# Scaling Algebras for Charge Carrying Quantum Fields and Superselection Structure in the Ultraviolet

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## Motivations

Main motivation: intrinsic understanding of charge confinement phenomenon in Quantum Field Theory.

Conventional notion of confinement is based on attaching a physical interpretation to unobservable degrees of freedom in the lagrangian (e.g. quark and gluon fields in QCD). But the same set of observables may be constructed starting from different lagrangians, as e.g. in the Schwinger model ($QED_2$ with massless fermions), whose observables are isomorphic to the massive scalar field.

D. Buchholz [Nucl. Phys. B **469** (1996)]: confined charges are those charges of the scaling limit theory which are not also charges of the underlying theory.

Intrinsic: both scaling limit construction [BV, RMP **7** (1995)] and superselection (i.e. charge) structure [DHR, CMP **23** (1971)] are canonically determined by the net of observables.

Example [BV, RMP **11** (1998)]: the scaling limit of the Schwinger model has nontrivial sectors, which are therefore confined in this sense.

But in general the two superselection structures are both nontrivial.

Problem: find a canonical way to identify (a subset of) charges of the scaling limit as coming from (a subset of) charges of the underlying theory which are *preserved* in the limit.

Then a confined charge is a charge of the scaling limit which doesn't come from a preserved charge.

Idea: try characterizing the preservance of charges by the scaling behaviour of the fields carrying them.

This leads one naturally to study the scaling limit for charge carrying fields.

## Scaling algebras and scaling limit

$(\mathscr{H}, \mathfrak{A}, U, \Omega)$ a Poincaré covariant local net, i.e.

(i) $\mathscr{O} \to \mathfrak{A}(\mathscr{O})$ local net of C*-algebras on $\mathscr{H}$, $\mathscr{O} \subset \mathbb{R}^4$ open double cone;

(ii) $U : \mathscr{P}_+^\uparrow \to \mathscr{U}(\mathscr{H})$ unitary repn such that

$$U(s)\mathfrak{A}(\mathscr{O})U(s)^* = \mathfrak{A}(s \cdot \mathscr{O}), \qquad s \in \mathscr{P}_+^\uparrow,$$

and $\mathrm{Sp}\, U(\mathbb{1}, \cdot) \subseteq \overline{V}_+$;

(iii) $\Omega \in \mathscr{H}$ vacuum (unique translation invariant vector), $\overline{\mathfrak{A}\Omega} = \mathscr{H}$;

such that $s \in \mathscr{P}_+^\uparrow \to \alpha_s(A) := U(s)AU(s)^*$ norm continuous for $A \in \mathfrak{A}(\mathscr{O})$ and $\mathfrak{A}(\mathscr{O})$ maximal.

The relevant feature of RG transformation is that RG orbits occupy a fixed phase space volume at all scales: lengths scale with $\lambda$ and 4-momenta scale with $\lambda^{-1}$ [BV95].

Definition. The *local scaling algebra* of $\mathscr{O}$ is

$$\underline{\mathfrak{A}}(\mathscr{O}) := \{\lambda \in \mathbb{R}_+^\times \to \underline{A}(\lambda) \in \mathfrak{A}(\lambda\mathscr{O}) : \\ \|\underline{A}\| < +\infty, \lim_{s \to e} \|\underline{\alpha}_s(\underline{A}) - \underline{A}\| = 0\}.$$

Here $\|\underline{A}\| := \sup_{\lambda > 0} \|\underline{A}(\lambda)\|$ and $\underline{\alpha}_s(\underline{A})(\lambda) := \alpha_{s_\lambda}(\underline{A}(\lambda))$, $(\Lambda, x)_\lambda = (\Lambda, \lambda x)$.

$(\underline{\mathfrak{A}}, \underline{\alpha})$ local Poincaré covariant net.

If $\varphi$ (locally normal) state on $\mathfrak{A}$:

$$\underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}(\lambda))$$
$$\mathrm{SL}_{\mathfrak{A}}(\varphi) := \left\{\text{weak* limit pts of } \left(\underline{\varphi}_\lambda\right)_{\lambda>0}\right\}.$$

Theorem.[BV95] $\mathrm{SL}_{\mathfrak{A}}(\varphi)$ is independent of $\varphi$. For $(\pi_0, \mathscr{H}_0, \Omega_0)$ and $U_0$ the GNS and $\mathscr{P}_+^\uparrow$ representations determined by $\underline{\omega}_0 \in \mathrm{SL}_{\mathfrak{A}}$, and with $\mathfrak{A}_0(\mathscr{O}) := \pi_0(\underline{\mathfrak{A}}(\mathscr{O}))$, $(\mathscr{H}_0, \mathfrak{A}_0, U_0, \Omega_0)$ is a Poincaré covariant local net (if $d = 2$ the vacuum may not be pure).

