Short Distance Analysis in Algebraic Quantum Field Theory II: Applications to Superselection Theory

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Outline

Introduction

Picture of hadronic matter as composed of charged particles (quarks and gluons) essentially free at high energy but confined into hadrons supported by several facts:

- quark model of hadronic spectrum
- parton picture of DIS
- perturbative treatment of QCD

Conceptual problem: confinement picture based on attaching a physical interpretation to unobservable gauge fields. Different set of fields may yield the same observable net. Examples:

- Schwinger model (2d massless QED)
- bosonization of 3d Chern-Simons QED
- Seiberg-Witten dualities in 4d SUSY YM

Proposed solution [Buchholz '96]: define confinement intrinsically through comparison of superselection structures of $\mathscr{A}$ and its scaling limit $\mathscr{A}_{0,\iota}$

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Superselection Theory 1/2

DHR (resp. BF) sectors described by unitary equivalence classes of localized morphisms of $\mathscr{A}$:

$$\Delta(O) := \{\rho : \mathscr{A} \to B(\mathscr{H}) \ : \ \rho(A) = A, \ A \in \mathscr{A}(O')\}$$

O double cone (resp. $O = C$ spacelike cone)
Intertwiners between $\rho, \sigma \in \Delta$:

$$(\rho : \sigma) := \{T \in \mathscr{A} \ : \ T\rho(A) = \sigma(A)T, \ A \in \mathscr{A}\}$$

Superselection category $\mathscr{T}$:

- objects of $\mathscr{T}$ = localized morphisms of $\mathscr{A}$
- morphisms of $\mathscr{T}$ = intertwiners

it's a tensor C*-category (tensor prodcut = composition of morphisms) with further properties (subobjects, direct sums, symmetry, conjugates)
It encodes unobservable charged fields and global gauge group

Superselection Theory 2/2

Theorem ([Doplicher, Roberts '90])

There exist unique:

- $O \mapsto \mathscr{F}(O) \subset B(\mathscr{H}_{\mathscr{F}})$ *field net with normal commutation relations*
- $g \in G \mapsto V(g) \in B(\mathscr{H}_{\mathscr{F}})$ *unitary representation of compact gauge group* G *with* $\beta_g(\mathscr{F}(O)) = \mathscr{F}(O)$ *(*$\beta_g := \mathrm{Ad}\, V(g)$*)*

such that:

- $\mathscr{A}(O) = \mathscr{F}(O)^G := \{F \in \mathscr{F}(O) \, : \, \beta_g(F) = F, \, g \in G\}$
- $\rho \in \Delta(O)$ *irreducible* $\implies \exists\, \psi_1, \dots, \psi_d \in \mathscr{F}(O)$ *d-dimensional irreducible G-tensor s.t.*

$$\psi_i^* \psi_j = \delta_{ij} \mathbb{1}, \quad \sum_j \psi_j \psi_j^* = \mathbb{1}, \quad \rho(A) = \sum_j \psi_j A \psi_j^*$$

$\mathscr{F}(\mathscr{A}) := \mathscr{F}$, $G(\mathscr{A}) := G$ canonical DR field net and gauge group
Similar result for BF sectors with field net $C \mapsto \mathscr{F}(C)$

Outline

Scaling of Field Net for DHR Sectors 1/2

Short distance behavior of superselection charges analyzed through extension of scaling algebra methods to canonical DR field net $\mathscr{F}$ of $\mathscr{A}$

Definition ([D'Antoni, M., Verch '04])

In the C*-algebra $B(\mathbb{R}_+, \mathscr{F})$ of bounded functions $\lambda \in \mathbb{R}_+ \mapsto \underline{F}_\lambda \in \mathscr{F}$, with norm and Γ action as for $\underline{\mathfrak{A}}$ and

- G action $\underline{\beta}_g(\underline{F})_\lambda := \beta_g(\underline{F}_\lambda)$, $g \in G$

the local field scaling algebra attached to O is the C*-algebra $\underline{\mathfrak{F}}(O)$ of functions $\underline{F}$ s.t.:

