Noncommutative Potential Theory 4

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Themes.

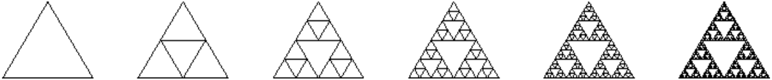
- Sierpinski Gasket K
- Harmonic structures and Dirichlet forms on K
- Dirac operators and Spectral Triples on K
- Volume functional dimensional spectrum
- Energy functional dimensional spectrum
- Dirichlet form as a residue
- Fredholm modules and pairing with K-theory
- de Rham cohomology and Hodge Harmonic decomposition on K
- Potentials of locally exact 1-forms on the projective covering of K

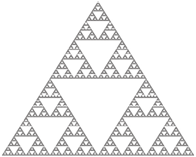
References.

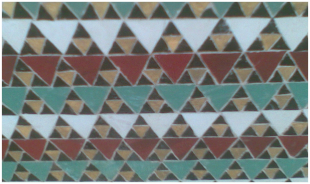
- Kigami, J. *Analysis on fractals* Cambridge Tracts in Mathematics, 143 (2001)
- Guido, D.; Isola, T. *Dimensions and singular traces for spectral triples, with applications to fractals* J. Funct. Anal. 203 (2003)
- A. Jonsson. *A trace theorem for the Dirichlet form on the Sierpinski gasket*, Math. Z., **250** (2005), 599–609.
- Christensen, E.; Ivan, C.; Lapidus, M. L. *Dirac operators and spectral triples for some fractal sets built on curves* Adv. Math. 217 (2008)
- Cipriani, F.; Sauvageot, J.-L. *Fredholm modules on P.C.F. self-similar fractals and their conformal geometry* Comm. Math. Phys. 286 (2009)
- Christensen, E.; Ivan, C.; Schrohe, E. *Spectral triples and the geometry of fractals* J. Noncommut. Geom. 6 (2012)
- Cipriani, F.; Guido, D.; Isola, T.; Sauvageot, J.-L. *Integrals and potentials of differential 1-forms on the Sierpinski gasket* Adv. Math. 239 (2013)
- Cipriani, F.; Guido, D.; Isola, T.; Sauvageot, J.-L. *Spectral triples for the Sierpinski gasket* J. Funct. Anal. 266 (2014)

Sierpinski gasket $K \subset \mathbb{C}$: self-similar compact set

- vertices of an equilateral triangle $\{p_1, p_2, p_3\}$
- contractions $F_i : \mathbb{C} \to \mathbb{C} \qquad F_i(z) := (z + p_i)/2$
- $K \subset \mathbb{C}$ is uniquely determined by $K = F_1(K) \cup F_2(K) \cup F_3(K)$ as the fixed point of a contraction of the Hausdorff distance on compact subsets of $\mathbb{C}$:







Duomo di Amalfi: Chiostro, sec. XIII

Geometric and analytic features of the Sierpinski gasket

- K is not a manifold
- the group of homeomorphisms is finite
- K is not semi-locally simply connected hence
- K does not admit a universal cover
- K-theory group

$$\mathbb{K}^1(K) = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$$

- K-homology group

$$\mathbb{K}_1(K) = \prod_{i\in\mathbb{N}} \mathbb{Z}$$

- Volume and Energy are distributed singularly on K
- existence of localized eigenfunctions

Self-similar volume measures and their Hausdorff dimensions

The natural measures on K are the self-similar ones

- for some fixed $(\alpha_1, \alpha_2, \alpha_3) \in (0,1)^3$ such that $\sum_{i=1}^3 \alpha_i = 1$

$$\int_K f\, d\mu = \sum_{i=1}^{3} \alpha_i \int_K (f \circ F_i)\, d\mu \qquad f \in C(K)$$

- when $\alpha_i = \frac{1}{3}$ for all $i = 1, 2, 3$ then μ is the normalized Hausdorff measure on K associated to the restriction of the Euclidean metric: its dimension is $d = \frac{\ln 3}{\ln 2}$

