A Model-Independent Approach to the Short Distance Limit of Quantum Fields

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Introduction

Short distance analysis of quantum fields is usually performed via the Renormalization Group (RG):

- Pass from ϕ to renormalized field at scale $\lambda > 0$
  $\phi_\lambda(x) = Z_\lambda \phi(\lambda x)$;
- Renormalization constants Z_λ fixed by requiring e.g. $\langle \Omega, \phi_\lambda(x)\phi_\lambda(y)\Omega \rangle$ has finite limit as $\lambda \to 0$;
- in good cases (e.g. asymptotically free theories) perturbation theory and RG equations show that

$$\exists \lim_{\lambda \to 0} \langle \Omega, \phi_\lambda(x_1) \ldots \phi_\lambda(x_n)\Omega \rangle = \langle \Omega_0, \phi_0(x_1) \ldots \phi_0(x_n)\Omega_0 \rangle;$$

- field ϕ_0 defines new theory, considered as scaling limit of the original one.

Introduction

Problems:

- how to define scaling limit if theory is not asymptotically free (or perturbation theory not useful)?
- many choices of Z_λ: all equivalent?
- why not more general renormalization prescription?
- fields do not have direct physical interpretation.

Algebraic approach to RG [Buchholz-Verch '95]:

- works for every theory;
- defined without using quantum fields $\implies$ Z_λ not needed;
- based only on observables.

How does it compare to the conventional approach? Can we use it to define scaling limit for quantum fields more generally?

Outline

Algebraic Quantum Field Theory

Theory defined by the assignment of $O \subset \mathbb{R}^4 \to \mathscr{A}(O) \subset B(\mathscr{H})$, local covariant net of observable algebras:

- $O_1 \subset O_2 \implies \mathscr{A}(O_1) \subset \mathscr{A}(O_2)$;
- O_1 spacelike to $O_2 \implies \mathscr{A}(O_1) \subset \mathscr{A}(O_2)'$;
- exists $(\Lambda, x) \to U(\Lambda, x)$ unitary representation of Poincaré group with positive energy such that

$$U(\Lambda, x)\mathscr{A}(O)U(\Lambda, x)^* = \mathscr{A}(\Lambda O + x);$$

- exists unique invariant $\Omega \in \mathscr{H}$ with $\overline{\mathscr{A}\Omega} = \mathscr{H}$ (vacuum).

Interpretation: $\mathscr{A}(O) = \{\text{observables measurable in } O\}''$.

Scaling Algebras

On C*-algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \to \underline{A}_\lambda \in \mathscr{A}$ define

- $\underline{\alpha}_{(\Lambda,x)}(\underline{A})_\lambda := U(\Lambda, \lambda x)\underline{A}_\lambda U(\Lambda, \lambda x)^*$;
- $\underline{\delta}_\mu(\underline{A})_\lambda := \underline{A}_{\lambda\mu}$;
- $\underline{\alpha}_g := \underline{\alpha}_{(\mu,\Lambda,x)} := \underline{\delta}_\mu \circ \underline{\alpha}_{(\Lambda,x)}$.

Local scaling algebra of O [Buchholz-Verch '95]:

$$\underline{\mathfrak{A}}(O) := \left\{ \underline{A} \,:\, \underline{A}_\lambda \in \mathscr{A}(\lambda O), \lim_{g \to \mathrm{id}} \|\underline{\alpha}_g(\underline{A}) - \underline{A}\| = 0 \right\}$$

where $\|\underline{A}\| := \sup_\lambda \|\underline{A}_\lambda\|$.

Scaling Algebras

- Continuity condition $\iff$ $\underline{A}_\lambda$ has a "phase space occupation" independent of λ $\iff$ $\hbar$ not rescaled.
- Typical elements

$$\underline{A}_\lambda = \int dx\, g(x) U(\lambda x) e^{i\phi_\lambda(f)} U(\lambda x)^*, \quad \mathrm{supp}\, f + \mathrm{supp}\, g \subset O,$$

bounded irrespective of Z_λ.
- We consider "all possible renormalization schemes" compatible with above requirements.

