Renormalization of Pointlike Quantum Fields from Scaling Algebras

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Outline

Introduction

Conventional approach to the Renormalization Group:

- Pass from ϕ to renormalized field $\phi_\lambda(x) = Z_\lambda \phi(\lambda x)$;
- Renormalization constants Z_λ fixed by requiring e.g. $\langle \Omega, \phi_\lambda(x)\phi_\lambda(y)\Omega \rangle$ has finite limit as $\lambda \to 0$.

Problems:

- many choices of Z_λ: all equivalent?
- why not more general renormalization?
- fields do not have direct physical interpretation.

Algebraic approach [Buchholz-Verch '95]:

- Z_λ not needed;
- based only on observables and model independent.

How does it compare to the conventional approach? How can we recover Z_λ?

Scaling Algebras

Data:

- $O \subset \mathbb{R}^4 \to \mathscr{A}(O) \subset B(\mathscr{H})$ net of observable algebras.
- $(\Lambda, x) \to U(\Lambda, x)$ unitary representation of Poincaré group.
- $\Omega \in \mathscr{H}$ vacuum.

On C*-algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \to \underline{A}_\lambda \in \mathscr{A}$ define

- $\underline{\alpha}_{(\Lambda,x)}(\underline{A})_\lambda := \mathrm{Ad}\, U(\Lambda, \lambda x)(\underline{A}_\lambda)$;
- $\underline{\delta}_\mu(\underline{A})_\lambda := \underline{A}_{\lambda\mu}$;
- $\underline{\alpha}_g := \underline{\alpha}_{(\mu,\Lambda,x)} := \underline{\delta}_\mu \circ \underline{\alpha}_{(\Lambda,x)}$.

Local scaling algebra of O:

$$\underline{\mathfrak{A}}(O) := \left\{ \underline{A} \,:\, \underline{A}_\lambda \in \mathscr{A}(\lambda O), \lim_{g \to \mathrm{id}} \|\underline{\alpha}_g(\underline{A}) - \underline{A}\| = 0 \right\}$$

Scaling Algebras

$\mathbf{m}$ mean on $(0, 1]$, i.e. state on $\mathscr{B}((0, 1])$,

$$\underline{\omega}_0(\underline{A}) := \mathbf{m}(\lambda \to \omega(\underline{A}_\lambda))$$

scaling limit state on $\underline{\mathfrak{A}}$.
Examples:

1. $\mathbf{m}_{\lambda_0}(f) = f(\lambda_0)$, then $\underline{\omega}_0$ vacuum state at scale λ_0;
2. $\mathbf{m}$ weak* limit point of $\mathbf{m}_{\lambda_0}$ as $\lambda_0 \to 0$, then $\underline{\omega}_0$ Buchholz-Verch limit state;
3. $\mathbf{m}$ invariant: $\mathbf{m}(f(\mu\cdot)) = \mathbf{m}(f)$.

2 and 3 generalizations of limit $\lambda \to 0$: $\mathbf{m}(f) = \lim_{\lambda\to 0} f(\lambda)$ if limit exists.

Scaling Algebras

Theorem

- $(\pi_0, \mathscr{H}_0, \Omega_0)$ *GNS representation of* $\underline{\omega}_0 \implies$ $\mathscr{A}_0(O) := \pi_0(\underline{\mathfrak{A}}(O))''$ *Poincaré covariant net in vacuum representation: scaling limit net;*
- *in case (1 and) 2* $\underline{\omega}_0$ *is pure, in 3 it is not;*
- *in case 3* $\mathscr{A}_0$ *also dilation covariant.*

Example: $\mathscr{A}$ massive free field in $d = 3 + 1$:

- $\mathscr{A}_0(O) \cong \mathscr{A}_{\text{massless}}(O) \bar{\otimes} \pi_0(\mathfrak{Z}(\underline{\mathfrak{A}}))''$;
- $\underline{\omega}_0$ pure $\implies \mathscr{A}_0(O) \cong \mathscr{A}_{\text{massless}}(O)$;
- $\underline{\omega}_0$ dilation invariant $\implies \overline{\pi_0(\mathfrak{Z}(\underline{\mathfrak{A}}))''\Omega_0}$ non-separable, $U_0(g) \cong U_{\text{massless}}(g) \otimes U_3(g)$.

Pointlike Fields from Local Algebras

Basic idea [Haag-Ojima '96]: assume

$$\Sigma_{E,r} = \{\sigma \restriction \mathscr{A}(O_r) \, : \, \sigma \in P(E)B(\mathscr{H})_* P(E)\}$$

is compact and "does not change" for small r

$\implies$ "finite" number of states describe short distance behaviour

$\implies$ basis (ϕ_j) of $\Sigma^*_{E,r}$ are pointlike fields.

Pointlike Fields from Local Algebras

Quantitative version:

- $\Sigma = B(\mathscr{H})_*$, $C^\infty(\Sigma) = \cap_{\ell>0} R^\ell \Sigma R^\ell$, $R = (1+H)^{-1}$;
- $\|\sigma\|^{(\ell)} = \|R^{-\ell}\sigma R^{-\ell}\|$, $\sigma \in C^\infty(\Sigma)$;
- $\Xi : \sigma \in C^\infty(\Sigma) \to \sigma \in \Sigma$.

Definition ([Bostelmann '05])

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition I if $\forall \gamma > 0$, $\exists \ell > 0$, $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi\|^{(\ell)} < \infty,$$

$$\|(\Xi - \psi)(\cdot) \restriction \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields from Local Algebras

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma := \text{span}\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ;
- $\cup_{\gamma>0}\Phi_\gamma = \{\phi : \exists \ell > 0,\ R^\ell \phi R^\ell \in \cap_{r>0}(R^\ell \mathscr{A}(O_r) R^\ell)^-\}$.