Harmonic structure

- word spaces: $\sum_0 := \emptyset\,, \quad \sum_m := \{1,2,3\}^m\,, \quad \sum := \bigcup_{m\geq 0} \sum_m$
- length of a word $\sigma \in \sum_m$: $\quad |\sigma| := m$
- iterated contractions: $\quad F_\sigma := F_{i_{|\sigma|}} \circ \ldots F_{i_1}$ if $\sigma = (i_1, \ldots, i_{|\sigma|})$
- vertices sets: $\quad V_\emptyset := \{p_1, p_2, p_3\}\,, \quad V_m := \bigcup_{|\sigma|=m} F_\sigma(V_0)$
- consider the quadratic form $\mathcal{E}_0 : C(V_0) \to [0,+\infty)$ of the Laplacian on V_0

$$\mathcal{E}_0[a] := (a(p_1) - a(p_2))^2 + (a(p_2) - a(p_3))^2 + (a(p_3) - a(p_1))^2$$

Theorem. (Kigami 1986)

The sequence of quadratic forms on $C(V_m)$ defined by

$$\mathcal{E}^m[a] := \sum_{|\sigma|=m} \left(\frac{5}{3}\right)^m \mathcal{E}_0[a \circ F_\sigma] \qquad a \in C(V_m)$$

is an harmonic structure in the sense that

$$\mathcal{E}^m[a] = \min\{\mathcal{E}^{m+1}[b] : b|_{V_m} = a\} \qquad a \in C(V_m)\,.$$

Dirichlet form

Theorem 1. (Kigami 1986)

The quadratic form $\mathcal{E} : C(K) \to [0, +\infty]$ *defined by*

$$\mathcal{E}[a] := \lim_{m \to +\infty} \mathcal{E}^m[a|_{V_m}] \qquad a \in C(K)$$

is a Dirichlet form, i.e. a l.s.c. quadratic form such that

$$\mathcal{E}[a \wedge 1] \leq \mathcal{E}[a] \qquad a \in C(K),$$

which is self-similar in the sense that

$$\mathcal{E}[a] = \frac{5}{3} \sum_{i=1}^{3} \mathcal{E}[a \circ F_i] \qquad a \in C(K).$$

It is closed in $L^2(K, \mu)$ *and the associated self-adjoint operator* H_μ *has discrete spectrum with spectral exponent* $d_S = \frac{\ln 9}{\ln 5/3}$:

$$\sharp\{\text{eigenvalue of } H_\mu \leq \lambda\} \asymp \lambda^{d_S/2} \qquad \lambda \to +\infty.$$

Volume and Energy measures

Theorem. (Kigami-Lapidus 2001)

The self-similar volume measure μ *with weights* $\alpha_i = 1/3$ *can be re-constructed as*

$$\int_K f\,d\mu = Trace_{Dix}(M_f \circ H_\mu^{-d_S/2}) = Res_{s=d_S} Trace(M_f \circ H_\mu^{-s/2})$$

Theorem. (M. Hino 2007)

The energy measures on K *defined by*

$$\int_K b\,d\Gamma(a) := \mathcal{E}(a|ab) - \frac{1}{2}\mathcal{E}(a^2|b) \qquad a, b \in \mathcal{F}$$

are singular with respect to all the self-similar measures on K.

Derivation and Fredholm module on K

Theorem. (FC-Sauvageot 2003)

There exists a symmetric derivation $(\mathcal{F}, \partial, \mathcal{H}, \mathcal{J})$, defined on the Dirichlet algebra $\mathcal{F}$, with values in a symmetric $C(K)$-monomodule $(\mathcal{H}, \mathcal{J})$ such that

$$\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}} \qquad a \in \mathcal{F}\,.$$

In other words, $(\mathcal{F}, \partial, \mathcal{H}, \mathcal{J})$ is a differential square root of H_μ:

$$H_\mu = \partial^* \circ \partial\,.$$

Theorem. (FC-Sauvageot 2009)

Let $P \in Proj(\mathcal{H})$ the projection onto the image $Im\partial$ of the derivation above

$$P\mathcal{H} = \overline{Im\partial}$$

and let $F := P - P^\perp$ the associated phase operator.
Then $(F, \mathcal{H})$ is a 2-summable (ungraded) Fredholm module over $C(K)$ and

$$Trace(|[F, a]|^2) \le const.\, \mathcal{E}[a] \qquad a \in \mathcal{F}\,.$$

Quasi-circles

We will need to consider on the 1-torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ structures of quasi-circle associated to the following Dirichlet forms and their associated Spectral Triples for any $\alpha \in (0, 1)$.