$\mathfrak{A}_0$ is called a scaling limit net of $\mathfrak{A}$. Possibilities:

(i) degenerate scaling limit: the various $\mathfrak{A}_0$ non-isomorphic (cfr. theories without ultraviolet fixed points);

(ii) unique (quantum) scaling limit: the various $\mathfrak{A}_0$ isomorphic and non-trivial (e.g. scalar field in $d = 3, 4$ [BV98], dilation invariant theories);

(iii) classical scaling limit: each $\mathfrak{A}_0(\mathscr{O}) = \mathbb{C}\mathbb{1}$, (linked to exceptional quantum behaviour of observables at small scales as in e.g. non-renormalizable theories, cfr. also [L, Diploma (1997)]).

## Superselection theory

Interesting states in particle physics describe localized excitation of the vacuum $\Longrightarrow$ criteria are needed to select the relevant states on $\mathfrak{A}$. Superselection sectors are equivalence classes of corresponding irreducible representations.

Definition.[DHR71] A representation $\pi$ is DHR (or describes a *localizable charge*) if

$$\pi \restriction \mathfrak{A}(\mathscr{O}') \cong \iota \restriction \mathfrak{A}(\mathscr{O}'), \qquad \forall \mathscr{O}, \tag{1}$$

with $\iota$ the vacuum representation.

Excludes states carrying electric charge, due to Gauss' law. Also, in massive theories the most general localization property for charges is

Definition.[BF, CMP **84** (1982)] A representation $\pi$ is BF (or describes a *topological charge*) if

$$\pi \restriction \mathfrak{A}(\mathscr{C}') \cong \iota \restriction \mathfrak{A}(\mathscr{C}')$$

for all spacelike cones $\mathscr{C} = a + \mathbb{R}_+^\times \mathscr{O}$, $\mathscr{O} \subseteq \{0\}'$, $\mathscr{C}$ causally complete.

Expected to show up in nonabelian gauge theories (cone = flux string).

Assuming Haag duality $\mathscr{A}(\mathscr{O}) = \mathfrak{A}(\mathscr{O}')'$ ($\mathscr{A}(\mathscr{O}) := \mathfrak{A}(\mathscr{O})''$, net of von Neumann algebras), if $\pi \in \mathrm{DHR}$ and $V \in \mathscr{U}(\mathscr{H}_\pi, \mathscr{H})$ realizes (1), with

$$\rho(A) := V\pi(A)V^*, \qquad A \in \mathfrak{A},$$

$\rho \in \mathrm{End}(\mathfrak{A})$ is localized: $\rho(A) = A$, $\forall A \in \mathfrak{A}(\mathscr{O}')$, and transportable: $\forall \mathscr{O}_1$, $\exists \rho_1 \cong \rho$, localized in $\mathscr{O}_1$. In symbols $\rho \in \Delta(\mathscr{O})$. If $\pi \in \mathrm{BF}$ then $\rho \in \mathrm{Hom}(\mathfrak{A}, B(\mathscr{H}))$ is localized in some $\mathscr{C}$ and transportable.

$$(\rho : \sigma) := \{T \in \mathfrak{A} \,:\, T\rho(A) = \sigma(A)T,\ A \in \mathfrak{A}\}$$

(global) intertwiners, the local ones, $(\rho : \sigma)_{\mathscr{O}}$, are defined by $T\rho(A) = \sigma(A)T$, $A \in \mathfrak{A}(\mathscr{O})$.