Scaling Algebras

Limit $\lambda \to 0$: take mean $\mathbf{m}$ on $(0, 1]$, i.e. state on bounded functions on $(0, 1]$, and define

$$\underline{\omega}_0(\underline{A}) := \mathbf{m}(\lambda \to \omega(\underline{A}_\lambda))$$

scaling limit state on $\underline{\mathfrak{A}}$. (Here $\omega := \langle \Omega, (\cdot)\Omega \rangle$.)
Examples:

1. $\mathbf{m}_{\lambda_0}(f) = f(\lambda_0)$, then $\underline{\omega}_0$ vacuum state at scale λ_0;
2. $\mathbf{m}$ weak* limit point of $\mathbf{m}_{\lambda_0}$ as $\lambda_0 \to 0$, then $\underline{\omega}_0$ Buchholz-Verch limit state;
3. $\mathbf{m}$ dilation invariant: $\mathbf{m}(f(\mu \cdot)) = \mathbf{m}(f)$.

2 and 3 generalizations of limit $\lambda \to 0$: $\mathbf{m}(f) = \lim_{\lambda \to 0} f(\lambda)$ if limit exists.

Scaling Algebras

Theorem

- $(\pi_0, \mathscr{H}_0, \Omega_0)$ *GNS representation of* $\underline{\omega}_0 \implies$ $\mathscr{A}_0(O) := \pi_0(\underline{\mathfrak{A}}(O))''$ *Poincaré covariant net in vacuum representation:* *scaling limit* *net;*
- *in case (1 and) 2* $\underline{\omega}_0$ *is pure, in 3 it is not;*
- *in case 3* $\mathscr{A}_0$ *also dilation covariant.*

- In good cases all $\mathscr{A}_0$ corresponding to $\underline{\omega}_0$ pure are isomorphic and nontrivial $\implies$ $\mathscr{A}$ has unique scaling limit.
- It can also be non-unique, or trivial ($\mathscr{A}_0 = \mathbb{C}\mathbb{1}$).

Scaling Algebras

Example: $\mathscr{A}$ massive free field in $d = 3 + 1$:

- $\mathscr{A}_0(O) \cong \mathscr{A}_{\text{massless}}(O) \bar{\otimes} \pi_0(\mathfrak{Z}(\underline{\mathfrak{A}}))''$;
- $\underline{\omega}_0$ pure $\implies$ $\mathscr{A}_0(O) \cong \mathscr{A}_{\text{massless}}(O)$;
- $\underline{\omega}_0$ dilation invariant $\implies$ $\overline{\pi_0(\mathfrak{Z}(\underline{\mathfrak{A}}))''\Omega_0}$ non-separable, $U_0(g) \cong U_{\text{massless}}(g) \otimes U_{\mathfrak{Z}}(g)$.

Summarizing:

- In this framework every theory has a scaling limit, but in an abstract way.
- How does it compare to conventional approach?
- In particular: Z_λ not needed. Can they be recovered?

Pointlike Fields from Local Algebras

Basic idea [Haag-Ojima '96]: assume

$$\Sigma_{E,r} = \{\sigma \restriction \mathscr{A}(O_r) \,:\, \sigma \in P(E)B(\mathscr{H})_* P(E)\}$$

is compact and "does not change" for small r

$\implies$ "finite" number of states describe short distance behaviour

$\implies$ basis (ϕ_j) of $\Sigma^*_{E,r}$ are pointlike fields.

Pointlike Fields from Local Algebras

Quantitative version:

- $\Sigma = B(\mathscr{H})_*$, $C^\infty(\Sigma) = \cap_{\ell>0} R^\ell \Sigma R^\ell$, $R = (1+H)^{-1}$;
- $\|\sigma\|^{(\ell)} = \|R^{-\ell}\sigma R^{-\ell}\|$, $\sigma \in C^\infty(\Sigma)$;
- $\Xi : \sigma \in C^\infty(\Sigma) \to \sigma \in \Sigma$.

