Secondary invariants for two-cocycle twists

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(joint work with Charlotte Wahl)

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Overview

- context: index theory of elliptic operators
  - primary: index, index class
  - secondary: eta, rho, analytic torsion...

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projectively invariant operators $\longleftrightarrow$ 2-cocycle twists

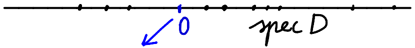
- in physics: magnetic fields, quantum Hall effect
- in geometry: main ideas (Gromov, Mathai)
  - $c \in H^2(B\Gamma, \mathbb{R}) \Rightarrow$ natural C^*-bundles of small curvature
  - pairing without the extension properties
- Joint with Charlotte Wahl:
  - define η and ρ for Dirac operators twisted by a 2-cocycle
  - use ρ to distinguish geometric structures (positive scalar curvature..)

Primary: the index

D elliptic, Dirac type on M, closed manifold
in particular:

1. $d + d^*$ on a Riemannian manifold M
2. $\not{D}$ "the Dirac" on a spin manifold M.

$\mathrm{spec}\, D = \{\lambda_j\}_{j \in \mathbb{N}}$



- primary $\quad \mathrm{ind}\, D := \dim \mathrm{Ker}\, D - \dim \mathrm{Coker}\, D \in \mathbb{Z}$

Theorem (Atiyah–Singer 1963)

$$\mathrm{ind}\, D^+ = \int_M \widehat{A}(M)\, \mathrm{ch}\, E/S$$

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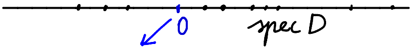

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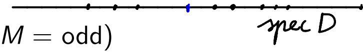
1. $d + d^*$ on $\Lambda^{+/-}$ $\mathrm{ind}(D^+) = \mathrm{sign}(M) = \int_M L(M)$ Hirzebruch's theorem
2. $\not{D}$ on spinors $\mathrm{ind}\, \not{D}^+ = \int_M \widehat{A}(M)$

Secondary: the eta invariant

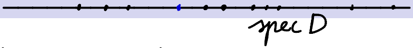


- Atiyah–Patodi–Singer, 1974: for $D = D^*$ $(\dim M = \text{odd})$

$$\eta(D,s) = \sum_{0\neq\lambda_j\in\operatorname{spec} D} \frac{\operatorname{sign}(\lambda_j)}{|\lambda_j|^s}$$

$$\eta(D) := \eta(D,s)|_{s=0} \text{ ``}=\text{''} \sum_{0\neq\lambda_j\in\operatorname{spec} D} \operatorname{sign}(\lambda_j)$$

Secondary: the eta invariant



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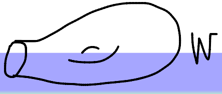
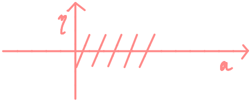
$M = \partial W$



Atiyah–Patodi–Singer theorem (1974)

$$\operatorname{ind} D_W = \int_W \hat{A}(W)\operatorname{ch} E/S - \frac{1}{2}\left(\eta(D_{\partial W}) + \dim \operatorname{Ker} D_{\partial W}\right)$$

- $\eta(D)$ spectral contribution, non-local!
- $M = S^1$, $\eta(-i\frac{d}{dx} + a) = \begin{cases} 0 & , a \in \mathbb{Z} \\ 2a-1, & a \in (0,1) \end{cases}$



Secondary: rho invariants

- Atiyah–Patodi–Singer: $\alpha\colon \Gamma \to U(k)$ flat bundle

$$\rho_\alpha(D) := \eta(D^{\oplus k}) - \eta(D \otimes \nabla^\alpha)$$

- Cheeger–Gromov:

$$\rho_\Gamma(D) := \eta_\Gamma(\tilde{D}) - \eta(D)$$

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Property: ρ can distinguish geometric structures

M closed spin

$$\not{D}_g^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{scal}\,g \quad \text{then} \quad \mathrm{scal}\,g > 0 \Rightarrow \not{D}_g \text{ invertible}$$