- $\underline{F}_\lambda \in \mathscr{F}(\lambda O)$
- $\lim_{\gamma \to e} \|\underline{\alpha}_\gamma(\underline{F}) - \underline{F}\| = 0$
- $\lim_{g \to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0$ ("dimensionless" charges)

$O \mapsto \underline{\mathfrak{F}}(O)$ normal covariant net with G-action $\underline{\beta}$
$\underline{\mathfrak{A}}(O) \subset \underline{\mathfrak{F}}(O)$

Scaling of Field Net for DHR Sectors 2/2

φ locally normal state on $\mathscr{F} \rightsquigarrow \underline{\varphi}_\lambda(\underline{F}) := \varphi(\underline{F}_\lambda)$ states on $\underline{\mathfrak{F}}$,

$$\mathrm{SL}^{\mathscr{F}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([DMV '04])

- $\mathrm{SL}^{\mathscr{F}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota\in I}$ *is independent of* φ.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{F}}$ *with GNS representation* $(\pi_{0,\iota}, \mathscr{H}_{0,\iota}, \Omega_{0,\iota})$. *Then* $\mathscr{F}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{F}}(O))''$ *is a field net in vacuum representation (irreducible if* $s = 2, 3$*).*
- $\exists G_{0,\iota} = G/N_{0,\iota}$ *and a representation* $V_{0,\iota} : G_{0,\iota} \to B(\mathscr{H}_{0,\iota})$ *defined through*
$$V_{0,\iota}(gN_{0,\iota})\pi_{0,\iota}(\underline{F})\Omega_{0,\iota} := \pi_{0,\iota}(\underline{\beta}_g(\underline{F}))\Omega_{0,\iota}$$
such that $\mathscr{A}_{0,\iota}(O) = \mathscr{F}_{0,\iota}(O)^{G_{0,\iota}}$.

$O \to \mathscr{F}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{F}$.

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Scaling Limit and DR Reconstruction

$\mathscr{F}_{0,\iota} = \mathscr{F}(\mathscr{A})_{0,\iota}$ is not in general the canonical DR field net for $\mathscr{A}_{0,\iota}$

General situation:

$$\begin{tikzcd}
& \mathscr{A} \arrow[dl, "\text{SL}"'] \arrow[dr, "\text{DR}"] & \\
\mathscr{A}_{0,\iota} \arrow[dr, "\text{DR}"'] & & \mathscr{F}(\mathscr{A}) \arrow[dl, "\text{SL}"] \\
& \mathscr{F}(\mathscr{A}_{0,\iota}) \supseteq \mathscr{F}(\mathscr{A}_{0,\iota})^{H(\mathscr{A}_{0,\iota})} = \mathscr{F}(\mathscr{A})_{0,\iota} &
\end{tikzcd}$$

$H(\mathscr{A}_{0,\iota}) \subset G(\mathscr{A}_{0,\iota})$ normal subgroup such that

$$G(\mathscr{A})_{0,\iota} = G(\mathscr{A})/N(\mathscr{A})_{0,\iota} = G(\mathscr{A}_{0,\iota})/H(\mathscr{A}_{0,\iota}).$$

$\mathscr{F}(\mathscr{A}_{0,\iota}) \supsetneq \mathscr{F}(\mathscr{A})_{0,\iota} \implies \mathscr{A}$ has confined charges [Buchholz '96]

Example: the Schwinger model (see below)

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Example: the Schwinger model (see below)

Presevation of DHR Sectors 1/2

Which DHR sectors *survive* the scaling limit?

Physical picture $\rightsquigarrow$ pointlike charges survive.

- $\psi_j(\lambda) \in \mathscr{F}(\lambda O)$ of class $[\rho] \implies \psi_j(\lambda)\Omega$ charge $[\rho]$ in λO.
- $[\rho]$ pointlike $\implies$ energy of $\psi_j(\lambda)\Omega \sim \lambda^{-1}$.