Definition ([Bostelmann '05])

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition I if $\forall \gamma > 0$, $\exists \ell > 0$, $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi\|^{(\ell)} < \infty,$$

$$\|(\Xi - \psi)(\cdot) \restriction \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields from Local Algebras

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma := \text{span}\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ*;*
- $\phi \in \Phi_\gamma \implies \exists A_r \in \mathscr{A}(O_r), \ell > 0$ *such that*

$$\|\phi - A_r\|^{(\ell)} = O(r).$$

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$.
ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 = \text{span}\{\mathbb{1}, \phi\}$, $\Phi_2 = \text{span}\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$,
...

Pointlike Fields from Local Algebras

According to phase space condition, if $A \in \mathscr{A}(O_r)$:

$$A \sim \sum_j \sigma_j(A)\phi_j \qquad \text{as } r \to 0.$$

Can be generalized to local fields by $\varepsilon/3$ argument.

Theorem ([Bostelmann '05])

$\phi, \phi' \in \Phi_\gamma$. For all $\beta > 0$ exist $\sigma_j \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ such that

$$\|\phi(f_d)\phi'(f'_d) - \sum_j \sigma_j(\phi(f_d)\phi'(f'_d))\phi_j\|^{(\ell)} = o(d^\beta),$$

where $f, f' \in \mathscr{S}$ and $f_d(x) = d^{-4}f(d^{-1}x)$.

Operator product expansion of $\phi(f)\phi'(f')$.

Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_0$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale $\lambda \implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Phase Space and Scaling Limit

Scaling: $r \to \lambda r$, $E \to \lambda^{-1} E \implies$ phase space condition needs sharpening:

Definition

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition II if $\forall \gamma > 0$, $\exists c, \varepsilon > 0$ and $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that for large E, small r,

$$\|\psi \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(1 + Er)^\gamma,$$
$$\|(\Xi - \psi) \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(Er)^{\gamma+\varepsilon}.$$

Satisfied by free fields in $d = 3 + 1$ [Bostelmann '00].
Reasonable for asymptotically free theories (logarithmic corrections).
Note: PSC II $\implies$ PSC I.

Phase Space and Scaling Limit 2/3

Consider $\lambda \to \underline{\phi}_\lambda \in C^\infty(\Sigma)^*$ such that

$$\sup_\lambda \|\underline{\phi}_\lambda \restriction \Sigma_{E/\lambda}\| < \infty$$

(and $g \to \underline{\alpha}_g(\underline{\phi})$ is continuous).

Theorem

Let $O \to \mathscr{A}(O)$ satisfy PSC II. Then:

- *π_0 extends to $\underline{\phi}$ and $\pi_0(\underline{\phi}) \in C^\infty(\Sigma_0)^*$ is a local field of $\mathscr{A}_0$.*

If $\underline{\omega}_0$ pure:

- *$O \to \mathscr{A}_0(O)$ satisfies PSC I;*
- $\dim \Phi_{0,\gamma} \leq \dim \Phi_\gamma$.

Phase Space and Scaling Limit

Idea of proof.

- For $\underline{B} \in \underline{\mathfrak{A}}$ with $\underline{B}_\lambda \Omega \in P(E/\lambda)\mathscr{H}$:

$$\langle \pi_0(\underline{B})\Omega_0, \pi_0(\underline{\phi})\pi_0(\underline{B})\Omega_0 \rangle := \mathbf{m}(\lambda \to \langle \underline{B}_\lambda \Omega, \underline{\phi}_\lambda \underline{B}_\lambda \Omega \rangle),$$

- Define $\psi_0^*(\underline{A}) := \pi_0(\psi^*(\underline{A})) \implies$ estimates on ψ^* pass to $\psi_0^* \implies$ PSC I for $\mathscr{A}_0$ if $\underline{\omega}_0$ pure.
- $\exists \underline{A}_r \in \underline{\mathfrak{A}}(O_r), \ell > 0$ such that

$$\sup_\lambda \|\underline{R}_\lambda^\ell (\underline{\phi}_\lambda - (\underline{A}_r)_\lambda)\underline{R}_\lambda^\ell\| = O(r)$$

where $\underline{R}_\lambda = (1 + \lambda H)^{-1} \implies \pi_0(\underline{\phi})$ local field.

