

Operator Algebras and the Renormalization Group in Quantum Field Theory

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Outline

- 1 Introduction
- 2 Algebraic Quantum Field Theory
- 3 Algebraic Renormalization Group
- 4 Superselection charges and their scaling limit
- 5 Further directions

Introduction

Renormalization Group (RG) is a method of analysis of the short distance limit of Quantum Field Theory (QFT - the mathematical framework of elementary particle physics).

It gave several important results:

- **asymptotic freedom** of nonabelian gauge theories
- corrections to **parton picture** of Deep Inelastic Scattering
- applications to **charge confinement**
- applications to Statistical Mechanics, Dynamical Systems...

Conceptual problem: formulated in Lagrangian approach, model-dependent

An **axiomatic approach to QFT** can be formulated in terms of Operator Algebras: **Algebraic Quantum Field Theory** (AQFT) [Haag-Kastler '64]

A model-independent version of RG adapted to AQFT, proposed in [Buchholz-Verch '94] gives several interesting structural results

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Algebras of Local Observables

Main assumption of [Haag-Kastler '64]: **A QFT is fixed by the knowledge of its local observables**

Basic input: assignment $O \mapsto \mathcal{A}(O)$, where

- O open bounded subset of Minkowski spacetime \mathbb{R}^{s+1}
- $\mathcal{A}(O)$ von Neumann algebras on a fixed Hilbert space \mathcal{H}
- if O_1 and O_2 spacelike separated (i.e. no signal can travel between them), then for all $A_1 \in \mathcal{A}(O_1)$, $A_2 \in \mathcal{A}(O_2)$

$$[A_1, A_2] = 0 \quad (\text{locality})$$

- $\exists U$ unitary cont. representation of Poincaré group $SO^\uparrow(1, s) \times \mathbb{R}^{s+1}$ on \mathcal{H} such that if $\alpha_{(\Lambda, x)} := \text{Ad } U(\Lambda, x)$ then

$$\alpha_{(\Lambda, x)}(\mathcal{A}(O)) = \mathcal{A}(\Lambda O + x) \quad (\text{covariance})$$

- $\exists ! \Omega \in \mathcal{H}$ $U(\mathbb{R}^{s+1})$ -invariant, s.t. $\overline{\bigcup_O \mathcal{A}(O)\Omega} = \mathcal{H}$ (**vacuum**)

This structure is a **local net of observable algebras**

Physical interpretation: self-adjoint $A \in \mathcal{A}(O)$ is an observable measurable in O

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Example: Free scalar field of mass $m \geq 0$

\mathfrak{W} **Weyl algebra**: C^* -algebra generated by unitaries $W(f)$, $f \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^{s+1})$, such that

$$W(f)W(g) = e^{i\sigma(f,g)} W(f+g)$$

$$\sigma(f,g) = \frac{1}{2} \operatorname{Im} \int_{\mathbb{R}^s} \frac{d^s p}{\omega_m(p)} \overline{\hat{f}(\omega_m(p), p)} \hat{g}(\omega_m(p), p)$$

$$\omega_m(p) = \sqrt{p^2 + m^2}$$

Fock vacuum: pos. norm. $\omega^{(m)} : \mathfrak{W} \rightarrow \mathbb{C}$ (**state**) determined by

$$\omega^{(m)}(W(f)) = \exp \left\{ -\frac{1}{4} \int_{\mathbb{R}^s} \frac{d^s p}{\omega_m(p)} |\hat{f}(\omega_m(p), p)|^2 \right\} \quad (m > 0 \text{ if } s = 1)$$

$(\pi^{(m)}, \mathcal{H}^{(m)}, \Omega^{(m)})$ corresponding GNS representation, i.e.

$$\omega^{(m)}(W) = \langle \Omega^{(m)}, \pi^{(m)}(W)\Omega^{(m)} \rangle, \quad W \in \mathfrak{W}$$