Theorem ([DMV '04])

With $\psi_j(\lambda)$ as above and

$$(\underline{\alpha}_h \psi_j)_\lambda := \int_{\mathbb{R}^4} dx\, h(x)\alpha_{\lambda x}(\psi_j(\lambda)),$$

there exists

$$\psi_j^0 = \text{s*-}\lim_{h\to\delta} \pi_{0,\iota}(\underline{\alpha}_h \psi_j) \in \mathscr{F}_{0,\iota}(O)$$

and ψ_j^0 is a $G_{0,\iota}$-multilplet which implements a DHR sector of $\mathscr{A}_{0,\iota}$.

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Local intertwiners between $\rho, \sigma \in \Delta$:

$$(\rho : \sigma)_O := \{T \in \mathscr{A} \ : \ T\rho(A) = \sigma(A)T, \ A \in \mathscr{A}(O)\}$$

Theorem ([DMV '04])

If

- *all algebras $\mathscr{F}(O)$ are factors, $\mathscr{F}(O) \cap \mathscr{F}(O)' = \mathbb{C}\mathbb{1}$,*
- *each sector is preserved in some limit*

then

$$(\rho : \sigma)_O = (\rho : \sigma)$$

Generalization of result of [Roberts '74] for dilation invariant theories
Interesting for superselection theory on curved spacetimes

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Scaling of BF Sectors 1/2

Study of short distance behavior of charges limited to DHR sectors too narrow: if quarks are non-confined they ought to be localized in cones

Problem: space-like cones not affected by rescaling. How to implement RG phase space?
At least in asymptotically free theories charges in cones should become localized in the scaling limit (flux string becomes weaker))

Definition ([DMV '04])

For $O \subset C$ define C*-algebra $\underline{\mathfrak{F}}(C, O) \subset B(\mathbb{R}_+, \mathscr{F})$ through:

- $\underline{F}_\lambda \in \mathscr{F}(\lambda C)$
- $\lim_{\gamma \to e} \|\underline{\alpha}_\gamma(\underline{F}) - \underline{F}\| = 0$
- $\lim_{g \to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0$ ("dimensionless" charges)
- $\lim_{\lambda \to 0} \sup_{\underline{A} \in \underline{\mathfrak{A}}(O')_1} \|[\underline{F}_\lambda, \underline{A}_\lambda]\| = 0$ ("asymptotic localizaiton" in O)

$\underline{\mathfrak{F}}$ C*-algebra generated by all $\underline{\mathfrak{F}}(C, O)$

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Scaling of BF Sectors 2/2

φ normal state on $B(\mathscr{H}_{\mathscr{F}}) \rightsquigarrow$ states $(\underline{\varphi}_\lambda)_{\lambda>0}$ on $\underline{\mathfrak{F}}$ and $\mathrm{SL}^{\mathscr{F}}(\varphi)$ defined as for DHR

Theorem ([DMV '04])

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$$O \mapsto \mathscr{F}_{0,\iota}(O) := \bigcap_{C \supset O} \pi_{0,\iota}(\underline{\mathfrak{F}}(C, O))''$$

covariant normal field net with action of $G_{0,\iota} = G/N_{0,\iota}$*, s.t.* $\mathscr{A}_{0,\iota}(O) = \mathscr{F}_{0,\iota}(O)^{G_{0,\iota}}$.

$\mathscr{F}_{0,\iota}$ indexed by double cones as expected in asymptotically free case

An intrinsic notion of charge confinement

Study of preservation of BF sectors similar to DHR case (technical details more involved)

Summarizing:

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, "\mathrm{SL}"] \arrow[d] & & \mathscr{A}_{0,\iota} \arrow[d] \\
\mathrm{BF}(\mathscr{A}) & \mathrm{BF}_0(\mathscr{A}) \arrow[l, hook'] \arrow[r, dashed] & \mathrm{DHR}(\mathscr{A}_{0,\iota})
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$\mathrm{BF}_0(\mathscr{A})$ = preserved BF sectors

Charge confinement

The sectors $\mathrm{DHR}(\mathscr{A}_{0,\iota}) \setminus \mathrm{BF}_0(\mathscr{A})$ are the confined sectors of $\mathscr{A}$, as they are sectors of $\mathscr{A}_{0,\iota}$ which cannot be created by operations at finite scales

