\Sigma), \forall x \in C_r^*(\Sigma)\}$$

These are all the maps $T : G \times A \to A$, linear in the second variable, s.t.

$$\sum_{g \in G} T_g(\widehat{x}(g))\lambda_\Sigma(g)$$

converges w.r.t. $\|\cdot\|$, for all $x \in C_r^*(\Sigma)$

Def. (1) A Fourier summing net for Σ is a net $\{T_i\} \subset MCF(\Sigma)$ s.t.

$$\lim_i \|M_{T_i}(x) - x\| = 0 \ , \quad \forall x \in C_r^*(\Sigma)$$

(2) A bounded Fourier summing net satisfies, in addition,

$$\sup_i |||T_i||| < \infty$$

Question: for which Σ there exists a Fourier summing net?
(unclear even for trivial A and σ)

A Fourier summing net $\{T_i\}$ for Σ preserves the invariant ideals of A if, for every α-invariant ideal $J \subset A$,

$$(T_i)_g(J) \subset J \ , \quad \forall i,\ g \in G$$

Useful notion to study the ideal structure of $C_r^*(\Sigma)$, cf.

- Zeller-Meier (for G amenable)

- Exel (for Σ with the approximation property)

Prop. Assume that there exists a Fourier summing net $\{T_i\}$ for Σ that preserves the invariant ideals of A. Then Σ is exact and $C_r^*(\Sigma)$ is exact iff A is exact.

For an invariant ideal $J \subset A$, set

$\langle J \rangle :=$ the ideal generated by J in $C_r^*(\Sigma)$

$\tilde{J} := \{x \in C_r^*(\Sigma) \mid \widehat{x}(g) \in J, \forall g \in G\}$
(Here, $\widehat{x}(g) = E(x\lambda(g)^*)$).

Then $E(\langle J \rangle) = J$ and $\langle J \rangle \subset \tilde{J}$.

Def. (Sierakowski 2010) Σ is exact whenever

$$\langle J \rangle = \tilde{J}$$

for all invariant ideals J of A.

Let $\mathcal{J}$ be an ideal of $C_r^*(\Sigma)$. Then $J := \overline{E(\mathcal{J})}$ is an invariant ideal of A s.t. $\mathcal{J} \subset \tilde{J}$. Hence, if Σ is exact, $\mathcal{J} \subset \langle J \rangle$.

An ideal $\mathcal{J}$ of $C_r^*(\Sigma)$ is

- induced, whenever it is generated by an invariant ideal of A;

- E-invariant, whenever

$$E(\mathcal{J}) \subset \mathcal{J}$$

(equivalently, $E(\mathcal{J}) = \mathcal{J} \cap A$).
In this case, $E(\mathcal{J})$ is a (closed) invariant ideal of A;

- δ_Σ-invariant, whenever

$$\delta_\Sigma(\mathcal{J}) \subset \mathcal{J} \otimes C_r^*(G)$$

where $\delta_\Sigma : C_r^*(\Sigma) \to C_r^*(\Sigma) \otimes C_r^*(G)$ denotes the (reduced) dual coaction of G on Σ

Remark: induced $\Rightarrow$ δ_Σ-invariant $\Rightarrow$ E-invariant

Prop. (cf. Exel, 2000) Assume that G is exact or that there exists a Fourier summing net for Σ that preserves the invariant ideals of A.
Then an ideal of $C_r^*(\Sigma)$ is E-invariant iff it is δ_Σ-invariant, iff it is induced.
Hence, the map $J \mapsto \langle J \rangle$ is a bijection between the set of all invariant ideals of A and the set of all E-invariant ideals of $C_r^*(\Sigma)$.

Rem. Indeed, under the given assumption, if $\mathcal{J}$ is E-invariant one has

$$\mathcal{J} = \langle E(\mathcal{J}) \rangle$$

Def. (1) Σ has the Fejér property if there exists a Fourier summing net $\{T_i\}$ for Σ s.t. each T_i has finite G-support;
(2) Σ has the bounded Fejér property if, in addition, such net can be chosen to be bounded

Remark. Zeller-Meier showed that Σ has the bounded Fejér property whenever G is amenable and σ is central. (In Davidsons' book can find a short proof for $G = \mathbb{Z}$ and σ trivial)

Theorem (E. Bédos, RC 2014): Assume that G is amenable. Then Σ has the bounded Fejér property.