$\square$

Uniform Operator Product Expansion

Define $\underline{\alpha}_f\underline{\phi} = \int dx\, f(x)\underline{\alpha}_x(\underline{\phi})$, unbounded operator $\forall\, \lambda > 0$.
Thanks to uniform approximation of $\underline{\alpha}_f\underline{\phi}$ by $\underline{\alpha}_f\underline{A}_r$,

$$\pi_0(\underline{\alpha}_f\underline{\phi}\,\underline{\alpha}_{f'}\underline{\phi}') = \alpha_{0,f}\pi_0(\underline{\phi})\alpha_{0,f'}\pi_0(\underline{\phi}'),$$

and furthermore:

Theorem

For all $\beta > 0$ *exist* $\sigma_{j,\lambda} \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ *such that*

$$\sup_\lambda \left\| \underline{R}^\ell_\lambda \left(\underline{\alpha}_{f_d}\underline{\phi}_\lambda\, \underline{\alpha}_{f'_d}\underline{\phi}'_\lambda - \sum_j \sigma_{j,\lambda}(\underline{\alpha}_{f_d}\underline{\phi}_\lambda\, \underline{\alpha}_{f'_d}\underline{\phi}'_\lambda)\phi_j\right)\underline{R}^\ell_\lambda \right\| = o(d^\beta).$$

Therefore OPE terms converge to OPE terms.

Renormalization Group

Renormalization constants:

- $\underline{\phi}_\lambda = \sum_j \sigma_j(\underline{A}_\lambda)\phi_j$ has well-defined limit $\phi_0 = \pi_0(\underline{\phi})$;
- therefore $Z_{j,\lambda} = \sigma_j(\underline{A}_\lambda)$ are renormalization constants.
- in particular for 2-point Wightman functions:

$$\langle \Omega_0, \phi_0(x)\phi_0'(x')\Omega_0\rangle = \mathbf{m}\Big(\lambda \to \sum_{j,k} Z_{j,\lambda} Z'_{k,\lambda}\langle \Omega, \phi_j(\lambda x)\phi_k(\lambda x')\Omega\rangle\Big),$$

where $Z'_{k,\lambda} = \sigma_k(\underline{A}'_\lambda)$, $\phi_0' = \pi_0(\psi^*(\underline{A}'))$.

Scaling transformations:

- $(\underline{\delta}_\mu \underline{\phi})_\lambda = \underline{\phi}_{\mu\lambda} = \sum_j Z_{j,\mu\lambda}\phi_j$: renormalization group;
- $\underline{\omega}_0$ invariant: $\pi_0(\underline{\delta}_\mu \underline{\phi}) = U_0(\mu)\pi_0(\underline{\phi})U_0(\mu)^*$.

Renormalization Group

Scaling of OPE:

- no Lagrangian in our approach $\Longrightarrow$ flow of coupling constants not visible;
- OPE coefficients are the “structure constants” of the algebra of quantum fields;
- scaling changes OPE coefficients.

Summary

Summary:

- short distance analysis of quantum fields performed in a model independent approach;
- multiplicative renormalization obtained in an axiomatic framework and renormalization constants provided automatically by general machinery;
- scaling of OPE coefficients is some substitute for coupling constant renormalization;
- renormalization group induces dilations in the limit theory;
- no new observable fields appear in the limit (not in contrast with QCD, where new unobservable fields should appear in the scaling limit).