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The Schwinger Model

Schwinger model:

- 2d QED with massless fermions
- exactly solvable, algebra of observables $\mathscr{A}$ generated by 2d neutral free field ϕ of mass $m > 0$ [Lowenstein, Swieca '71]
- no charged states $\Rightarrow \mathscr{F} = \mathscr{A}$ [Fröhlich, Morchio, Strocchi '79]
- interpreted as confinement of fermions: 2d electric potential rises linearly
- interpretation questionable form the point of view of observables

Schwinger Model: Scaling limit and Charged States 1/2

Scaling limit of $\mathscr{A}$ [Buchholz, Verch '97]:

- algebra $\mathscr{A}_{0,\iota} \supset \mathscr{W}_0 \otimes \mathscr{C}_0$, with $\mathscr{W}_0$ (Weyl) algebra generated by $e^{i\phi_0(f)}$, ϕ_0 2d massless free field, and $\mathscr{C}_0$ a suitable abelian algebra not visible in conventional approach
- vacuum (on $\mathscr{W}_0$)

$$\langle \Omega_{0,\iota}, e^{i\phi_0(f)}\Omega_{0,\iota}\rangle = \begin{cases} \exp\left[-\pi\int_{\mathbb{R}} \frac{dp}{2|p|}|\hat{f}(|p|,p)|^2\right] & \text{if } \hat{f}(0) = 0 \\ 0 & \text{if } \hat{f}(0) \neq 0 \end{cases}$$

(non-regular because of infrared divergence)

$\exists$ Charged states on $\mathscr{A}_{0,\iota}$ s.t.

$$\omega_q(e^{i\phi_0(f)}) = e^{iL(f)}\langle \Omega_{0,\iota}, e^{i\phi_0(f)}\Omega_{0,\iota}\rangle$$

where

- $L(f) = -\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}} dx(h(x) - h(-x))\int_{-\infty}^{x} dy\,\rho(y)$
- $h(x) = \frac{1}{2}\int_{\mathbb{R}} dx_1[f(x_1 + x, x_1) + f(-x_1 - x, x_1)]$
- ρ function with support in $[-r, r]$ and such that $\int_{\mathbb{R}} dx\,\rho(x) = q$

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Schwinger Model: Scaling limit and Charged States 2/2

Properties of ω_q:

- ω_q is (spacelike cone) localized: for supp f in the left/right spacelike complement of $[-r, r] \times \{0\}$

$$L(f) = \mp \pi q \hat{f}(0) \Rightarrow \omega_q(e^{i\phi_0(f)}) = \langle \Omega_{0,\iota}, e^{i\phi_0(f)} \Omega_{0,\iota} \rangle$$

- ω_q has charge q: with $j_\mu(x) = \varepsilon_{\mu\nu} \partial^\nu \phi_0(x)$

$$\int_{\mathbb{R}} dx_1 \, \omega_q(j_0(x_0, x_1)) = q$$

  while the integral vanishes for all states induced by vectors in $\overline{\mathscr{W}_0 \Omega_{0,\iota}}$

Thus $\mathscr{F}(\mathscr{A}_{0,\iota}) \supsetneq \mathscr{F}(\mathscr{A})_{0,\iota} = \mathscr{A}_{0,\iota}$ and the Schwinger model has a confined charge

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Models with Non-Preserved Sectors

It is not difficult to build examples of non-preserved sectors

Theorem ([D'Antoni, M. '06])

For each Lie group G and closed normal subgroup $N \subset G$, there exists $\mathscr{F}$ with gauge group G such that only sectors of $\mathscr{A} := \mathscr{F}^G$ corresponding to representations of G/N are preserved.