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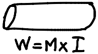
- $(g_t)_{t\in[0,1]} \in \mathcal{R}^+(M) := \{g \text{ metric on } TM \,|\, \operatorname{scal} g > 0\}$



$$0 = \int_W \hat{A}(W) + \tfrac{1}{2}\eta(\not{D}_{g_0}) - \tfrac{1}{2}\eta(\not{D}_{g_1})$$

$$0 = \int_W \hat{A}(W)\operatorname{ch}\nabla^\alpha + \tfrac{1}{2}\eta(\not{D}^\alpha_{g_0}) - \tfrac{1}{2}\eta(\not{D}^\alpha_{g_1})$$

- $\Rightarrow$ it gives a map $\rho(\not{D})\colon \pi_0(\mathcal{R}^+(M)) \to \mathbb{R}$

Role of index and rho: positive scalar curvature

M closed spin manifold $\qquad \not{D}_g^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{scal}\, g.$

$$\mathcal{R}^+(M) = \{g \text{ metric on } TM \,|\, \mathrm{scal}\, g > 0\}$$

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- $\operatorname{ind} \not{D}$ is an obstruction ($\operatorname{ind} \not{D} \neq 0 \quad \Rightarrow \quad \mathcal{R}^+(M) = \emptyset$)
- $\rho(\not{D})$ can distinguish non-cobordant metrics, assuming $\mathcal{R}^+(M) \neq \emptyset$,

Theorem: (Piazza–Schick, Botvinnik–Gilkey)

- $\mathcal{R}^+(M) \neq \emptyset$
- $\dim M = 4k+3$, $k > 0$
- $\pi_1(M)$ has torsion

$\Rightarrow$ $\exists$ infinitely many non-cobordant $\{g_j\}_{j\in\mathbb{N}} \subset \mathcal{R}^+(M)$ $\rho(\not{D}_{g_i}) \neq \rho(\not{D}_{g_j})\ \forall i \neq j$

Theorem: (Piazza–Schick)

- $\pi_1(M)$ torsion free, and satisfies Baum–Connes $\Rightarrow \rho(\not{D}_g) = 0$ for $g \in \mathcal{R}^+(M)$

$\pi\colon \tilde{M} \to M$ universal covering, $\Gamma = \pi_1(M)$

Exemple:
$\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathcal{T}^n$

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- Γ-invariant: $\tilde{D}\gamma = \gamma\tilde{D} \quad \forall\gamma \in \Gamma$
- projectively Γ-invariant : $BT_\gamma = T_\gamma B \ \forall\gamma \in \Gamma$, where

$$T_\gamma T_{\gamma'} = \sigma(\gamma, \gamma') T_{\gamma\gamma'} \quad , \quad T_e = I$$

- then $\sigma : \Gamma \times \Gamma \to U(1)$ is a multiplier, i.e. $[\sigma] \in H^2(\Gamma, U(1))$

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$
$$\sigma(e, \gamma) = \sigma(\gamma, e) = 1$$

Typical construction

$\pi\colon \tilde{M} \to M$ universal covering, $\Gamma = \pi_1(M)$.

On the trivial line $L = \tilde{M} \times \mathbb{C} \to \tilde{M}$ consider

- $\nabla = d + iA$, where $dA \in \Omega^2(\tilde{M}, \mathbb{R})$ is Γ-invariant: $dA = \pi^*\omega$, $\omega \in \Omega^2(M, \mathbb{R})$

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Then:

- $\gamma^* A - A = d\psi_\gamma$
- $\sigma(\gamma, \gamma') = \exp(i\psi_\gamma(\gamma' \tilde{x}_0))$ is a multiplier
- $H_A = (d + iA)^*(d + iA)$, called the magnetic Laplacian,
  is projectively invariant with respect to $T_\gamma u = (\gamma^{-1})^* e^{-i\psi_\gamma} u$, $u \in L^2(\tilde{M})$

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- Remark: the construction can be done starting from $c \in H^2(B\Gamma, \mathbb{R})$