Lemma. *Fractional Dirichlet forms on a circle* (CGIS 2010)

Consider the Dirichlet form on $L^2(\mathbb{T})$ defined on the fractional Sobolev space

$$\mathcal{E}_\alpha[a] := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|a(z) - a(w)|^2}{|z - w|^{2\alpha+1}} \, dzdw \qquad \mathcal{F}_\alpha := \{a \in L^2(\mathbb{T}) : \mathcal{E}_\alpha[a] < +\infty\} \, .$$

Then $\mathcal{H}_\alpha := L^2(\mathbb{T} \times \mathbb{T})$ is a symmetric Hilbert $C(K)$-bimodule w.r.t. actions and involutions given by

$$(a\xi)(z, w) := a(z)\xi(z, w) \, , \quad (\xi a)(z, w) := \xi(z, w)a(w) \, , \quad (\mathcal{J}\xi)(z, w) := \overline{\xi(w, z)} \, .$$

The derivation $\partial_\alpha : \mathcal{F}_\alpha \to \mathcal{H}_\alpha$ associated to $\mathcal{E}_\alpha$ is given by

$$\partial_\alpha(a)(z, w) := \frac{a(z) - a(w)}{|z - w|^{\alpha+1/2}} \, .$$

Proposition. *Spectral Triples on a circle* (CGIS 2010)

Consider on the Hilbert space $\mathcal{K}_\alpha := L^2(\mathbb{T}\times\mathbb{T}) \bigoplus L^2(\mathbb{T})$, *the left* $C(\mathbb{T})$*-module structure resulting from the sum of those of* $L^2(\mathbb{T}\times\mathbb{T})$ *and* $L^2(\mathbb{T})$ *and the operator*

$$D_\alpha := \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix}.$$

Then $\mathcal{A}_\alpha := \{a \in C(\mathbb{T}) : \sup_{z\in\mathbb{T}} \int_{\mathbb{T}} \frac{|a(z)-a(w)|^2}{|z-w|^{2\alpha+1}} < +\infty\}$ *is a uniformly dense subalgebra of* $C(\mathbb{T})$ *and* $(\mathcal{A}_\alpha, D_\alpha, \mathcal{K}_\alpha)$ *is a densely defined Spectral Triple on* $C(\mathbb{T})$.

Dirac operators on K.

Identifying isometrically the main lacuna $\ell_\emptyset$ of the gasket with the circle $\mathbb{T}$, consider the Dirac operator $(C(K), D_\emptyset, \mathcal{K}_\emptyset)$ where

- $\mathcal{K}_\emptyset := L^2(\ell_\emptyset \times \ell_\emptyset) \oplus L^2(\ell_\emptyset)$
- $D_\emptyset := D_\alpha$
- the action of $C(K)$ is given by restriction $\pi_\emptyset(a)b := a|_{\ell_\emptyset}$.

Fix $c > 1$ and for $\sigma \in \sum$ consider the Dirac operators $(C(K), \pi_\sigma, D_\sigma, \mathcal{K}_\sigma)$ where

- $\mathcal{K}_\sigma := \mathcal{K}_\emptyset$
- $D_\sigma := c^{|\sigma|} D_\alpha$
- the action of $C(K)$ is given by contraction/restriction $\pi_\sigma(a)b := (a \circ F_\sigma)|_{\ell_\emptyset}\, b$.

Finally, consider the Dirac operator $(C(K), \pi, D, \mathcal{K})$ where

- $\mathcal{K} := \oplus_{\sigma \in \sum} \mathcal{K}_\sigma$
- $\pi := \oplus_{\sigma \in \sum} \pi_\sigma$
- $D := \oplus_{\sigma \in \sum} D_\sigma$

Notice that $dim\, Ker\, D = +\infty$ and that D^{-1} will be defined to be zero on $Ker\, D$.