Structures on the set of sectors [DHR71, DHR74, BF82]:

(i) composition, induced by $(\rho, \sigma) \in \Delta \times \Delta \to \rho\sigma \in \Delta$;

(ii) statistics, described by unitary repns $p \in S_n \to \varepsilon_\rho^{(n)}(p) \in (\rho^n : \rho^n)$ classified by $\lambda_\rho \in \{0\} \cup \{\pm 1/d, d \in \mathbb{N}\}$; $\lambda_\rho \neq 0$ finite statistics, $\implies$ : $\rho \in \Delta_f$;

(iii) conjugation: $\forall \rho \in \Delta_f(\mathscr{O})$, $\exists \overline{\rho} \in \Delta_f(\mathscr{O})$ and $R \in (\iota : \overline{\rho}\rho)$, $\overline{R} \in (\iota : \rho\overline{\rho})$ isometries.

In the Wightman approach to QFT, unobservable fields transfer superselection quantum numbers between sectors. This is actually encoded in the structure of the net of observables.

Definition. A *normal, $\tilde{\mathscr{P}}_+^\uparrow$-covariant QFT with gauge group action* (QFTGA) is a quintuple $(\mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ such that

(i) $(\mathscr{F}, U_{\mathscr{F}})$ is a $\tilde{\mathscr{P}}_+^\uparrow$-covariant net of von Neumann algebras on a Hilbert space $\mathscr{H}_{\mathscr{F}}$, and $\operatorname{Sp} U_{\mathscr{F}}(\mathbb{1}, \cdot) \subseteq \overline{V}_+$;

(ii) $V$ is a faithful unitary repn of a compact group $G$ on $\mathscr{H}_{\mathscr{F}}$, such that $\beta_g(\mathscr{F}(\mathscr{O})) = \mathscr{F}(\mathscr{O})$, and $\alpha_s := \operatorname{Ad} U_{\mathscr{F}}(s)$, $\beta_g := \operatorname{Ad} V(g)$ commute;

(iii) $V(g)\Omega = \Omega$, $\Omega$ is the unique translation invariant vector in $\mathscr{H}_{\mathscr{F}}$, and $\overline{\mathscr{F}\Omega} = \mathscr{H}_{\mathscr{F}}$;

(iv) $k \in Z(G)$, $k^2 = e$, and $\mathscr{F}$ obeys local $\mathbb{Z}_2$-graded commutativity w.r.t $\gamma := \beta_k$, i.e. has *normal commutation relations*: if $F_i \in \mathscr{F}(\mathscr{O}_i)$, $i = 1, 2$, $\mathscr{O}_1 \subset \mathscr{O}_2'$, and $F_{i,\pm} := \frac{1}{2}(F_i \pm \gamma(F_i))$,

$$[F_{1,+}, F_{2,+}] = [F_{1,+}, F_{2,-}] = \{F_{1,-}, F_{2,-}\} = 0.$$

Theorem.[DR, CMP **131** (1990)] There exist a QFTGA $(\mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ and a repn $\pi_{\mathscr{F}}$ of $\mathfrak{A}$ on $\mathscr{H}_{\mathscr{F}}$, such that

(i) $\mathscr{H} \subset \mathscr{H}_{\mathscr{F}}$ and $\pi_{\mathscr{F}}(\cdot) \restriction \mathscr{H} = \iota$;

(ii) $\mathscr{F}(\mathscr{O})^G = \pi_{\mathscr{F}}(\mathscr{A}(\mathscr{O}))$;

(iii) $\forall \rho \in \Delta_f(\mathscr{O})$ irreducible and covariant, $\exists\, \psi_1, \ldots, \psi_d \in \mathscr{F}(\mathscr{O})$ and $v_{[\rho]}$ $d$-dim. irrepn of $G$ such that

$$\psi_i^* \psi_j = \delta_{ij} \mathbb{1}, \qquad \sum_{j=1}^{d} \psi_j \psi_j^* = \mathbb{1},$$

$$\beta_g(\psi_i) = \sum_{j=1}^{d} v_{[\rho]}(g)_{ij} \psi_j,$$

$$\rho(A) = \sum_{j=1}^{d} \psi_j A \psi_j^*, \qquad A \in \mathfrak{A}.$$

(iv) $\mathscr{F}(\mathscr{O})$ is generated as a von Neumann algebra by $\mathscr{A}(\mathscr{O})$ and the multiplets $\psi_j$.

$(\pi_{\mathscr{F}}, \mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ is unique up to unitary equivalence.

There is an analogous theorem for topological charges [DR90], the main difference being that the field net $\mathscr{C} \to \mathscr{F}(\mathscr{C})$ is indexed by spacelike cones.