$\mathscr{F} = \mathscr{F}_1 \otimes \mathscr{F}_2$ with

- $\mathscr{F}_1$ a G_1-multiplet of Lutz models
- $\mathscr{F}_2$ a G_2-multiplet of free scalar fields
- $G_1, G_2 = G/N$ such that canonically $G \subset G_1 \times G_2$

$\implies \mathscr{F}_{0,\iota} = (\mathscr{F}_2)_{0,\iota}$ and sectors corresponding to representations of G non-trivial on N disappear in the scaling limit

In particular: all sectors of a multiplet of free scalar fields are preserved

Models with Non-Preserved Sectors

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Outline

Asymptotic morphisms and KK-theory 1/2

Notion of aysmptotic morphism of C*-algebras introduced by [Connes-Higson '90] in connection with KK-theory and Baum-Connes conjecture.

Definition

$\mathfrak{A}, \mathfrak{B}$ C*-algebras. An asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a family a functions $\rho_\lambda : \mathfrak{A} \to \mathfrak{B}$, $\lambda \in (0, 1]$, such that

- $\lambda \mapsto \rho_\lambda(A)$ is continuous for every $A \in \mathfrak{A}$;
- for every $A, A' \in \mathfrak{A}$, $\alpha \in \mathbb{C}$,

$$\lim_{\lambda \to 0} \rho_\lambda(A + \alpha A') - \rho_\lambda(A) - \alpha \rho_\lambda(A') = 0$$

$$\lim_{\lambda \to 0} \rho_\lambda(A^*) - \rho_\lambda(A)^* = 0$$

$$\lim_{\lambda \to 0} \rho_\lambda(AA') - \rho_\lambda(A)\rho_\lambda(A') = 0$$

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Asymptotic morphisms and KK-theory 2/2

Definition above has several important consequences:

- asymptotic morphism (ρ_λ) induces a homomorphism $\rho_* : K_*(\mathfrak{A}) \to K_*(\mathfrak{B})$
- there is a natural notion of homotopy, homotopy classes can be composed in a way consistent with K-theory
- homotopy classes of asymptotic morphisms $C_0(\mathbb{R}) \otimes \mathfrak{A} \otimes \mathfrak{K} \to C_0(\mathbb{R}) \otimes \mathfrak{B} \otimes \mathfrak{K}$ form an abelian group $E(\mathfrak{A}, \mathfrak{B})$ (E-theory)
- E-theory is a bifunctor, with product structure analogous to Kasparov product, and there is a natural transformation $KK(\mathfrak{A}, \mathfrak{B}) \to E(\mathfrak{A}, \mathfrak{B})$
- $KK(\mathfrak{A}, \mathfrak{B}) \simeq E(\mathfrak{A}, \mathfrak{B})$ if $\mathfrak{A}$ is K-nuclear

Moral: $E(\mathfrak{A}, \mathfrak{B})$ is another description of $KK(\mathfrak{A}, \mathfrak{B})$
KK-theory is in turn a key tool for constructing C^*-algebras invariants and an important ingredient in noncommutative geometry (index theorems, Novikov and Baum-Connes conjecture...)

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Asymptotic morphisms and QFT

Intriguing question [Doplicher '99]: Is it possible to associate to DHR morphisms ρ_0 of $\mathscr{A}_{0,\iota}$, some sort of asymptotic morphism of $\mathscr{A}$?

We try to use two remarks:

- Connes-Higson asymptotic morphisms (from $\mathfrak{A}$ to $\mathfrak{A}$) can be equivalently described as morphisms $\rho : \mathfrak{A} \to \mathfrak{A}_0$ where $\mathfrak{A}_0 := C_b((0,1],\mathfrak{A})/C_0((0,1],\mathfrak{A})$, using a (set-theoretic) section $s : \mathfrak{A}_0 \to C_b((0,1],\mathfrak{A})$
- Analogy: $\underline{\mathfrak{A}} \leftrightarrow C_b((0,1],\mathfrak{A})$, $\ker \pi_{0,\iota} \leftrightarrow C_0((0,1],\mathfrak{A})$, $\mathscr{A}_{0,\iota} \leftrightarrow \mathfrak{A}_0$

Problems:

1. How to relate morphisms of $\mathscr{A}_{0,\iota}$ with morphisms $\mathscr{A} \to \mathscr{A}_{0,\iota}$?
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Relating $\mathscr{A}$ and $\mathscr{A}_{0,\iota}$