Local algebras

$$\mathcal{A}^{(m)}(\mathcal{O}) := \{\pi^{(m)}(W(f)) : \operatorname{supp} f \subset \mathcal{O}\}''$$

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Scaling Algebras

On C^* -algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \mapsto \underline{A}_\lambda \in \mathcal{A}$ define:

$$\|\underline{A}_\lambda\| := \sup_{\lambda} \|\underline{A}_\lambda\|,$$

$$\underline{\alpha}_{(\Lambda, X)}(\underline{A})_\lambda := \text{Ad } U(\Lambda, \lambda X)(\underline{A}_\lambda),$$

Definition ([Buchholz-Verch '94])

Local scaling algebra of \mathcal{O} :

$$\underline{\mathfrak{A}}(\mathcal{O}) := \left\{ \underline{A} : \underline{A}_\lambda \in \mathcal{A}(\lambda\mathcal{O}), \lim_{(\Lambda, X) \rightarrow (1, 0)} \|\underline{\alpha}_{(\Lambda, X)}(\underline{A}) - \underline{A}\| = 0 \right\}$$

Scaling algebra $\underline{\mathfrak{A}}$: C^* -inductive limit of $\{\underline{\mathfrak{A}}(\mathcal{O})\}$, i.e. $\underline{\mathfrak{A}} := \overline{\bigcup_{\mathcal{O}} \underline{\mathfrak{A}}(\mathcal{O})}$
 Continuity w.r.t. Poincaré group action is equivalent to the requirement that \underline{A}_λ occupies a phase-space volume independent of λ

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Scaling Limits

1/2

φ (locally normal) state on $\mathcal{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathcal{A}}$,

$SL^{\mathcal{A}}(\varphi) := \{\text{weak}^* \text{ limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \rightarrow 0\}$.

Theorem ([Buchholz-Verch '94])

- $SL^{\mathcal{A}}(\varphi) = (\underline{\omega}_{0,l})_{l \in I}$ is independent of φ .
- $\underline{\omega}_{0,l} \in SL^{\mathcal{A}}$ with GNS representation $(\pi_{0,l}, \mathcal{H}_{0,l}, \Omega_{0,l})$. Then $\mathcal{A}_{0,l}(O) := \pi_{0,l}(\underline{\mathcal{A}}(O))''$ is a net of local algebras with Poincaré group action defined by

$$U_{0,l}(\Lambda, x)\pi_{0,l}(\underline{A})\Omega_{0,l} = \pi_{0,l}(\underline{\alpha}_{(\Lambda, x)}(\underline{A}))\Omega_{0,l}$$

(If $s = 1$ the vacuum may be not unique)

$O \mapsto \mathcal{A}_{0,l}(O)$ is the **scaling limit net** of \mathcal{A} .

Physical interpretation: $\mathcal{A}_{0,l}$ describes the short-distance (i.e. high-energy) behaviour of \mathcal{A} .

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Scaling Limits

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Classification:

- \mathcal{A} has **trivial scaling limit** if $\mathcal{A}_{0,\ell} = \mathbb{C}\mathbb{1}$ for all $\omega_{0,\ell}$
- \mathcal{A} has **unique scaling limit** if all nets $\mathcal{A}_{0,\ell}$ are isomorphic and $\neq \mathbb{C}\mathbb{1}$
- \mathcal{A} has **degenerate scaling limit** otherwise

Examples:

- $\mathcal{A}^{(m)}$ free scalar field of mass $m \geq 0$ in $s = 2, 3$ spatial dimensions
 $\implies \mathcal{A}_{0,\ell}^{(m)} \simeq \mathcal{A}^{(0)}$: $\mathcal{A}^{(m)}$ has unique limit [Buchholz-Verch '97]
- \mathcal{A} Lutz model (suitable subnet of a generalized free field) has trivial limit [Lutz '98]

Open problem: construct nets with a degenerate scaling limit

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States and representations

Physical states are represented by linear, positive, normalized functionals $\varphi : \mathcal{A} \rightarrow \mathbb{C}$.

