Algebraic Renormalization Group and Pointlike Fields

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Introduction

Conventional approach to the Renormalization Group:

- Pass from ϕ to renormalized field $\phi_\lambda(x) = Z_\lambda \phi(\lambda x)$;
- Renormalization constants Z_λ fixed by requiring e.g. $\langle \Omega, \phi_\lambda(x)\phi_\lambda(y)\Omega \rangle \sim$ const as $\lambda \to 0$.

Problems:

- need to have detailed information on short-distance behaviour to calculate Z_λ;
- fields do not have direct physical interpretation.

Algebraic approach [Buchholz-Verch '95]:

- Z_λ not needed;
- based only on observables and model independent.

How does it compare to the conventional approach? How can we recover Z_λ?

Scaling Algebras

Data:

- $O \subset \mathbb{R}^4 \to \mathscr{A}(O) \subset B(\mathscr{H})$ net of observable algebras.
- $x \in \mathbb{R}^4 \to U(x)$ unitary representation of translations.
- $\Omega \in \mathscr{H}$ vacuum.

On C*-algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \to \underline{A}_\lambda \in \mathscr{A}$

$$\underline{\alpha}_x(\underline{A})_\lambda := \mathrm{Ad}\, U(\lambda x)(\underline{A}_\lambda), \qquad x \in \mathbb{R}^4,$$

Definition ([Buchholz-Verch '95])

Local scaling algebra of O:

$$\underline{\mathfrak{A}}(O) := \left\{ \underline{A} \,:\, \underline{A}_\lambda \in \mathscr{A}(\lambda O),\, \lim_{x \to 0} \|\underline{\alpha}_x(\underline{A}) - \underline{A}\| = 0 \right\}$$

Scaling Algebras

φ locally normal state on $\mathscr{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{A}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([Buchholz-Verch '95])

- $\mathrm{SL}^{\mathscr{A}}(\varphi)$ *is independent of* φ.
- $\underline{\omega}_0 \in \mathrm{SL}^{\mathscr{A}}$ *with GNS representation* π_0. *Then* $\mathscr{A}_0(O) := \pi_0(\underline{\mathfrak{A}}(O))''$ *is a covariant net in vacuum representation.*

$O \to \mathscr{A}_0(O)$ is the scaling limit net of $\mathscr{A}$.
Physical interpretation: $\mathscr{A}_0$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

Scaling Algebras

$\exists(\mathscr{F}, G)$ such that $\mathscr{A} = \mathscr{F}^G$. [Doplicher-Roberts '90]
Field scaling algebra $\underline{\mathfrak{F}}$ and scaling limit field net $\mathscr{F}_0$ defined in analogy to $\underline{\mathfrak{A}}$, $\mathscr{A}_0$.

$$\mathscr{A}_0 = \mathscr{F}_0^{G_0} \subset \mathscr{F}_0 \subseteq \mathscr{F}^0, \qquad \mathscr{F}^0 \text{ DR net of } \mathscr{A}_0$$

$\mathscr{F}^0 \supsetneq \mathscr{F}_0 \implies \mathscr{A}$ has confined charges.
E.g. in the Schwinger model $\mathscr{F} = \mathscr{A} \implies \mathscr{F}_0 = \mathscr{A}_0 \subsetneq \mathscr{F}^0$.

Charge preservation: identifies sectors of $\mathscr{A}_0$ which are scaling limit of sectors of $\mathscr{A}$ [D'Antoni-Verch-M. '03, D'Antoni-M. '06].

Pointlike Fields from Local Algebras

Basic idea [Haag-Ojima '96]:

$$\Sigma_{E,r} = \{\sigma \restriction \mathscr{A}(O_r) \ : \ \sigma \in P(E)B(\mathscr{H})_* P(E)\}$$

is compact for small r
$\Longrightarrow$ "finite" number of states describe short distance behaviour
$\Longrightarrow$ basis (ϕ_j) of $\Sigma^*_{E,r}$ are pointlike fields.

Pointlike Fields from Local Algebras

Quantitative version:

- $\Sigma = B(\mathscr{H})_*$, $C^\infty(\Sigma) = \cap_{\ell>0} R^\ell \Sigma R^\ell$, $R = (1+H)^{-1}$;
- $\|\sigma\|^{(\ell)} = \|R^{-\ell}\sigma R^{-\ell}\|$, $\sigma \in C^\infty(\Sigma)$;
- $\Xi : \sigma \in C^\infty(\Sigma) \to \sigma \in \Sigma$.