Recall: $\mathscr{A}$ satisfies the split property if for all $O_1 \Subset O_2$ (i.e. $\bar{O}_1 \subset O_2$) there exists $\mathscr{N}$ type I factor such that

$$\mathscr{A}(O_1) \subset \mathscr{N} \subset \mathscr{A}(O_2)$$

Equivalently $\mathscr{A}(O_1) \vee \mathscr{A}(O_2)' \simeq \mathscr{A}(O_1) \otimes \mathscr{A}(O_2)'$

Theorem ([Conti, M. '12])

- $\mathscr{A}$, $\mathscr{B}$ *nets of type III_1 factors with split property*
- $O_1 \Subset O_2 \Subset \cdots \Subset O_n \Subset \cdots$ *with* $\bigcup_n O_n = \mathbb{R}^4$

There exists isomorphism $\phi : \mathscr{A} \to \mathscr{B}$ of quasi-local C-algebras such that $\phi(\mathscr{A}(O_n)) = \mathscr{B}(O_n)$.*

Hypotheses above:

- satisfied in free field theory, and expected to hold quite generally
- conditions are known which imply (some of) them for $\mathscr{A}$, $\mathscr{A}_{0,\iota}$ (nuclearity)

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A “Weak Closure” of the Scaling Algebra

To exploit analogy $\mathscr{A}_{0,\iota} \leftrightarrow \mathfrak{A}_0$: need to lift elements $A \in \mathscr{A}_{0,\iota}(O) \setminus \pi_{0,\iota}(\underline{\mathfrak{A}}(O))$ to some kind of functions $\lambda \mapsto A_\lambda \in \mathscr{A}$

Definition ([DMV '04])

$\lambda \in \mathbb{R}_+ \mapsto A_\lambda \in \mathscr{A}(\lambda O)$ norm bounded is asymptotically contained in $\mathscr{A}_{0,\iota}(\hat{O})$ if $\forall \varepsilon > 0$, $\exists \underline{A}, \underline{A}' \in \underline{\mathfrak{A}}(\hat{O})$ s.t.

$$\limsup_{\kappa} \left(\|(A_{\lambda_\kappa} - \underline{A}_{\lambda_\kappa})\Omega\| + \|(A^*_{\lambda_\kappa} - \underline{A}'_{\lambda_\kappa})\Omega\| \right) < \varepsilon$$

$\underline{\mathfrak{A}}^\bullet(O) := \{\lambda \mapsto A_\lambda$ asymptotically contained in $\mathscr{A}_{0,\iota}(\hat{O})$ for all $\hat{O} \Supset O\}$

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$O \mapsto \underline{\mathfrak{A}}^\bullet(O)$ is a local net of C^*-algebras, $\underline{\mathfrak{A}}(O) \subset \underline{\mathfrak{A}}^\bullet(O)$

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Extending $\pi_{0,\iota}$ to $\underline{\mathfrak{A}}^\bullet$

We need to extend $\pi_{0,\iota}$ to $\underline{\mathfrak{A}}^\bullet$

Definition ([Bostelamnn, D'Antoni, M. '09])

$\mathscr{A}$ has convergent scaling limit if $\exists$ subalgebra $\hat{\underline{\mathfrak{A}}} \subset \underline{\mathfrak{A}}$ such that

- $\underline{A} \in \hat{\underline{\mathfrak{A}}} \implies \exists \lim_{\lambda\to 0} \omega(\underline{A}_\lambda)$;
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Satisfied by free massive scalar field in $s = 2, 3$ ($\hat{\underline{\mathfrak{A}}}$ generated by scaled Weyl operators)

Implies uniqueness of scaling limit

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$\mathscr{A}$ with convergent scaling limit. There exists a morphism $\pi^\bullet_{0,\iota} : \underline{\mathfrak{A}}^\bullet \to \mathscr{A}^d_{0,\iota}$ which extends $\pi_{0,\iota}$ and such that $\mathscr{A}_{0,\iota}(O) \subset \pi^\bullet_{0,\iota}(\underline{\mathfrak{A}}^\bullet(O))$ for all O