GNS Theorem: $\{ \text{states on } \mathcal{A} \} \leftrightarrow \{ \text{representations (i.e. } * \text{-homomorphisms on some } B(\mathcal{H}) \text{) of } \mathcal{A} \}$

Problem: identify representations of interest in particle physics

Definition (DHR selection criterion [Doplicher-Haag-Roberts '71])

A representation π is DHR if

$$\pi \upharpoonright \mathcal{A}(O') \cong \pi_0 \upharpoonright \mathcal{A}(O') \quad \forall O$$

where $\pi_0(A) = A$ (vacuum representation)

Superselection sectors: unitary equivalence classes of DHR rep. Interpreted as a physical charge localized in some bounded region

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Doplicher-Roberts reconstruction

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One would like to have a more explicit description of DHR states

Theorem (DR reconstruction [Doplicher-Roberts '90])

If $s \geq 2$, there exist unique

- net $O \mapsto \mathcal{F}(O)$ on Hilbert space $\mathcal{H} \supset \mathcal{H}$ (*canonical field net*)
- V unitary cont. rep. of compact group G on \mathcal{H} (*canonical gauge group*)

such that:

- $\mathcal{A}(O) = \mathcal{F}(O)^G := \{F \in \mathcal{F}(O) : V(g)FV(g)^* = F, \forall g \in G\}$
- for all superselection sectors ξ of \mathcal{A} and all O there exist $\psi_1, \dots, \psi_d \in \mathcal{F}(O)$ orthogonal isometries ($\psi_j^* \psi_k = \delta_{jk}$) transforming like a d -dimensional irreducible representation of G such that $\pi_\xi(A) = \sum_j \psi_j A \psi_j^*$ is a DHR representation of class ξ
- \mathcal{F} satisfies \mathbb{Z}_2 -graded commutation relations

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Main result of AQFT superselection theory, remarkable both from physical viewpoint:

- in the conventional approach to QFT charge carrying fields, their commutation relations, and gauge group are put in by assumption, and the observables are derived as the invariant part
- here, all this structure is fixed by the knowledge of the local observables

and from mathematical viewpoint:

- (suitable) DHR representation form the objects of a C^* -category with additional properties
- above correspondence between sectors and representations of G leads to the identification of the abstract dual of a compact group as a symmetric tensor C^* -category with subobjects, direct sums and conjugates, generalizing Tannaka-Krein duality

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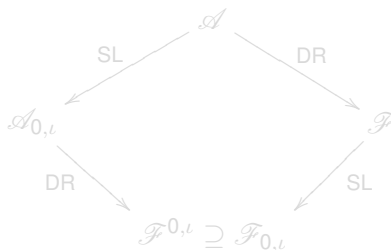
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Scaling limit of superselection sectors

1/2

Field scaling algebra \mathfrak{F} and **scaling limit field net** $\mathcal{F}_{0,\iota}$ defined in analogy to \mathcal{A} , $\mathcal{A}_{0,\iota}$ + continuity w.r.t. action of G [D'Antoni-M-Verch '04]
 $\exists G_{0,\iota} = G/N_{0,\iota}$ acting on $\mathcal{F}_{0,\iota}$ such that $\mathcal{A}_{0,\iota}(O)$ is the invariant part of $\mathcal{F}_{0,\iota}(O)$

General situation:



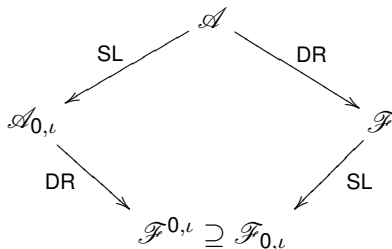
There may be sectors of $\mathcal{A}_{0,\iota}$ which cannot be reached using fields from $\mathcal{F}_{0,\iota}$ (but only from $\mathcal{F}^{0,\iota}$). These should correspond to “confined charges” of particle physics, supposed to appear e.g. in QCD.