## Scaling algebras and scaling limit for (DHR) charge carrying fields

$(\mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ a QFTGA. Construction proceeds along the lines of the observable case. For $\underline{F} : \mathbb{R}_+^\times \to \mathfrak{F}$, $s \in \tilde{\mathscr{P}}_+^\uparrow$, $g \in G$,

$$\|\underline{F}\| := \sup_{\lambda>0} \|\underline{F}(\lambda)\|, \quad \underline{\alpha}_s(\underline{F})(\lambda) := \alpha_{s_\lambda}(\underline{F}(\lambda)),$$
$$\underline{\beta}_g(\underline{F})(\lambda) := \beta_g(\underline{F}(\lambda)).$$

Definition. The *local scaling field algebra* of $\mathscr{O}$ is

$$\underline{\mathfrak{F}}(\mathscr{O}) := \{\lambda \in \mathbb{R}_+^\times \to \underline{F}(\lambda) \in \mathscr{F}(\lambda\mathscr{O}) \ : \ \|\underline{F}\| < +\infty,$$
$$\lim_{s \to e} \|\underline{\alpha}_s(\underline{F}) - \underline{F}\| = 0, \lim_{g \to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0\}.$$

$\implies$ We restrict to "dimensionless" charges.

For $\varphi$ locally normal state on $\mathfrak{F}$, $(\underline{\varphi}_\lambda)_{\lambda>0}$ and $\mathrm{SL}_{\mathfrak{F}}(\varphi)$ defined as for observables.

Theorem. $\mathrm{SL}_{\mathfrak{F}}(\varphi)$ is independent of $\varphi$. For $\underline{\omega}_0 \in \mathrm{SL}_{\mathfrak{F}}$ there holds

$$\underline{\omega}_0 \circ \underline{\alpha}_s = \underline{\omega}_0, \qquad \underline{\omega}_0 \circ \underline{\beta}_g = \underline{\omega}_0,$$

and let $(\pi_0, \mathscr{H}_0, \Omega_0)$ and $U_0$, $\tilde{V}_0$ be the corresponding GNS, $\tilde{\mathscr{P}}_+^\uparrow$ and $G$ representations. Then if

$$N_0 := \{g \in G : \tilde{V}_0(g)\Phi = \Phi, \forall \Phi \in \mathscr{H}_0\},$$

$$V_0(gN_0) := \tilde{V}_0(g),$$

$V_0$ is a faithful unitary representation of $G_0 := G/N_0$, and with

$$\mathscr{F}_0(\mathscr{O}) := \pi_0(\underline{\mathfrak{F}}(\mathscr{O}))'',$$

$(\mathscr{F}_0, U_0, V_0, kN_0, \Omega_0)$ is a QFTGA with the property that $\mathscr{F}_0(\mathscr{O})^G = \mathscr{A}_0(\mathscr{O})$, and is called a *scaling limit* QFTGA of $\mathscr{F}$.

# Preservance of DHR charges

Some charges may disappear in the scaling limit. E.g. if the scaling limit is classical, $G^0 = \{e\}$ (canonical DR gauge group of $\mathscr{A}_0$) $\implies$ all charges disappear. Conditions are then needed to select charges which are preserved.

$(\mathscr{H}, \mathscr{A}, U, \Omega)$ local covariant net, and $(\pi_{\mathscr{F}}, \mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ corresponding canonical DR net determined by DHR sectors.

Idea: study the behaviour, for $\lambda \to 0$, of *scaled multiplets* $\psi_1(\lambda), \dots \psi_d(\lambda) \in \mathscr{F}(\lambda\mathscr{O})$ associated to a fixed sector $\xi$.

But in general $\psi_j(\cdot) \notin \underline{\mathfrak{F}}(\mathscr{O})$ because of lacking of the right phase space properties.

Definition. A finite statistics covariant DHR sector $\xi$ is *preserved in the scaling limit QFTGA* $\mathscr{F}_0$ if $\forall \mathscr{O}_1$, $\exists \psi_1(\lambda), \dots, \psi_d(\lambda) \in \mathscr{F}(\lambda\mathscr{O}_1)$ multiplet of class $\xi$, such that $\forall \varepsilon > 0$, $\exists \underline{F}_j, \underline{F}'_j \in \underline{\mathfrak{F}}$ such that

$$\limsup_{\lambda\to 0} \|[\psi_j(\lambda) - \underline{F}_j(\lambda)]\Omega\| + \|[\psi_j(\lambda) - \underline{F}'_j(\lambda)]^*\Omega\| < \varepsilon.$$

Amounts essentially to requiring that the states $\psi_j(\lambda)\Omega$, describing a charge roughly localized in $\lambda\mathscr{O}$, have energy scaling at most as $\lambda^{-1}$, i.e. a charge is preserved if it doesn't require too much energy in order to be localized in smaller regions. It is satisfied in the Majorana-Dirac free field model in all scaling limits.

