# Operator Algebras and the Renormalization Group in Quantum Field Theory

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# Outline

# Introduction

Renormalization Group (RG) is a method of analysis of the short distance limit of Quantum Field Theory (QFT - the mathematical framework of elementary particle physics).
It gave several important results:

- asymptotic freedom of nonabelian gauge theories
- corrections to parton picture of Deep Inelastic Scattering
- applications to charge confinement
- applications to Statistical Mechanics, Dynamical Systems...

Conceptual problem: formulated in Lagrangian approach, model-dependent
An axiomatic approach to QFT can be formulated in terms of Operator Algebras: Algebraic Quantum Field Theory (AQFT) [Haag-Kastler '64]
A model-independent version of RG adapted to AQFT, proposed in [Buchholz-Verch '94] gives several interesting structural results

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# Algebras of Local Observables

Main assumption of [Haag-Kastler '64]: A QFT is fixed by the knowledge of its local observables

Basic input: assignment $O \mapsto \mathscr{A}(O)$, where

- $O$ open bounded subset of Minkowski spacetime $\mathbb{R}^{s+1}$
- $\mathscr{A}(O)$ von Neumann algebras on a fixed Hilbert space $\mathscr{H}$
- if $O_1$ and $O_2$ spacelike separated (i.e. no signal can travel between them), then for all $A_1 \in \mathscr{A}(O_1)$, $A_2 \in \mathscr{A}(O_2)$

$$[A_1, A_2] = 0 \qquad \text{(locality)}$$

- $\exists U$ unitary cont. representation of Poincaré group $SO^{\uparrow}(1,s) \rtimes \mathbb{R}^{s+1}$ on $\mathscr{H}$ such that if $\alpha_{(\Lambda,x)} := \mathrm{Ad}\, U(\Lambda, x)$ then

$$\alpha_{(\Lambda,x)}(\mathscr{A}(O)) = \mathscr{A}(\Lambda O + x) \qquad \text{(covariance)}$$

- $\exists! \Omega \in \mathscr{H}$ $U(\mathbb{R}^{s+1})$-invariant, s.t. $\overline{\bigcup_O \mathscr{A}(O)\Omega} = \mathscr{H}$ (vacuum)

This structure is a local net of observable algebras

Physical interpretation: self-adjoint $A \in \mathscr{A}(O)$ is an observable measurable in $O$

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# Example: Free scalar field of mass $m \geq 0$

$\mathfrak{W}$ Weyl algebra: C*-algebra generated by unitaries $W(f)$, $f \in \mathscr{D}_{\mathbb{R}}(\mathbb{R}^{s+1})$, such that

$$W(f)W(g) = e^{i\sigma(f,g)}W(f+g)$$

$$\sigma(f,g) = \frac{1}{2}\,\mathrm{Im}\int_{\mathbb{R}^s}\frac{d^sp}{\omega_m(p)}\overline{\hat{f}(\omega_m(p),p)}\hat{g}(\omega_m(p),p)$$

$$\omega_m(p) = \sqrt{p^2+m^2}$$

Fock vacuum: pos. norm. $\omega^{(m)} : \mathfrak{W} \to \mathbb{C}$ (state) determined by

$$\omega^{(m)}(W(f)) = \exp\left\{-\frac{1}{4}\int_{\mathbb{R}^s}\frac{d^sp}{\omega_m(p)}|\hat{f}(\omega_m(p),p)|^2\right\} \qquad (m > 0 \text{ if } s = 1)$$

$(\pi^{(m)}, \mathscr{H}^{(m)}, \Omega^{(m)})$ corresponding GNS representation, i.e.

$$\omega^{(m)}(W) = \langle \Omega^{(m)}, \pi^{(m)}(W)\Omega^{(m)}\rangle, \qquad W \in \mathfrak{W}$$

Local algebras

$$\mathscr{A}^{(m)}(O) := \{\pi^{(m)}(W(f)) \,:\, \mathrm{supp}\, f \subset O\}''$$

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# Scaling Algebras

On C*-algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \mapsto \underline{A}_\lambda \in \mathscr{A}$ define:

$$\|\underline{A}_\lambda\| := \sup_\lambda \|\underline{A}_\lambda\|,$$
$$\underline{\alpha}_{(\Lambda,x)}(\underline{A})_\lambda := \mathrm{Ad}\, U(\Lambda, \lambda x)(\underline{A}_\lambda),$$