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Asymptotic Morphisms and Scaling Limit Morphisms1/2

From now on, $\mathscr{A}$ has convergent scaling limit.
We introduce a weak notion of asymptotic morphism

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An asymptotic morphism relative to $\underline{\omega}_{0,\iota}$ is a family of maps $\rho_\lambda : \mathscr{A} \to \mathscr{A}$, $\lambda > 0$, such that

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(ρ_λ) is called tame if furhermore

- $\rho^\bullet(A) : \lambda \in \mathbb{R}_+ \mapsto \rho_\lambda(A) \in \mathscr{A}$ belongs to $\underline{\mathfrak{A}}^\bullet$
- $A \in \mathscr{A} \mapsto \rho^\bullet(A) \in \underline{\mathfrak{A}}^\bullet$ is continuous
- $\pi^\bullet_{0,\iota}(\rho^\bullet(\mathscr{A}_{\mathrm{loc}})) \subset (\mathscr{A}_{0,\iota})_{\mathrm{loc}}$

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$\mathrm{AMor}_\iota(\mathscr{A}) := \{(\rho_\lambda)$ asymptotic morphism relative to $\underline{\omega}_{0,\iota}\}$
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Isomorphism $\phi : \mathscr{A} \to \mathscr{A}_{0,\iota}$ used to lift morphisms $\rho_0 : \mathscr{A}_{0,\iota} \to \mathscr{A}_{0,\iota}$ to morphisms $\rho : \mathscr{A} \to \mathscr{A}_{0,\iota}$

Theorem ([CM '12])

There is a bijective correspondence between asymptotic equivalence classes of pairs $((\rho_\lambda), (\phi_\lambda)) \in \mathrm{AMor}_\iota(\mathscr{A}) \times \mathrm{AIso}_\iota(\mathscr{A})$ *and unitary equivalence classes of morphisms* $\rho_0 : \mathscr{A}_{0,\iota} \to \mathscr{A}_{0,\iota}$

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With $[\rho]$ preserved define:

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  DHR morphism of $\mathscr{A}$ localized in λO
- $\rho_0(A) = \sum_{j=1}^{d} \psi_j^0 A (\psi_j^0)^*, \ A \in \mathscr{A}_{0,\iota}$
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Theorem ([CM '12])

- $\rho(A)^\bullet : \lambda \in \mathbb{R}_+ \mapsto \rho(\lambda)(A_\lambda) \in \mathscr{A}$ *belongs to* $\mathfrak{A}^\bullet$ *for all* $A \in \mathfrak{A}^\bullet$
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Question: Can one characterize confinement/preservation of charges in terms of a relation as above between asymptotic morphism and families of scaled morphisms of $\mathscr{A}$?

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Open Problems

Summary

Scaling algebras and scaling limits are a very useful structural tool for model-independent, intrinsic analysis of short-distance properties of QFT and their superselection sectors

Some further questions:

- extend analysis of charge scaling to charges with anomalous dimension (e.g. electric charge in Schwinger model)
- is it possible to show that $\mathscr{A}$ generated by pointlike fields $\Longrightarrow$ $\mathscr{A}_{0,\iota}$ generated by pointlike fields?
- is it possible to characterize asymptotically free theories, maybe in terms of scaling of 2-point functions of their pointlike fields?
- can one define noncommutative geometry invariants of $\mathscr{A}$ with interesting physical interpretation in terms of asymptotic morphisms?
- relations with local gauge theories?

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Some further questions:

- extend analysis of charge scaling to charges with anomalous dimension (e.g. electric charge in Schwinger model)
- is it possible to show that $\mathscr{A}$ generated by pointlike fields $\implies$ $\mathscr{A}_{0,\iota}$ generated by pointlike fields?
- is it possible to characterize asymptotically free theories, maybe in terms of scaling of 2-point functions of their pointlike fields?
- can one define noncommutative geometry invariants of $\mathscr{A}$ with interesting physical interpretation in terms of asymptotic morphisms?
- relations with local gauge theories?

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