For $h \in C_c(\tilde{\mathscr{P}}_+^\uparrow)$, with $h \geq 0$, $\int_{\tilde{\mathscr{P}}_+^\uparrow} h = 1$, define

$$\underline{\alpha}_h \psi_j(\lambda) := \int_{\tilde{\mathscr{P}}_+^\uparrow} ds\, h(s) \alpha_{s_\lambda}(\psi_j(\lambda)),$$

then $\underline{\alpha}_h \psi_j \in \underline{\mathfrak{F}}(\mathscr{O})$, $\forall \mathscr{O} \supset \overline{\mathscr{O}_1}$.

Theorem. With $\xi$ and $\psi_j(\lambda)$ as above, there exists

$$\boldsymbol{\psi}_j = {}^*s\!-\!\lim_{h\to\delta} \pi_0(\underline{\alpha}_h \psi_j) \in \mathscr{F}_0(\mathscr{O})$$

for each $\mathscr{O} \supset \overline{\mathscr{O}}_1$, and $\boldsymbol{\psi}_j$ is a $G_0$-multiplet of class $\xi$ of orthogonal isometries of support $\mathbb{1}$, i.e.

$$\boldsymbol{\psi}_i^* \boldsymbol{\psi}_j = \delta_{ij}\mathbb{1}, \qquad \sum_{j=1}^{d} \boldsymbol{\psi}_j \boldsymbol{\psi}_j^* = \mathbb{1},$$

$$V_0(gN_0)\boldsymbol{\psi}_i V_0(gN_0)^* = \sum_{j=1}^{d} v_\xi^0(gN_z)_{ij}\boldsymbol{\psi}_j,$$

with $v_\xi^0(gN_0) := v_\xi(g)$ a well-defined irrep of $G_0$.

Furthermore,

$$\boldsymbol{\rho}(\boldsymbol{a}) := \sum_{j=1}^{d} \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^*, \qquad \boldsymbol{a} \in \mathfrak{A}_0,$$

is a localized, transportable irreducible endomorphism of $\mathfrak{A}_0$ which is covariant with positive energy and finite statistics.

If $\mathscr{F}_0$ satisfies a mild additional assumption (*geometric modular action*), a charge is preserved if and only if the conjugate charge is also preserved, and the charges obtained as above in the scaling limit are conjugate one to the other.

If all sectors are preserved, much of the superselection structure can be determined locally.

Theorem. If all the $\mathscr{F}(\mathscr{O})$ are factors, $\mathscr{F}(\mathscr{O}) \cap \mathscr{F}(\mathscr{O})' = \mathbb{C}\mathbb{1}$, and each sector is preserved in some scaling limit, then local intertwiners are equivalent to global ones

$$(\rho : \sigma) = (\rho : \sigma)_{\mathscr{O}}, \qquad \forall \mathscr{O}$$

This is a generalization of a result of Roberts for dilation invariant theories [R, CMP **37** (1974)], and it is useful on curved spacetimes.

## Scaling algebras for fields in cones and preservance of BF charges

Topological charges expected to appear in nonabelian gauge theories, which should also exhibit confinement $\Longrightarrow$ above analysis too narrow, if quarks are non-confined, they are localized in cones.

Spacelike cones are not affected by rescaling $\Longrightarrow$ unclear how to implement the RG phase space.

But at least in asymptotically free theories charges in cones should become localized in the scaling limit (the flux string vanishes) $\Longrightarrow$ phase space recoverd asymptotically.