What about non-amenable groups?

Theorem (E. Bédos, RC 2014): Assume that G is weakly amenable, or that Σ has the weak approximation property. Then Σ has the bounded Fejér property.

On maximal ideals in twisted crossed products

Def. A discrete group Γ is called C^*-simple if its reduced C^*-algebra $C_r^*(\Gamma)$ is simple.

Many classes of groups are known to be C^*-simple!

Theorem (de la Harpe-Skandalis, 1986) If Γ is a Powers group acting on the C^*-algebra A and A is Γ-simple then $A \rtimes_r \Gamma$ is simple.

(later generalized to weak Powers groups and twisted actions)

More generally, what can be said about the ideal structure of $C_r^*(\Gamma)$ and of $A \rtimes_r \Gamma$?

Theorem (E. Bedos, RC, 2014): Σ discrete twisted C^*-dynamical system. If Σ is exact and has property (DP) then there are one-to-one correspondences between:

- the set of maximal ideals of $C_r^*(\Sigma)$ and the set of maximal invariant ideals of A;
- the set of all tracial states of $C_r^*(\Sigma)$ and the set of invariant tracial states of A

Rem. Σ is exact whenever Γ is exact. Also, Σ is exact whenever there exists a Fourier summing net for Σ preserving the invariant ideals of A. The latter condition is satisfied when Σ has Exel approximation property, e.g. when the associated action of Γ on the center $Z(A)$ is amenable (as in Brown-Ozawa book).

Def. Σ has property DP if

$$0 \in \overline{\mathrm{co}}\{vyv^* \mid v \in \mathcal{U}(C_r^*(\Sigma))\}$$

for every $y = y^* \in C_r^*(\Sigma)$ with $E(y) = 0$

Remark. If $C_r^*(\Sigma)$ as the Dixmier property then it has (DP).

Let $(\mathcal{P})$ be the class of groups consisting of Promislow PH groups and groups satisfying the property (P_{com}) of Bekka-Cowling-de la Harpe (1994).

Theorem (E. Bédos, RC): If Γ belongs to $(\mathcal{P})$ the associated system Σ has (strong) DP.

Remark: Powers groups are weak Powers group, which in turn are PH groups.

Def. a group Γ has Powers property if, for any finite subset $F \subset \Gamma \setminus 1$ and for any integer $N \geq 1$, there exists a partition $\Gamma = C \cup D$ and elements $g_1, \ldots, g_N \in \Gamma$ s.t. $fC \cap C = \emptyset$ for all $f \in F$ and $g_i D \cap g_j D = \emptyset$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$.

A group Γ is a weak Powers group if the above holds only for every finite subset F in a nontrivial conjugacy class of Γ.

A group Γ has property (P_{com}) if, for any non-empty subset $G \subset \Gamma \setminus \{e\}$, there exists $N \geq 1$, $g_0 \in \Gamma$ and subsets $U, D_1, \ldots, D_N$ of Γ, s.t.