Properties and the L^2-index

Properties of $B = \tilde{D} \otimes \nabla$: (Gromov, Mathai)

- $BT_\gamma = T_\gamma B \Rightarrow B$ is affiliated to $B(L^2(\tilde{M}))^T \simeq \mathcal{N}(\Gamma, \sigma) \otimes L^2(\mathcal{F})$
- $Q \in B(L^2(\tilde{M}))^T$, $\mathrm{tr}_{\Gamma,\sigma}(Q) = \int_{\mathcal{F}} k_Q(x,x)$

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$$e^{-i\psi_\gamma(x)} k_Q(\gamma x, \gamma y) e^{i\psi_\gamma(y)} = k_Q(x, y)$$

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Atiyah's L^2-theorem (Gromov)

$$\mathrm{ind}_{\Gamma,\sigma} B = \int_M \hat{A}(M)\,\mathrm{ch}(E/S)\; e^{\omega}$$

where $\omega = f^* c$, $f\colon M \to B\Gamma$ classifying map

Making the curvature small

Mathai's constructions: given $c \in H^2(B\Gamma, \mathbb{R})$

- Mischchenko-type $C^*(\Gamma, \sigma)$-bundle $\mathcal{V}_{C^*(\Gamma,\sigma)} \to M$ whose curvature is $\omega \otimes I$
- idea: pass to $\sigma^s = e^{is\psi}$ corresponds to curvature $= s\omega \otimes I$

Theorem: (Mathai)

For $c \in H^2(B\Gamma, \mathbb{R})$, $\quad \mathrm{sign}(M, c) := \int_M L(M) \wedge f^*c$ are homotopy invariants.

Proof:

- $$\mathrm{tr}_{\Gamma,\sigma} \, \mathrm{ind}(D^{sign} \otimes \nabla_{\mathcal{V}_{C^*(\Gamma,\sigma)}}) = \sum_j \frac{s^j}{j!} \int_M L(M) \wedge \omega^j$$
- Hilsum–Skandalis's theorem: $\mathrm{ind}(D^{sign} \otimes \nabla_{\mathcal{V}_{C^*(\Gamma,\sigma)}})$ is a homotopy invariant, for s small enough

Eta and rho for 2-cocycle twists

Questions:

1. define $\eta^c(D)$? (easy)
2. prove an index theorem for manifolds with boundary
3. find interesting $\rho^c(D)$
4. relate them with higher rho-invariants (defined by Lott, Leichtnam–Piazza)

Definitions

- Let τ_s be a family of positive (finite) traces on $C^*(\Gamma, \sigma^s)$ s.t. $\tau_s(\delta_\gamma) = \tau_s(\chi(\gamma)\delta_\gamma)$, for any homomorphism $\chi\colon \Gamma \to U(1)$

$$\eta^c_{\tau_s}(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{\,\tau}(D_{\mathcal{V}_{C^*(\Gamma,\sigma)}} e^{-tD_{\mathcal{V}_{C^*(\Gamma,\sigma)}}}) \frac{dt}{\sqrt{t}}$$

- $\rho^c_{\tau_s}(D) =$ same expression, with τ_s delocalized $(\tau_s(\delta_e) = 0)$ assume here $D_{\mathcal{V}_{C^*(\Gamma,\sigma^s)}}$ is invertible

Examples of such traces on $C^*(\Gamma, \sigma)$

1. $\mathrm{tr}_{\Gamma,\sigma}(\sum a_\gamma \gamma) = a_e$
2. on $\Gamma = \Gamma_1 \times \Gamma_2$, with Γ_1 perfect and $\sigma = \pi_2^* \sigma'$, for $\sigma' \in H^2(\Gamma_2, U(1))$, take any $\tau_1 \otimes \mathrm{tr}_{\Gamma,\sigma}$

Positive scalar curvature

Lemma (Mathai)