Definition ([Bostelmann '05])

$O \to \mathscr{A}(O)$ satisfies the asymptotic phase space condition if $\forall \gamma > 0$, $\exists \ell > 0$, $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi\|^{(\ell)} < \infty,$$

$$\|(\Xi - \psi) \restriction \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields from Local Algebras

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma :=$ span$\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ;
- $\cup_{\gamma>0}\Phi_\gamma = \{\phi : \; \exists \ell > 0, \; R^\ell \phi R^\ell \in \cap_{r>0}(R^\ell \mathscr{A}(O_r) R^\ell)^-\}$.

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$ [Fredenhagen-Hertel '81].
ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 =$ span$\{\mathbb{1}, \phi\}$, $\Phi_2 =$ span$\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$,...

Basic Idea

- Is the asymptotic phase space condition valid for $\mathscr{A}_0$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda \to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma > 0} \Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale $\lambda \implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good objects in the scaling limit.

Phase Space and Scaling Limit

Scaling: $r \to \lambda r$, $E \to \lambda^{-1} E \implies$ phase space condition needs sharpening:

Definition

$O \to \mathscr{A}(O)$ satisfies the homogeneous asymptotic phase space condition if $\forall \gamma > 0$, $\exists c, \ell > 0$ and $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(1 + Er)^\ell,$$
$$\|(\Xi - \psi) \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(Er)^\gamma.$$

Satisfied by free fields.
Also by asymptotically free theories (logarithmic corrections)?
Note: HAPSC $\implies$ APSC.

Phase space and Scaling Limit

Theorem

- *$O \to \mathscr{A}(O)$ satisfies HAPSC $\implies$ $O \to \mathscr{A}_0(O)$ satisfies APSC;*
- *$\dim \Phi_{0,\gamma} \leq \dim \Phi_\gamma$.*

Proof.

- Define $\chi_0 : C^\infty(\Sigma_0) \to \underline{\mathfrak{A}}^*$ by

$$\chi_0(\langle \pi_0(\underline{B})\Omega_0, (\cdot)\pi_0(\underline{B}')\Omega_0\rangle)(\underline{A}) = \lim_\lambda \psi(\langle \underline{B}_\lambda\Omega, (\cdot)\underline{B}'_\lambda\Omega\rangle)(\underline{A}_\lambda),$$

  where spectral support of $\underline{B}, \underline{B}' = [0, E]$;
- $\text{rank}\chi_0 \leq \text{rank}\psi$;
- $\Phi_{0,\gamma} := (\ker\chi_0)^\perp$, $\psi_0 := \Xi_0 \circ (p_{\Phi_{0,\gamma}})_*$. □

Renormalization Constants

- $\mathrm{rank}\chi_0 < \infty \implies \Phi_{0,\gamma} = \mathrm{img}(\chi_0^* \restriction \underline{\mathfrak{A}})$;
- $\forall \phi_0 \in \Phi_{0,\gamma},\ \exists \underline{A} \in \underline{\mathfrak{A}} : \phi_0 = \chi_0^*(\underline{A})$;
- Compute:

$$\begin{aligned}\langle \pi_0(\underline{B})\Omega_0, &\phi_0(x)\pi_0(\underline{B})\Omega_0\rangle \\ &= \chi_0(\langle \pi_0(\underline{\alpha}_{-x}\underline{B})\Omega_0, (\cdot)\pi_0(\underline{\alpha}_{-x}\underline{B})\Omega_0\rangle)(\underline{A}) \\ &= \lim_\lambda \psi(\langle (\underline{\alpha}_{-x}\underline{B})_\lambda\Omega, (\cdot)(\underline{\alpha}_{-x}\underline{B})_\lambda\Omega\rangle)(\underline{A}_\lambda) \\ &= \lim_\lambda \sum_j \sigma_j(\underline{A}_\lambda)\langle \underline{B}_\lambda\Omega, \phi_j(\lambda x)\underline{B}_\lambda\Omega\rangle;\end{aligned}$$

- $Z_{j,\lambda} := \sigma_j(\underline{A}_\lambda)$ renormalization constants.

This should be generalized to n-point functions.

Summary & Outlook

In the algebraic approach to the Renormalization Group, the field strength renormalization constants are obtained in a model-independent way.

Outlook:

- Coupling constants renormalization: Can be replaced by scaling of OPE coefficients?
- Asymptotic freedom: Under which conditions the limit fields are free? Perhaps conditions on the 2-point functions?
- Completness: $\mathscr{A}$ generated by pointlike fields $\implies$ $\mathscr{A}_0$ generated by pointlike fields?