$(\mathscr{H}, \mathscr{A}, U, \Omega)$ local covariant net, and $(\pi_{\mathscr{F}}, \mathscr{F}, U_{\mathscr{F}}, V, k, \Omega)$ corresponding canonical DR net determined by BF sectors. Additional assumptions:

(i) weak additivity for $\mathfrak{A}$: $\bigvee_{x\in\mathbb{R}^4} \mathscr{A}(\mathscr{O}+x) = B(\mathscr{H})$;

(ii) $\forall \mathscr{O} \ni 0$, $\mathscr{A}(\mathscr{O}) = \bigvee_{\mathscr{O}_0 \ni 0} \mathscr{A}(\mathscr{O} \setminus \mathscr{O}_0)$, suggested by $\bigcap_{\mathscr{O}_0\ni 0} \mathscr{A}(\mathscr{O}_0) = \mathbb{C}\mathbb{1}$ and satisfied in models;

(iii) geometric modular action for $\mathscr{F}$: $\Delta^{it} = U_{\mathscr{F}}(\Lambda_{\mathscr{W}}(2\pi t))$, with $\Delta$ modular operator of $(\mathscr{F}(\mathscr{W})^-, \Omega)$, and $\Lambda_{\mathscr{W}}(\cdot)$ boosts leaving the wedge $\mathscr{W}$ invariant.

A bounded function $F : \mathbb{R}_+^\times \to \mathscr{F}$ is asymptotically localized in $\mathscr{O}$ if

$$\lim_{\lambda \to 0} \sup_{A \in \underline{\mathfrak{A}}(\mathscr{O}')_1} \|[F(\lambda), \underline{A}(\lambda)]\| = 0.$$

Actions $\underline{\alpha}_s$ and $\underline{\beta}_g$ of $s \in \tilde{\mathscr{P}}_+^\uparrow$, $g \in G$ defined as above.

Definition. We introduce the C*-algebras, for $\mathscr{O} \subset \mathscr{C}$,

$$\begin{aligned}\underline{\mathfrak{F}}(\mathscr{C}, \mathscr{O}) := \{\lambda \in \mathbb{R}_+^\times \to \underline{F}(\lambda) \in \mathscr{F}(\lambda\mathscr{C}) \ : \ &\|\underline{F}\| < +\infty \\ &\lim_{s \to e} \|\underline{\alpha}_s(\underline{F}) - \underline{F}\| = 0, \ \lim_{g \to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0 \\ &\underline{F} \text{ asymptotically localized in } \mathscr{O}\},\end{aligned}$$

$$\underline{\mathfrak{F}}^\times(\mathscr{O}) := C^*\Big(\bigcup_{\mathscr{C} \supset \mathscr{O}} \underline{\mathfrak{F}}(\mathscr{C}, \mathscr{O})\Big).$$

$\underline{\mathfrak{F}}^\times$ is the *extended scaling algebra of asymptotically localized fields.*

$(\underline{\mathfrak{F}}^\times, \underline{\alpha})$ is a covariant net, but not normal. For $\varphi$ normal state on $B(\mathscr{H}_\mathscr{F})$, states $(\underline{\varphi}_\lambda)_{\lambda > 0}$ on $\underline{\mathfrak{F}}^\times$ and $\mathrm{SL}_{\underline{\mathfrak{F}}^\times}(\varphi)$ defined as above.

Theorem. $\mathrm{SL}_{\underline{\mathfrak{F}}^\times}(\varphi)$ is independent of $\varphi$. For $\underline{\omega}_0 \in \mathrm{SL}_{\underline{\mathfrak{F}}^\times}$, let $(\pi_0^\times, \mathscr{H}_0^\times, \Omega_0)$ and $U_0^\times$, $\tilde{V}_0^\times$ be as in the DHR case, and let

$$\mathscr{F}_0^\times(\mathscr{O}) := \pi_0^\times(\underline{\mathfrak{F}}^\times(\mathscr{O})).$$

Then $\Omega_0$ is cyclic and separating for $\mathscr{F}_0^\times(\mathscr{W})^-$. Furthermore if

$$\mathscr{F}_0(\mathscr{O}) := \bigcap_{\mathscr{C} \supset \mathscr{O}} \pi_0^\times(\underline{\mathfrak{F}}(\mathscr{C}, \mathscr{O}))'',$$

and $\mathscr{H}_0 := \overline{\bigcup_{\mathscr{O}} \mathscr{F}_0(\mathscr{O})\Omega_0}$, with $U_0 := U_0^\times(\cdot) \upharpoonright \mathscr{H}_0$, $\tilde{V}_0 := \tilde{V}_0^\times(\cdot) \upharpoonright \mathscr{H}_0$ and $N_0$, $G_0$ and $V_0$ defined as in the DHR case, $(\mathscr{F}_0, U_0, V_0, kN_0, \Omega_0)$ is a QFTGA, called a *scaling limit* QFTGA of $\mathscr{F}$.