**Definition ([Buchholz-Verch '94])**

Local scaling algebra of $O$:

$$\underline{\mathfrak{A}}(O) := \left\{ \underline{A} : \underline{A}_\lambda \in \mathscr{A}(\lambda O), \lim_{(\Lambda,x)\to(\mathbb{1},0)} \|\underline{\alpha}_{(\Lambda,x)}(\underline{A}) - \underline{A}\| = 0 \right\}$$

Scaling algebra $\underline{\mathfrak{A}}$: C*-inductive limit of $\{\underline{\mathfrak{A}}(O)\}$, i.e. $\underline{\mathfrak{A}} := \overline{\bigcup_O \underline{\mathfrak{A}}(O)}$

Continuity w.r.t. Poincaré group action is equivalent to the requirement that $\underline{A}_\lambda$ occupies a phase-space volume independent of $\lambda$

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# Scaling Limits 1/2

$\varphi$ (locally normal) state on $\mathscr{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{A}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

**Theorem ([Buchholz-Verch '94])**

- $\mathrm{SL}^{\mathscr{A}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota \in I}$ *is independent of* $\varphi$.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{A}}$ *with GNS representation* $(\pi_{0,\iota}, \mathscr{H}_{0,\iota}, \Omega_{0,\iota})$. *Then* $\mathscr{A}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{A}}(O))''$ *is a net of local algebras with Poincaré group action defined by*

$$U_{0,\iota}(\Lambda, x)\pi_{0,\iota}(\underline{A})\Omega_{0,\iota} = \pi_{0,\iota}(\underline{\alpha}_{(\Lambda,x)}(\underline{A}))\Omega_{0,\iota}$$

*(If $s = 1$ the vacuum may be not unique)*

$O \mapsto \mathscr{A}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{A}$.
Physical interpretation: $\mathscr{A}_{0,\iota}$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

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# Scaling Limits 2/2

Classification:

- $\mathscr{A}$ has trivial scaling limit if $\mathscr{A}_{0,\iota} = \mathbb{C}\mathbb{1}$ for all $\underline{\omega}_{0,\iota}$
- $\mathscr{A}$ has unique scaling limit if all nets $\mathscr{A}_{0,\iota}$ are isomorphic and $\neq \mathbb{C}\mathbb{1}$
- $\mathscr{A}$ has degenerate scaling limit otherwise

Examples:

- $\mathscr{A}^{(m)}$ free scalar field of mass $m \geq 0$ in $s = 2, 3$ spatial dimensions $\implies \mathscr{A}_{0,\iota}^{(m)} \simeq \mathscr{A}^{(0)}$: $\mathscr{A}^{(m)}$ has unique limit [Buchholz-Verch '97]
- $\mathscr{A}$ Lutz model (suitable subnet of a generalized free field) has trivial limit [Lutz '98]

Open problem: construct nets with a degenerate scaling limit

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# States and representations

Physical states are represented by linear, positive, normalized functionals $\varphi : \mathscr{A} \to \mathbb{C}$.
GNS Theorem: { states on $\mathscr{A}$ } $\leftrightarrow$ { representations (i.e. *-homomorphisms on some $B(\mathscr{K})$) of $\mathscr{A}$ }
Problem: identify representations of interest in particle physics

**Definition (DHR selection criterion [Doplicher-Haag-Roberts '71])**

A representation $\pi$ is DHR if

$$\pi \restriction \mathscr{A}(O') \cong \pi_0 \restriction \mathscr{A}(O') \qquad \forall O$$

where $\pi_0(A) = A$ (vacuum representation)

Superselection sectors: unitary equivalence classes of DHR rep.
Interpreted as a physical charge localized in some bounded region

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# Doplicher-Roberts reconstruction 1/2

One would like to have a more explicit description of DHR states

**Theorem (DR reconstruction [Doplicher-Roberts '90])**

*If $s \geq 2$, there exist unique*

- *net $O \mapsto \mathscr{F}(O)$ on Hilbert space $\mathscr{K} \supset \mathscr{H}$ (canonical field net)*
- *$V$ unitary cont. rep. of compact group $G$ on $\mathscr{K}$ (canonical gauge group)*