Volume functionals and their Spectral dimensions

Theorem. (CGIS 2010)

The zeta function $\mathcal{Z}_D$ of the Dirac operator $(C(K), D, \mathcal{K})$, i.e. the meromorphic extension of the function $\mathbb{C} \ni s \mapsto Trace(|D|^{-s})$ is given by

$$\mathcal{Z}_D(s) = \frac{4}{1-3c^{-s}} z(\alpha s)$$

where z denotes the Riemann zeta function. The dimensional spectrum is given by

$$\mathcal{S}_{dim} = \{\frac{1}{\alpha}\} \cup \{\frac{\ln 3}{\ln c}\Big(1 + \frac{2\pi i}{\ln 3}k\Big) : k \in \mathbb{Z}\}$$

and the abscissa of convergence is $d_D = \max(\alpha^{-1}, \frac{\ln 3}{\ln c})$. When $1 < c < 3^\alpha$ there is a simple pole in $d_D = \frac{\ln 3}{\ln c}$ and the residue of the meromorphic extension of $\mathbb{C} \ni s \mapsto Trace(f|D|^{-s})$ gives the Hausdorff measure of dimension $d = \frac{\ln 3}{\ln 2}$

$$Trace_{Dix}(f|D|^{-s}) = Res_{s=d_D} Trace(f|D|^{-s}) = \frac{4d}{\ln 3}\frac{z(d)}{(2\pi)^d}\int_K f\, d\mu\,.$$

Notice the complex dimensions and the independence of the residue Hausdorff measure upon $c > 1$.

Spectral Triples and Connes metrics on the Sierpinski gasket

Theorem. (CGIS 2010)

$(C(K), D, \mathcal{K})$ is a Spectral Triple for any $1 < c \leq 2$. In particular we have the commutator estimate for Lipschiz functions with respect to the Euclidean metric

$$\|[D,a]\| \leq \frac{(1/2)^{(1-\alpha)}}{(1-\alpha)^{1/2}} \sup_{\sigma\in\sum} \left(\frac{c}{2}\right)^{\sigma} \|a\|_{Lip(l_\sigma)} \qquad a \in Lip(K)\,.$$

For $c = 2$ the Connes distance is bi-Lipschitz w.r.t. the geodesic distance on K induced by the Euclidean metric

$$(1-\alpha)^{1/2} 2^{(1-\alpha)} d_{geo}(x,y) \leq d_D(x,y) \leq (1+\alpha)^{-1/2} 2^{(3/2)} 3^{-\alpha} d_{geo}(x,y)\,.$$

Energy functionals and their Spectral dimensions

By the Spectral Triple it is possible to recover, in addition to dimension, volume measure and metric, also the energy form of K

Theorem. (CGIS 2010)

Consider the Spectral Triple $(C(K), D, \mathcal{K})$ *for* $\alpha \leq \alpha_0 := \frac{\ln 5}{\ln 4} - \frac{1}{2}\left(\frac{\ln 3}{\ln 2} - 1\right) \sim 0,87$ *and assume* $a \in \mathcal{F}$*. Then the abscissa of convergence of*

$$\mathbb{C} \ni s \mapsto Trace(|[D, a]|^2 |D|^{-s})$$

is $\delta_D := \max(\alpha^{-1}, 2 - \frac{\ln 5/3}{\ln c})$.

- *If* $\delta_D > \alpha^{-1}$ *then* $s = \delta_D$ *is a simple pole and the residue is proportional to the Dirichlet form*

$$Res_{s=\delta_D} Trace(|[D, a]|^2 |D|^{-s}) = const. \mathcal{E}[a] \qquad a \in \mathcal{F};$$

- *if* $\delta_D = \alpha^{-1}$ *then* $s = \delta_D$ *is a pole of order 2 but its residue of order 2 is still proportional to the Dirichlet form.*

Pairing with K-Theory

Trying to construct a Fredholm module from the Dirac operator one may consider

- $F := D|D|^{-1}$ to be the phase of the Dirac operator.
- $\varepsilon_0 := \oplus_\sigma \varepsilon_0^\sigma$ where $\varepsilon_0^\sigma := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ on $\mathcal{K}_\sigma = L^2(\ell_\emptyset \times \ell_\emptyset) \oplus L^2(\ell_\emptyset)$
- $\varepsilon_1 := \oplus_\sigma \varepsilon_1^\sigma$ where $\varepsilon_1^\sigma := \begin{pmatrix} 0 & -iV_\sigma \\ -iV_\sigma^* & 0 \end{pmatrix}$ on $\mathcal{K}_\sigma = L^2(\ell_\emptyset \times \ell_\emptyset) \oplus L^2(\ell_\emptyset)$ where $V_\sigma : L^2(l_\emptyset) \to L^2(l_\emptyset \times l_\emptyset)$ is the partial isometry between $Im\,(\partial_\sigma^* \circ \partial_\sigma) \subset L^2(\ell_\emptyset)$ and $Im\,(\partial_\sigma \circ \partial_\sigma^*) \subset L^2(\ell_\emptyset \times \ell_\emptyset)$.