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$ [Fredenhagen-Hertel '81].
ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 = \text{span}\{\mathbb{1}, \phi\}$, $\Phi_2 = \text{span}\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$,...

Pointlike Fields from Local Algebras

According to phase space condition, if $A \in \mathscr{A}(O_r)$:

$$A \sim \sum_j \sigma_j(A)\phi_j \qquad \text{as } r \to 0.$$

Can be generalized to unbounded objects by $\varepsilon/3$ argument.

Theorem ([Bostelmann '05])

$\phi, \phi' \in \Phi_\gamma$. For all $\beta > 0$ exist $\sigma_j \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ such that

$$\|\phi(f)\phi'(f') - \sum_j \sigma_j(\phi(f)\phi'(f'))\phi_j\|^{(\ell)} = o(d^\beta),$$

where $d = \max\{\text{diam supp } f, \text{diam supp } f'\}$.

Operator product expansion of $\phi(f)\phi'(f')$.

Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_0$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale $\lambda \implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Phase Space and Scaling Limit

Scaling: $r \to \lambda r$, $E \to \lambda^{-1} E \implies$ phase space condition needs sharpening:

Definition

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition II if $\forall \gamma > 0$, $\exists c, \varepsilon > 0$ and $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that for large E, small r,

$$\|\psi \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(1 + Er)^\gamma,$$
$$\|(\Xi - \psi) \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(Er)^{\gamma+\varepsilon}.$$

Satisfied by free fields in $d = 3 + 1$ [Bostelmann '00].
Reasonable for asymptotically free theories (logarithmic corrections).
Note: PSC II $\implies$ PSC I.

Phase Space and Scaling Limit

Theorem

Let $O \to \mathscr{A}(O)$ satisfy PSC II, $\underline{\phi}_\lambda := \psi^*(\underline{A}_\lambda)$, $\underline{A} \in \underline{\mathfrak{A}}(O_r)$.

- *π_0 extends to $\underline{\phi}$ and $\pi_0(\underline{\phi})$ is a local field of $\mathscr{A}_0$.*

If $\underline{\omega}_0$ pure:

- *$O \to \mathscr{A}_0(O)$ satisfies PSC I;*
- $\dim \Phi_{0,\gamma} \leq \dim \Phi_\gamma$.

For $\underline{B} \in \underline{\mathfrak{A}}$ with $\underline{B}_\lambda \Omega \in P(E/\lambda)\mathscr{H}$:

$$\langle \pi_0(\underline{B})\Omega_0, \pi_0(\underline{\phi})\pi_0(\underline{B})\Omega_0 \rangle = \mathbf{m}(\lambda \to \langle \underline{B}_\lambda \Omega, \underline{\phi}_\lambda \underline{B}_\lambda \Omega \rangle).$$

Uniform Operator Product Expansion

Consider $\underline{\alpha}_f \underline{\phi} = \int dx\, f(x) \underline{\alpha}_x(\underline{\phi})$.
$\pi_0(\underline{\phi})$ local $\implies \pi_0(\underline{\alpha}_f \underline{\phi}\, \underline{\alpha}_{f'} \underline{\phi}') = \alpha_{0,f}\pi_0(\underline{\phi})\alpha_{0,f'}\pi_0(\underline{\phi}')$.

Define $\underline{R}_\lambda = (1 + \lambda H)^{-1}$.

Theorem

For all $\beta > 0$ exist $\sigma_{j,\lambda} \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ such that

$$\sup_\lambda \left\| \underline{R}_\lambda^\ell \left(\underline{\alpha}_f \underline{\phi}_\lambda\, \underline{\alpha}_{f'} \underline{\phi}'_\lambda - \sum_j \sigma_{j,\lambda}(\underline{\alpha}_f \underline{\phi}_\lambda\, \underline{\alpha}_{f'} \underline{\phi}'_\lambda)\phi_j \right) \underline{R}_\lambda^\ell \right\| = o(d^\beta),$$

where $d = \max\{\mathrm{diam\,supp}\, f, \mathrm{diam\,supp}\, f'\}$.

Therefore OPE terms converge to OPE terms.

Renormalization Group

Renormalization constants:

- Let $\psi = \sum_j \sigma_j \phi_j$, then $\underline{\phi}_\lambda = \sum_j \sigma_j(\underline{A}_\lambda)\phi_j = \sum_j Z_{j,\lambda}\phi_j$;
- 2-point Wightman functions:

$$\langle \Omega_0, \phi_0(x)\phi_0'(x')\Omega_0\rangle = \mathbf{m}\Big(\lambda \to \sum_{j,k} Z_{j,\lambda} Z'_{k,\lambda}\langle \Omega, \phi_j(\lambda x)\phi_k(\lambda x')\Omega\rangle\Big);$$

- therefore $Z_{j,\lambda} = \sigma_j(\underline{A}_\lambda)$ are renormalization constants.

Scaling transformations:

- $(\underline{\delta}_\mu \underline{\phi})_\lambda = \underline{\phi}_{\mu\lambda} = \sum_j Z_{j,\mu\lambda}\phi_j$: renormalization group;
- $\underline{\omega}_0$ invariant: $\pi_0(\underline{\delta}_\mu \underline{\phi}) = U_0(\mu)\pi_0(\underline{\phi})U_0(\mu)^*$.

Summary & Outlook

Summary:

- no new observable fields appear in the limit;
- multiplicative renormalization obtained in an axiomatic framework;
- scaling of OPE coefficients is some substitute for coupling constant renormalization;
- renormalization group induces dilations in the limit theory.

Outlook:

- Structure of limit theory in general case (in particular for **m** invariant);
- Extension to charge carrying fields;
- Conditions for asymptotic freedom, completeness...