Scaling Limit of Quantum Fields from Local Algebras

Gerardo Morsella

Università di Roma “Tor Vergata”

joint work with H. Bostelmann, C. D’Antoni

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Introduction

Short distance analysis of quantum fields is usually performed via the Renormalization Group (RG):

- Pass from ϕ to renormalized field at scale $\lambda > 0$: $\phi_\lambda(x) := Z_\lambda \phi(\lambda x)$;
- Renormalization constants Z_λ fixed by requiring e.g. $\langle \Omega, \phi_\lambda(x)\phi_\lambda(y)\Omega\rangle$ has finite limit as $\lambda \to 0$;
- in good cases (e.g. asymptotically free theories) perturbation theory and RG equations show that

$$\exists \lim_{\lambda\to 0} \langle \Omega, \phi_\lambda(x_1)\ldots\phi_\lambda(x_n)\Omega\rangle = \langle \Omega_0, \phi_0(x_1)\ldots\phi_0(x_n)\Omega_0\rangle;$$

- field ϕ_0 defines new theory, considered as scaling limit of the original one.

Introduction

Problems:

- how to define scaling limit if theory is not asymptotically free (or perturbation theory not useful)?
- many choices of Z_λ: all equivalent?
- why not more general renormalization prescription?
- fields do not have direct physical interpretation.

Algebraic approach to RG [Buchholz-Verch '95]:

- works for every theory;
- defined without using quantum fields $\implies$ Z_λ not needed;
- based only on observables.

How does it compare to the conventional approach? Can we use it to define scaling limit for quantum fields more generally?

Outline

Scaling Limit in Algebraic Quantum Field Theory

Setting as in previous talk:

- $O \subset \mathbb{R}^4 \mapsto \mathscr{A}(O) \subset B(\mathscr{H})$ net of observable algebras;
- $(\Lambda, x) \mapsto U(\Lambda, x)$ unitary representation of Poincaré group on $\mathscr{H}$;
- $\Omega \in \mathscr{H}$ vacuum;
- $O \mapsto \underline{\mathfrak{A}}(O)$ scaling algebra of $\mathscr{A}$;
- typical element:

$$\underline{A}_\lambda = \int dx\, g(x) U(\lambda x) e^{i\phi_\lambda(f)} U(\lambda x)^*, \quad \operatorname{supp} f + \operatorname{supp} g \subset O,$$

  bounded irrespective of choice of Z_λ;
- $\underline{\omega}_0$ scaling limit state on $\underline{\mathfrak{A}}$;
- $O \mapsto \mathscr{A}_0(O) = \pi_0(\underline{\mathfrak{A}}(O))''$ corresponding scaling limit theory;

Pointlike Fields from Local Algebra 1/4

There should be an expansion for $A_r \in \mathscr{A}(O_r)$:

$$A_r \underset{r\to 0}{\sim} \sum_j \sigma_j(A_r)\phi_j(0).$$

Consider the classical Taylor expansion:

- Let f be a smooth function on $J = [-1, 1]$, and p_γ its γ-order Taylor polynomial at 0. Then:

$$|f(x) - p_\gamma(x)| \leq \|f^{(\gamma+1)}\|_\infty O(|x|^{\gamma+1}) \quad \text{as } |x| \to 0.$$

- Note that $\psi_\gamma : C^\infty(J) \to L_1(J)$, $f \mapsto p_\gamma[f]$ is linear, continuous (in the Schwartz topology), and of finite rank.
- ψ_γ approximates the inclusion map $\Xi : C^\infty(J) \to L_1(J)$.

Now generalize this to quantum field theory.

Pointlike Fields from Local Algebras 2/4

Consider net of C^* algebras, $O \mapsto \mathscr{A}(O) \subset B(\mathscr{H})$ as above.

- $\Sigma := B(\mathscr{H})_*$ normal functionals, with the norm topology.
- $H \geq 0$ generator of time translations (the Hamiltonian), and $R := (1 + H)^{-1}$.
- For $\sigma \in \Sigma$, set $\|\sigma\|^{(\ell)} = \|\sigma(R^{-\ell} \cdot R^{-\ell})\|$ (where it exists).
- $C^\infty(\Sigma) := \{\sigma \in \Sigma \mid \|\sigma\|^{(\ell)} < \infty \text{ for all } \ell > 0\}$, equipped with the Fréchet topology induced by all those norms.
- Ξ the inclusion map $C^\infty(\Sigma) \to \Sigma$.

Definition ([Bostelmann '05])

$\mathscr{A}$ satisfies the microscopic phase space condition if there is a family $\{\psi_\gamma\}_{\gamma \geq 0}$ of continuous linear finite-rank maps $\psi_\gamma : C^\infty(\Sigma) \to \Sigma$ with the following property: for every $\gamma \geq 0$, there exists $\ell > 0$ such that

$$\|(\Xi - \psi_\gamma)(\cdot) \restriction \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields form Local Algebras 3/4

For real scalar free field ϕ in physical space-time, expansion exists (h a specific test function in the time-0 plane):

$$
\begin{aligned}
\Xi = {} & \langle\Omega,(\cdot)\Omega\rangle && \cdot\, \mathbb{1} && \big\}\ \gamma = 0 \\
& + i\langle\Omega,[\dot\phi(h),\cdot\,]\Omega\rangle && \cdot\, \phi(0) && \big\}\ \gamma = 1 \\
& \left.\begin{aligned}
& + i\sum_{k=1}^{3}\langle\Omega,[\dot\phi(x_k h),\cdot\,]\Omega\rangle && \cdot\, \partial_k\phi(0) \\
& - i\langle\Omega,[\phi(h),\cdot\,]\Omega\rangle && \cdot\, \partial_0\phi(0) \\
& - \frac{1}{2}\langle\Omega,[\dot\phi(h),[\dot\phi(h),\cdot\,]]\Omega\rangle && \cdot\, :\!\phi^2\!:(0)
\end{aligned}\right\}\ \gamma = 2 \\
& + \cdots
\end{aligned}
$$