However $(C(K), F, \mathcal{K}, \varepsilon_0, \varepsilon_1)$ is not a 1-graded Fredholm module because

- $[F, \pi(a)] \in \mathbb{K}(\mathcal{K})$ for all $a \in C(K)$
- $F = F^*$, $\varepsilon_0 F + F\varepsilon_0 = 0$, but $dim\,Ker\,F = +\infty$ and $(F^2 - I) \notin \mathbb{K}(\mathcal{K})$
- ε_0 is unitary but ε_1 is a partial isometry only
- $\varepsilon_0\varepsilon_1 + \varepsilon_1\varepsilon_0 = 0$, $\varepsilon_0^2 = I$ but $\varepsilon_1^2 \neq -I$
- $[\varepsilon_0, \pi(a)] = 0$ but $[\varepsilon_1, \pi(a)] \notin \mathbb{K}(\mathcal{K})$ in general

Theorem. (CGIS 2010)

Consider the operator $F_1 := i\varepsilon_1 F$ *and the projection* $P := \frac{F_1 + F_1^2}{2}$. *Then*

- $[F_1, \pi(a)]$ *is compact for all* $a \in C(K)$
- PaP *is a Fredholm operator for all invertible* $u \in C(K)$
- *a nontrivial homomorphism on* $\mathbb{K}^1(K)$ *is determined by*

$$u \mapsto Index\, PuP$$

- *the nontrivial element of* $\mathbb{K}_1(K) = \prod_{\sigma \in \sum} \mathbb{Z}$ *determined by* F_1 *is* $(1, 1, \dots)$:

$$Index\, Pu_\sigma P = +1 \qquad \sigma \in \sum$$

where u_σ *is the unitary associated to the lacuna* ℓ_σ.

Proofs are based on

- the harmonic structure defining the Dirichlet form $\mathcal{E}$
- a result

  A. Jonsson: A trace theorem for the Dirichlet form on the Sierpinski gasket, Math. Z. 250 (2005), no. 3, 599-609

  by which the restriction to a lacuna ℓ_σ of a finite energy function $a \in \mathcal{F}$ belongs to the fractional Sobolev space $\mathcal{F}_\alpha$, if $\alpha < \alpha_0$.

These properties allow to develop

- integration of 1-forms $\omega \in \mathcal{H}$ in the tangent module associated to $(\mathcal{E}, \mathcal{F})$ along paths $\gamma \subset K$
- characterize exact and locally exact 1-forms in terms of their periods around lacunas
- define a de Rham cohomology of 1-forms and prove a separating duality with the Cech homology group.
- represent the integrals of 1-forms around cycles in K by potentials, i.e. affine functions on the abelian universal projective covering L of K

Elementary 1-forms and elementary paths

Let $(\mathcal{E},\mathcal{F})$ be the standard Dirichlet form on K and $(\mathcal{F},\partial,\mathcal{H})$ the associated derivation with values in the tangent module $\mathcal{H}$ (whose elements are understood as square integrable 1-forms).

- *exact* 1-forms $\Omega^1_e(K) := \mathrm{Im}\,\partial = \{\partial a \in \mathcal{H} : a \in \mathcal{F}\}$
- *elementary* 1-forms $\Omega^1(K) := \{\sum_{i=1}^n a_i\partial b_i \in \mathcal{H} : a_i, b_i \in \mathcal{F}\}$
- *locally exact* 1-forms $\Omega^1_{\mathrm{loc}}(K)$ are those $\omega \in \Omega^1(K)$ which admit a *primitive* $U_A \in \mathcal{F}$ on a suitable neighborhood $A \subset K$ of a any point of K

$$\omega = \partial U_A \qquad \text{on} \quad A \subset K\,.$$

- *n-exact* 1-forms are those $\omega \in \Omega^1(K)$ which are exact on any cell C_σ with $|\sigma|$=n
- an *elementary path* $\gamma \subset K$ is a path which is a finite union of edges of K. If it is contained in a cell $C_\sigma \subset K$ we say that γ has depth $|\sigma| \in \mathbb{N}$.