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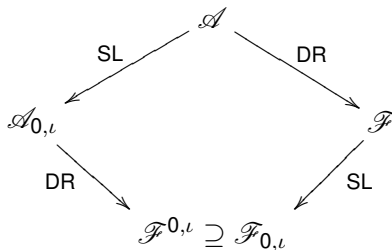
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 $\exists G_{0,\ell} = G/N_{0,\ell}$ acting on $\mathcal{F}_{0,\ell}$ such that $\mathcal{A}_{0,\ell}(O)$ is the invariant part of $\mathcal{F}_{0,\ell}(O)$

General situation:



There may be sectors of $\mathcal{A}_{0,\ell}$ which cannot be reached using fields from $\mathcal{F}_{0,\ell}$ (but only from $\mathcal{F}^{0,\ell}$). These should correspond to “confined charges” of particle physics, supposed to appear e.g. in QCD.

Scaling limit of superselection sectors

2/2

Problem: find a canonical way of identifying charges of \mathcal{A} which are preserved in the limit

Theorem ([D'Antoni-M-Verch '04])

The preserved charges are those associated to orthogonal isometries $\psi_j(\lambda) \in \mathcal{F}(\lambda O)$ such that $\psi_j(\lambda)^ \Omega$ has energy scaling as λ^{-1}*

Physical interpretation: preserved charges are pointlike (their localization only requires energy according to uncertainty principle, no internal structure) \Rightarrow they survive in the limit

This gives an intrinsic (i.e. only based on observables) notion of confinement:

Definition ([D'Antoni-M-Verch '04])

A **confined charge** of the theory defined by \mathcal{A} is a charge of \mathcal{A}_0 which does not come from a preserved charge of \mathcal{A}

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Example: the Schwinger model

Schwinger model:

- 2d QED with massless fermions
- algebra of observables: $\mathcal{A}^{(m)}$ with $m > 0$, $s = 1$ [Lowenstein-Swieca '71]
- no charged states $\Rightarrow \mathcal{F} = \mathcal{A}^{(m)}$ [Fröhlich-Morchio-Strocchi '79]
- interpreted as confinement of fermions

Scaling limit [Buchholz-Verch '97]:

- $\mathcal{A}^{(0)} \subset \mathcal{A}_{0,t}^{(m)}$
- \exists states ω_q on $\mathcal{A}_{0,t}^{(m)}$ s.t. $\omega_q(W(f)) = e^{iL(f)}\omega^{(0)}(W(f))$ where $L(f) = \mp\pi q\hat{f}(0)$ if $\text{supp } f$ is in the left/right spacelike complement of $\{0\} \times [-r, r] \implies \omega_q$ induces non-trivial (BF) sectors

Thus $\mathcal{F}^{0,t} \supsetneq \mathcal{F}_{0,t} = \mathcal{A}_{0,t}^{(m)}$ and **the Schwinger model has confined sectors**

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Outline

- 1 Introduction
- 2 Algebraic Quantum Field Theory
- 3 Algebraic Renormalization Group
- 4 Superselection charges and their scaling limit
- 5 Further directions**

Further directions

Summary:

- in AQFT the (ultraviolet) scaling limit is defined in an intrinsic, model independent fashion
- can be used to formulate an **intrinsic notion of charge confinement**

Further results:

- given $(G, N \triangleleft G)$, examples in which sectors corresponding to G -rep. trivial on N preserved, while others are not, can be constructed [D'Antoni-M '06]
- provides a model independent framework for **pointlike field renormalization** [Bostelmann-D'Antoni-M '09 '10]
- scaling limit of **interacting** models in 2d can be studied [Bostelmann-Lechner-M '11]
- connections with **noncommutative geometry** (in particular with Connes-Higson asymptotic morphisms and KK-theory) can be established [Conti-M, to appear]

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