Pointlike Fields from Local Algebras 4/4

In the general case, take microscopic phase space condition as an additional axiom (noncommutative Taylor expansion).
Then:

Theorem ([Bostelmann '05])

- *If the rank of ψ_γ is minimized, then $\Phi_\gamma := img\,\psi_\gamma^* \subset C^\infty(\Sigma)^*$ consists of pointlike fields, i.e. smeared $\phi(f)$ is a Wightman field and $\phi(f)\,\eta\,\mathscr{A}(O)$ if* $\mathrm{supp}\, f \subset O$.
- $\Phi := \bigcup_{\gamma>0} \Phi_\gamma$ *set of all pointlike fields locally associated to $O \mapsto \mathscr{A}(O)$.*
- $\phi, \phi' \in \Phi$ *have an OPE.*

Scaling Limit of Pointlike Fields 1/3

Question: Can we recover Z_λ such that there exists

$$\phi_0(x) = \lim_{\lambda \to 0} Z_\lambda \phi(\lambda x)?$$

Idea:

- Consider free field ϕ, $\psi^*_\gamma = \sigma_1(\cdot)\phi$ of rank one $\implies$ $\|\sigma_1 \restriction \mathscr{A}(\lambda O)\| = O(\lambda)$;
- $\underline{A} \in \underline{\mathfrak{A}}(O) \implies \underline{\phi}_\lambda := \psi^*_\gamma(\underline{A}_\lambda) = \sigma_1(\underline{A}_\lambda)\phi$ has to be thought as a field at scale λ;
- we can choose $Z_\lambda := \sigma_1(\underline{A}_\lambda) \sim \lambda$.

In general: fields $\underline{\phi}_\lambda := \psi^*_\gamma(\underline{A}_\lambda)$ should have well behaved scaling limit as fields of $\mathscr{A}_0$.

Scaling Limit of Pointlike Fields 2/3

Theorem

Assume $\underline{\omega}_0$ is a pure scaling limit state. Then:

- *if $\mathscr{A}$ satisfies (slightly strengthened) phase space condition $\implies$ $\mathscr{A}_0$ satisfies phase space condition and $\dim \Phi_{0,\gamma} \leq \dim \Phi_\gamma$;*
- *if $\underline{\phi}_\lambda := \psi^*_\gamma(\underline{A}_\lambda) = \sum_j \sigma_j(\underline{A}_\lambda)\phi_j = \sum_j Z_{j,\lambda}\phi_j \implies \pi_0$ extends to $\underline{\phi}$ and $\phi_0 := \pi_0(\underline{\phi})$ is a pointlike field associated to $\mathscr{A}_0$;*
- *given $\underline{\phi}'_\lambda = \psi^*_{\gamma'}(\underline{A}'_\lambda) = \sum_k Z'_{k,\lambda}\phi_k$, $\phi'_0 = \pi_0(\underline{\phi}')$, there holds:*

$$\langle \Omega_0, \phi_0(x)\phi'_0(x')\Omega_0\rangle = \mathbf{m}\Big(\lambda \mapsto \sum_{j,k} Z_{j,\lambda}Z'_{k,\lambda}\langle \Omega, \phi_j(\lambda x)\phi_k(\lambda x')\Omega\rangle\Big);$$

- *the OPE coefficients of $\underline{\phi}_\lambda, \underline{\phi}'_\lambda$ converge to the OPE coefficients of ϕ_0, ϕ'_0;*

Scaling Limit of Pointlike Fields 3/3

Idea of proof.

- Definition of $\pi_0(\underline{\phi})$: For $\underline{B} \in \underline{\mathfrak{A}}$ with $\underline{B}_\lambda\Omega \in P(E/\lambda)\mathscr{H}$:

$$\langle \pi_0(\underline{B})\Omega_0, \pi_0(\underline{\phi})\pi_0(\underline{B})\Omega_0 \rangle := \mathbf{m}(\lambda \mapsto \langle \underline{B}_\lambda\Omega, \underline{\phi}_\lambda \underline{B}_\lambda\Omega \rangle),$$

- Phase space condition for $\mathscr{A}_0$: Define $\psi^*_{0,\gamma}(\underline{A}) := \pi_0(\psi^*_\gamma(\underline{A})) \implies$ estimates on ψ^* pass to ψ^*_0
- Locality of $\pi_0(\underline{\phi})$: $\exists \underline{A}_r \in \underline{\mathfrak{A}}(O_r), \ell > 0$ such that

$$\sup_\lambda \| \underline{R}^\ell_\lambda(\underline{\phi}_\lambda - (\underline{A}_r)_\lambda)\underline{R}^\ell_\lambda \| = O(r)$$

where $\underline{R}_\lambda = (1 + \lambda H)^{-1}$.

□

Conclusions

In summary:

- in the algebraic approach to QFT, the short distance limit exists in a model independent framework, but in an abstract sense;
- from this, we can derive an intrinsic, model independent procedure to perform the scaling limit for the quantum fields canonically associated to the theory;
- multiplicative renormalization is therefore a consequence of first principles, and renormalization constants Z_λ are given automatically by the general machinery;
- scaling of OPE coefficients can be seen as the counterpart of coupling constant renormalization.
- no new observable fields appear in the limit (not in contrast with QCD, where new unobservable fields should appear in the scaling limit).

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