Lemma. (CGIS 2010)

A 1-form is locally exact if and only if it is n-exact form some $n \in \mathbb{N}$.

Universal 1-forms and line integrals

Let $(\mathcal{E}, \mathcal{F})$ be the standard Dirichlet form on K and $(\mathcal{F}, \partial, \mathcal{H})$ the associated derivation with values in the tangent module $\mathcal{H}$ (whose elements are understood as square integrable 1-forms).

- Let $\Omega^1(\mathcal{F})$ be the $\mathcal{F}$-bimodule of universal 1-forms on the Dirichlet algebra $\mathcal{F}$
- pairing with edges $f, g \in \mathcal{F}$

$$(f \otimes g)(e) := f(e_+)g(e_-) \quad dg(e) := g(e_+) - g(e_-)$$

$$(f(dg))(e) = f(e_+)dg(e) \quad ((dg)f) = f(e_-)dg(e)$$

The integral of the 1-form $\omega = \sum_{\in} If_i dg_i$ along the elementary path $\gamma \subset K$ is defined by

$$I_n(\gamma)(\omega) := \sum_{e \in E_n(\gamma)} \omega(e) \qquad \int_\gamma \omega := \lim_{n \to \infty} I_n(\gamma)(\omega)$$

where $E_n(\gamma)$ denotes the set of edges of γ.

Integration of elementary 1-forms along elementary paths

Theorem. (CGIS 2010)

- *The integral of an elementary 1-forms is well defined.*
- *The energy seminorm on* $\Omega^1(\mathcal{F})$ *specified by* $fdg \mapsto (f\partial g|f\partial g)_{\mathcal{H}}$ *and the collection of seminorms given integrals along edges have same kernel* Ω^1_0
- *consequently the quotient* $\Omega^1(K) := \Omega^1(\mathcal{F})/\Omega^1_0$ *can be identified with a subspace of the tangent module* $\mathcal{H}$ *and on it integrals make sense*

Proof is a based on an embedding $\mathcal{F} \hookrightarrow H^\alpha(\gamma)$ of the Dirichlet space into a fractional Sobolev space with $\alpha > 1/2$.

Definition. *Potentials* (CGIS 2010)

- A continuous function $U \in C(A)$ defined on subset $A \subset K$ is a *local potential on* A of a 1-form $\omega \in \Omega^1(K)$ if for all elementary path $\gamma \subset A$

$$\int_\gamma \omega = U(\gamma(1)) - U(\gamma(0))\,.$$

Proposition. (CGIS 2010)

- *Local potentials of a 1-form on $A \subset K$ have finite energy on A*
- *the class of potentials $U \in C(K)$ of an exact 1-form $\omega \in \Omega^1_e(K)$ coincides with the class of its primitives $U \in \mathcal{F}$ on K*

$$\int_\gamma \omega = U(\gamma(1)) - U(\gamma(0)) \qquad \Leftrightarrow \qquad \omega = \partial U\,.$$

A system of locally exact 1-forms associated with lacunas

Cells and lacunas

- The lacuna $\ell_\emptyset \subset K$ (depth 0) is defined as the boundary of the triangle $K \setminus (F_1(K) \cup F_2(K) \cup F_3(K))$
- the lacunas ℓ_σ (depth n) are defined as its successive contraction: $\ell_\sigma := F_\sigma(\ell_\emptyset)$

Theorem. (CGIS 2010)

- *For any $\sigma \in \sum$ there exists only one $(|\sigma| + 1)$-exact 1-form $dz_\sigma \in \Omega^1_{\text{loc}}(K)$ having minimal norm $\|\omega\|_{\mathcal{H}}$ among those with unit period on the lacuna ℓ_σ*

$$\int_{\ell_\sigma} \omega = 1$$

- $\exists\, \omega_\sigma \in \text{Lin}\,\{dz_\tau : |\tau| \leq |\sigma|\}$ *such that for all elementary paths $\gamma \subset K$*

$$\int_\gamma \omega_\sigma = \text{ winding number of } \gamma \text{ around } \ell_\sigma \,.$$

Theorem. *(a la de Rham)* (CGIS 2010)

- *Any elementary 1-form* $\omega \in \Omega^1(K)$ *can be uniquely decomposed as*

$$\omega = \partial U + \sum_{\sigma} k_\sigma dz_\sigma$$

the convergence taking place in a topology where integrals are continuous

- *the coefficients* k_σ *only depend upon the periods* $\int_{\ell_\sigma} \omega$ *around lacunas*
- *the form is locally exact* $\omega \in \Omega^1_{\text{loc}}(K)$ *iff the* k_σ*'s are eventually zero*
- *the form is exact* $\omega \in \Omega^1_e(K)$ *iff all the periods, hence all* k_σ*'s, are zero*

Cells and lacunas

There exist elementary forms which are not closed: $f_0 \partial f_1$.