Cyclicity of $\Omega_0$ is not completely trivial, and uses analyticity of boosts, consequence of geometric modular action, as in [BB, AHP **70** (1999)].

We obtain a net indexed by double cones in the scaling limit, as expected in the asymptotically free case. The problem is that it may be too small.

The direction of the cone is irrelevant, so that if the charges are asymptotically localizable in bounded regions, each choice of direction should give the same states in the limit.

**Definition.** A finite statistics covariant BF sector $\xi$ is *preserved in the scaling limit QFTGA* $\mathscr{F}_0$ if $\forall \mathscr{O}_1$, $\forall \mathscr{C}_1 \supset \mathscr{O}_1$, $\exists \psi_1(\lambda), \ldots, \psi_d(\lambda) \in \mathscr{F}(\lambda \mathscr{C}_1)$ multiplet of class $\xi$, such that $\lambda \to \psi_j(\lambda)$ is asymptotically localized in $\mathscr{O}_1$, and $\forall \mathscr{C} \supset \mathscr{O}_1$ $\exists \lambda \to \psi_j^{\mathscr{C}}(\lambda) \in \mathscr{F}(\lambda \mathscr{C})$ which is bounded, asymptotically localized in $\mathscr{O}_1$ and with $\lim_{g \to e} \|\beta_g(\psi_j^{\mathscr{C}}(\lambda)) - \psi_j^{\mathscr{C}}(\lambda)\| = 0$, such that

(i)

$$\lim_{\lambda \to 0} \|[\psi_j^{\mathscr{C}}(\lambda) - \psi_j(\lambda)]\Omega\| + \|[\psi_j^{\mathscr{C}}(\lambda) - \psi_j(\lambda)]^*\Omega\| = 0;$$

(ii) $\forall \varepsilon > 0$, $\exists \underline{F}_j^{\mathscr{C}}, \underline{F}_j^{\mathscr{C}\prime} \in \underline{\mathfrak{F}}^\times$ such that

$$\limsup_{\lambda \to 0} \|[\psi_j^{\mathscr{C}}(\lambda) - \underline{F}_j^{\mathscr{C}}(\lambda)]\Omega\| + \|[\psi_j^{\mathscr{C}}(\lambda) - \underline{F}_j^{\mathscr{C}\prime}(\lambda)]^*\Omega\| < \varepsilon.$$

Generalizes the preservance condition for DHR charges (take $\psi_j^{\mathscr{C}}(\cdot) = \psi_j(\cdot)$, $\underline{F}_j^{\mathscr{C}} = \underline{F}_j$).

Theorem. With $\xi$ and $\psi_j(\lambda)$ as above, there exists

$$\boldsymbol{\psi}_j = {}^*s\text{-}\lim_{h\to\delta} \pi_0(\underline{\alpha}_h \psi_j^{\mathscr{C}})$$

independent of $\mathscr{C}$, so that $\boldsymbol{\psi}_j \in \mathscr{F}_0(\mathscr{O})$, for each $\mathscr{O} \supset \overline{\mathscr{O}}_1$, and $\boldsymbol{\psi}_j$ is a $G_0$-multiplet of class $\xi$ of orthogonal isometries of support $\mathbb{1}$. Furthermore if

$$\omega_\xi(\boldsymbol{a}) := \sum_{j=1}^{d} (\Omega_0 | \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^* \Omega_0), \qquad \boldsymbol{a} \in \mathfrak{A}_0,$$

then $\omega_\xi \restriction \mathfrak{A}_0(\mathscr{O}') = \omega_0 \restriction \mathfrak{A}_0(\mathscr{O}')$ and the GNS representation $(\pi_\xi, \mathscr{H}_\xi, \Omega_\xi)$ of $\omega_\xi$ is translation covariant with positive energy.

Argument in [DHR71] shows then that if $\pi_\xi$ is irreducible and locally normal, then it is DHR. There are indications that this should hold quite generally (e.g. if the modular group of $\overline{V}_+$ acts by dilations in the scaling limit, in which case $\mathscr{A}_0(\mathscr{O})$ is a factor [BDLR, RMP **SI1** (1992)]).