M spin, $g \in \mathcal{R}^+(M) \neq \emptyset$.
For s small enough, $D_{\mathcal{V}_{C^*(\Gamma,\sigma^s)}}$ is invertible

Definition:

Let M be spin, $g \in \mathcal{R}^+(M) \neq \emptyset$. Call $\rho^c_{\tau_s}(\not{D})$ the direct limit of $s \mapsto \rho^c_{\tau_s}(\not{D})$, $s \in U$

$$\rho^c_{\tau_s} \in \lim_{0 \in U} \mathrm{Maps}(U \to \mathbb{C})$$

Theorem: bordism invariance of $\rho^c_{\tau_s}$

If (M, g_0) and (M, g_1) are Γ-cobordant in $\mathcal{R}^+(M)$, then $\rho^c_\tau(\not{D}_{g_0}) = \rho^c_\tau(\not{D}_{g_1})$

Proof: immediate, applying a C^*-algebraic Atiyah–Patodi–Singer index theorem, and the Lemma.

Theorem (with C. Wahl 2013)

M closed spin, connected, $\pi_1(M) = \Gamma_1 \times \Gamma_2$, $\dim M = 4k+1$

- Γ_1 has torsion
- $\exists\ N$, $\pi_1(N) = \Gamma_2$ s.t. $\int_N \hat{A}(N) \wedge f_N^* c \neq 0$, $c \in H^2(B\Gamma_2, \mathbb{Q})$

If $\mathcal{R}^+(M) \neq \emptyset$ then
$\exists \{g_j\}_{j\in\mathbb{N}} \subset \mathcal{R}^+(M)$ non cobordant $\rho^c_{\tau_s}(\not{D}_{g_i}) \neq \rho^c_{\tau_s}(\not{D}_{g_j})$.

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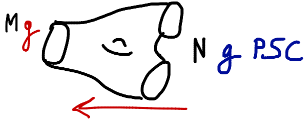
Idea of the proof (after Botvinnik–Gilkey, Piazza–Schick, Piazza–Leichtnam):

- Start with $g_0 \in \mathcal{R}^+(M)$. Apply a machine to generate new non-cobordant metrics on M:
  a) Bordism theorem (by Gromov–Lawson, Rosenberg–Stolz, Botvinnik–Gilkey)
  b) $\exists X^{4k+1}$ closed spin, $\pi_1(X) = \Gamma$,

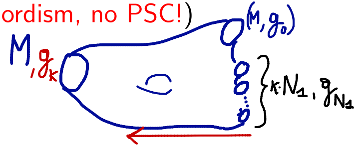
$$\rho^c_{\tau_s}(\not{D}_X) \neq 0\ , \quad j \cdot [X] = 0 \in \Omega_k^{spin,+}(B\Gamma)$$

The machine:

a) Bordism theorem (Gromov–Lawson): M, N spin Γ-cobordant manifold, M connected.
If $\exists g \in \mathcal{R}^+(N)$, then $\exists G \in \mathcal{R}^+(W)$ (of product structure near the boundary)



b) Basic case: $\Gamma = \mathbb{Z}_n$ (Botvinnik–Gilkey):
$\dim M = 7$. There exists $N_1 = S^7/\mathbb{Z}_n$ lens space, $g_{N_1} \in \mathcal{R}(N_1)$, and $\rho_\Gamma(\not{D}_{N_1, g_{N_1}}) \neq 0$.
Moreover: $\Omega_k^{spin,+}(B\mathbb{Z}_n)$ is finite $\Rightarrow \quad \exists j \cdot [N_1] = 0 \in \Omega_k^{spin,+}(B\Gamma)$
(here it is pure bordism, no PSC!)



then $\forall k$: $\rho_\Gamma(M, g_k) = \rho_\Gamma\left((M, g_0) \sqcup k(jN_1, g_{N_1})\right) = \rho_\Gamma(M, g_0) + kj\rho_\Gamma(N_1, g_{N_1})$

$$\rho_\Gamma(M, g_k) \neq \rho_\Gamma(M, g_h) \quad \forall k, h.$$