Theorem. *(a la Hodge)* (CGIS 2010)

- *The forms* $\{dz_\sigma : \sigma \in \sum\} \subset \mathcal{H}$ *are pairwise orthogonal*
- *Any 1-form* $\omega \in \mathcal{H}$ *can be orthogonally decomposed as*

$$\omega = \partial U + \sum_\sigma k_\sigma dz_\sigma$$

- *the forms* $\{dz_\sigma . \sigma \in \sum\}$ *are co-closed*

$$\partial^*(dz_\sigma) = 0 \qquad \text{so that} \qquad \partial^*\omega = \partial^*\partial U = \Delta U$$

Theorem. *(a de Rham cohomology Theorem)* (CGIS 2010)

The pairing $\langle\gamma,\omega\rangle = \int_\gamma \omega$ *between elementary paths* $\gamma \subset K$ *and locally exact forms* $\omega \in \Omega^1_{\mathrm{loc}}(K)$ *gives rise to a nondegenerate pairing between the Cech homology group* $H_1(K,\mathbb{R})$ *and the cohomology group*

$$H^1_{\mathrm{dR}}(K) := \frac{\Omega^1_{\mathrm{loc}}(K)}{\Omega^1_e(K)}$$

in which the classes of the co-closed forms $\{dz_\sigma : \sigma \in \sum\} \subset \mathcal{H}$ *is a system of generators parameterized by lacunas.*

Coverings

Let $T \subset \mathbb{C}$ the closed triangle envelope of the gasket K and $T_n := \bigcup_{|\sigma|=n} F_\sigma(T)$. The embedding $i_n : K \to T_n$ give rise to regular coverings $(\widetilde{K}_n, K)$ whose group of deck transformations can identified with $\pi_1(T_n)$. Its abelianization Γ_n can be identified with the homology group $H_1(T_n)$. Defining $\widetilde{L}_n := \widetilde{K}_n/[\Gamma_n, \Gamma_n]$ one has that $(\widetilde{L}_n, K)$ form coverings whose group of deck transformations can be identified with Γ_n.

Theorem. *()* (CGIS 2010)

The family $\{(\widetilde{L}_n, K) : n \geq 0\}$ *is projective and the group* Γ *of deck transformations of its limit* $\widetilde{L} = \lim_{\leftarrow} \widetilde{L}_n$ *can be identified with the Cech homology group of the gasket*

$$H_1(K) \simeq \Gamma := \lim_{\leftarrow} \Gamma_n \, .$$

The covering $\widetilde{L}$ *has the unique lifting property.*

Potentials of 1-forms

Definition. *Affine functions*

A continuous function $f \in C(\widetilde{L})$ *is said to be affine if*

$$f(gx) = f(x) + \phi(g) \qquad x \in \widetilde{L}, \quad g \in H_1(K)$$

for some continuous group homomorphisms $\phi : H_1(K) \to \mathbb{C}$. *Denote by* $A(\widetilde{L}, \Gamma)$ *the space of affine functions.*

Theorem. *(Potentials of locally exact forms)* (CGIS 2010)

- *Any locally exact form* $\omega \in \Omega^1_{\text{loc}}(K)$ *admits a potential* $U \in A(\widetilde{L}, \Gamma)$ *in the sense that*

$$\int_\gamma \omega = U(\widetilde{\gamma}(1)) - U(\widetilde{\gamma}(0)).$$

- *An affine function* $U \in A(\widetilde{L}, \Gamma)$ *is a potential of a locally exact form iff it has finite energy*

$$\mathcal{E}_\Gamma[U] := \lim_n \left(\frac{5}{3}\right)^n \sum_{e \in E_n} |U(\widetilde{e}_+) - U(\widetilde{e}_-)|^2 .$$