(i) $\Gamma \setminus U \subset D_1 \cup \ldots \cup D_N$

(ii) $gU \cap U = \emptyset$ for all $g \in F$

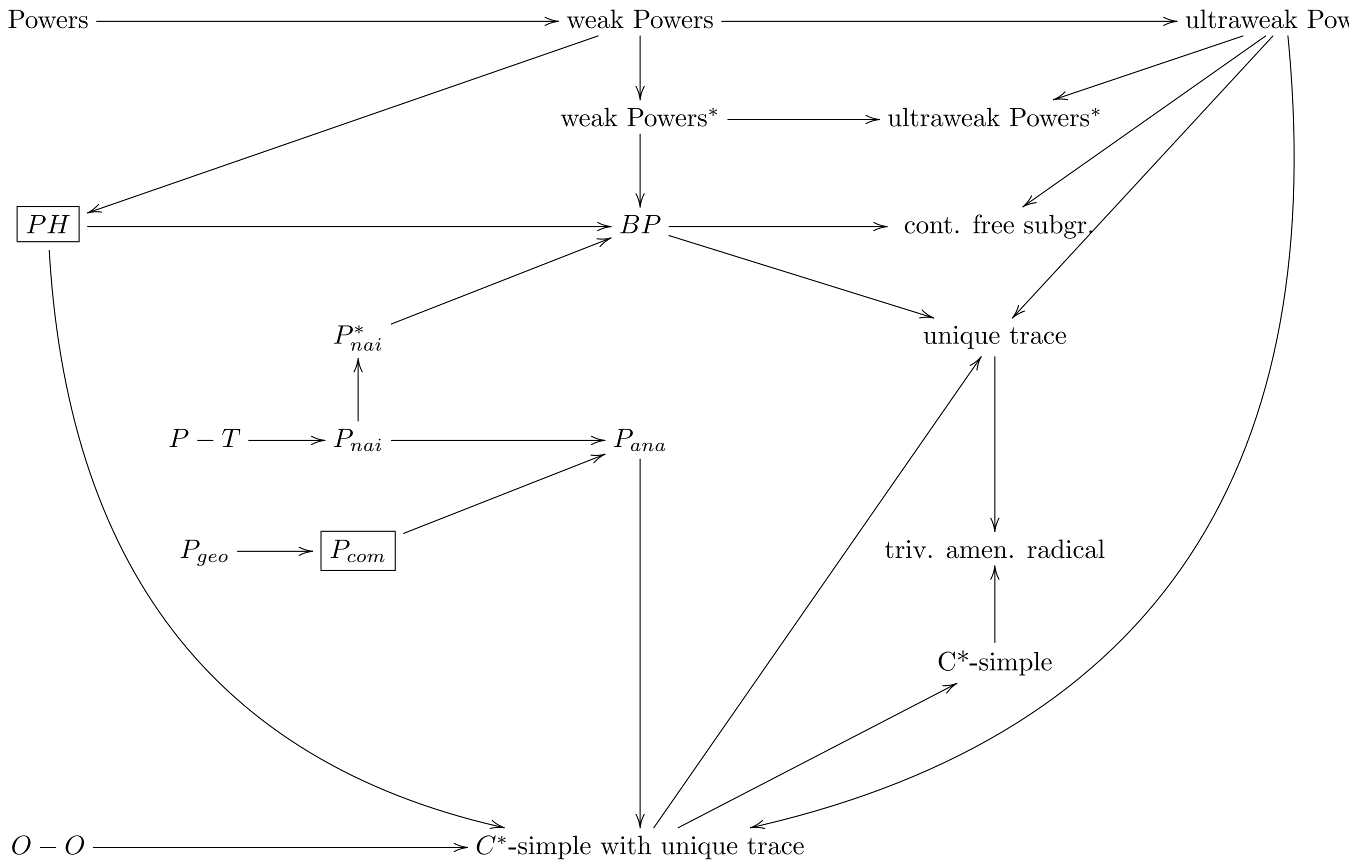
(iii) $g_0^{-j} D_k \cap D_k = \emptyset$ for all $j \geq 1$ and $k = 1, \ldots, N$

Applications/examples:

a description of the unique simple quotient of the twisted Roe algebra $C_r^*(\ell^\infty(\Gamma), \Gamma, \mathsf{lt}, \sigma)$ for Γ exact in $(\mathcal{P})$ and scalar σ

an explicit description of maximal ideals/simple quotients of $C_r^*(\Gamma)$ for $\Gamma = \mathbb{Z}^3 \rtimes \mathsf{SL}(3, \mathbb{Z})$;

a description of the ideals of $C_r^*(\Gamma)$, where Γ is an exact group s.t. $G := \Gamma/Z \in (\mathcal{P})$; e.g. $\Gamma = \mathsf{SL}(2n, \mathbb{Z})$, $n \geq 1$, P_n (pure braid group on n strands), $\mathbb{B}_3$.


Powers
weak Powers
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weak Powers*
ultraweak Powers*
PH
BP
cont. free subgr.
P*nai
unique trace
P − T
Pnai
Pana
Pgeo
Pcom
triv. amen. radical
C*-simple
O − O
C*-simple with unique trace