*such that:*

- $\mathscr{A}(O) = \mathscr{F}(O)^G := \{F \in \mathscr{F}(O) \, : \, V(g)FV(g)^* = F, \forall g \in G\}$
- *for all superselection sectors $\xi$ of $\mathscr{A}$ and all $O$ there exist $\psi_1, \dots, \psi_d \in \mathscr{F}(O)$ orthogonal isometries ($\psi_j^* \psi_k = \delta_{jk}$) transforming like a $d$-dimensional irreducible representation of $G$ such that $\pi_\xi(A) = \sum_j \psi_j A \psi_j^*$ is a DHR representation of class $\xi$*
- *$\mathscr{F}$ satisfies $\mathbb{Z}_2$-graded commutation relations*

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# Doplicher-Roberts reconstruction 2/2

Main result of AQFT superselection theory, remarkable both from physical viewpoint:

- in the conventional approach to QFT charge carrying fields, their commutation relations, and gauge group are put in by assumption, and the observables are derived as the invariant part
- here, all this structure is fixed by the knowledge of the local observables

and from mathematical viewpoint:

- (suitable) DHR representation form the objects of a C*-category with additional properties
- above correspondence between sectors and representations of $G$ leads to the identification of the abstract dual of a compact group as a symmetric tensor C*-category with subobjects, direct sums and conjugates, generalizing Tannaka-Krein duality

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# Scaling limit of superselection sectors 1/2

Field scaling algebra $\underline{\mathfrak{F}}$ and scaling limit field net $\mathscr{F}_{0,\iota}$ defined in analogy to $\underline{\mathfrak{A}}$, $\mathscr{A}_{0,\iota}$ + continuity w.r.t. action of $G$ [D'Antoni-M-Verch '04]

$\exists G_{0,\iota} = G/N_{0,\iota}$ acting on $\mathscr{F}_{0,\iota}$ such that $\mathscr{A}_{0,\iota}(O)$ is the invariant part of $\mathscr{F}_{0,\iota}(O)$

General situation:

$$\begin{tikzcd}
 & \mathscr{A} \arrow[dl, "\mathrm{SL}"'] \arrow[dr, "\mathrm{DR}"] & \\
\mathscr{A}_{0,\iota} \arrow[dr, "\mathrm{DR}"'] & & \mathscr{F} \arrow[dl, "\mathrm{SL}"] \\
 & \mathscr{F}^{0,\iota} \supseteq \mathscr{F}_{0,\iota} &
\end{tikzcd}$$

There may be sectors of $\mathscr{A}_{0,\iota}$ which cannot be reached using fields from $\mathscr{F}_{0,\iota}$ (but only from $\mathscr{F}^{0,\iota}$). These should correspond to "confined charges" of particle physics, supposed to appear e.g. in QCD.

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# Scaling limit of superselection sectors 2/2

Problem: find a canonical way of identifying charges of $\mathscr{A}$ which are preserved in the limit

**Theorem ([D'Antoni-M-Verch '04])**

*The preserved charges are those associated to orthogonal isometries $\psi_j(\lambda) \in \mathscr{F}(\lambda O)$ such that $\psi_j(\lambda)^*\Omega$ has energy scaling as $\lambda^{-1}$*

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Schwinger model:

- 2d QED with massless fermions
- algebra of observables: $\mathscr{A}^{(m)}$ with $m > 0$, $s = 1$ [Lowenstein-Swieca '71]
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Scaling limit [Buchholz-Verch '97]:

- $\mathscr{A}^{(0)} \subset \mathscr{A}_{0,\iota}^{(m)}$
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# Outline

# Further directions

Summary:

- in AQFT the (ultraviolet) scaling limit is defined in an intrinsic, model independent fashion
- can be used to formulate an intrinsic notion of charge confinement

Further results:

- given $(G, N \triangleleft G)$, examples in which sectors corresponding to $G$-rep. trivial on $N$ preserved, while others are not, can be constructed [D'Antoni-M '06]
- provides a model independent framework for pointlike field renormalization [Bostelmann-D'Antoni-M '09 '10]
- scaling limit of interacting models in 2d can be studied [Bostelmann-Lechner-M '11]
- connections with noncommutative geometry (in particular with Connes-Higson asymptotic morphisms and KK-theory) can be established [Conti-M, to appear]

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