Asymptotic Morphisms and Scaling Limit of Quantum Field Theories

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work in progress with R. Conti

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Outline

1. Introduction
2. Superselection Theory
3. Scaling Limit of QFT
4. Asymptotic Morphisms and Scaling Limit Superselection Theory
5. Conclusions and Outlook

Introduction 1/3

Notion of aysmptotic morphism of C*-algebras introduced by [Connes-Higson '90] in connection with KK-theory and Baum-Connes conjecture.

Definition

$\mathfrak{A}, \mathfrak{B}$ C*-algebras. An asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a family a functions $\rho_\lambda : \mathfrak{A} \to \mathfrak{B}$, $\lambda \in (0,1]$, such that

- $\lambda \mapsto \rho_\lambda(A)$ is continuous for every $A \in \mathfrak{A}$;
- for every $A, A' \in \mathfrak{A}$, $\alpha \in \mathbb{C}$,

$$\lim_{\lambda \to 0} \rho_\lambda(A + \alpha A') - \rho_\lambda(A) - \alpha\rho_\lambda(A') = 0$$

$$\lim_{\lambda \to 0} \rho_\lambda(A^*) - \rho_\lambda(A)^* = 0$$

$$\lim_{\lambda \to 0} \rho_\lambda(AA') - \rho_\lambda(A)\rho_\lambda(A') = 0$$

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Introduction 2/3

Definition above has several important consequences:

- asymptotic morphism (ρ_λ) induces a homomorphism $\rho_* : K_*(\mathfrak{A}) \to K_*(\mathfrak{B})$
- there is a natural notion of homotopy, homotopy classes can be composed in a way consistent with K-theory
- homotopy classes of asymptotic morphisms $C_0(\mathbb{R}) \otimes \mathfrak{A} \otimes \mathfrak{K} \to C_0(\mathbb{R}) \otimes \mathfrak{B} \otimes \mathfrak{K}$ form an abelian group $E(\mathfrak{A}, \mathfrak{B})$ (E-theory)
- E-theory is a bifunctor, with product structure analogous to Kasparov product, and there is a natural transformation $KK(\mathfrak{A}, \mathfrak{B}) \to E(\mathfrak{A}, \mathfrak{B})$
- $KK(\mathfrak{A}, \mathfrak{B}) \simeq E(\mathfrak{A}, \mathfrak{B})$ if $\mathfrak{A}$ is K-nuclear

Moral: $E(\mathfrak{A}, \mathfrak{B})$ is another description of $KK(\mathfrak{A}, \mathfrak{B})$

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KK-theory is in turn a key tool for constructing C^*-algebras invariants and an important ingredient in noncommutative geometry (index theorems, Novikov and Baum-Connes conjecture...)

Intriguing question [Doplicher '99]: Does asymptotic morphisms (or related objects) play a role in (the operator algebraic approach to) Quantum Field Theory?

Motivation:

- morphisms of the algebra of observables are fundamental elements in the description of the theory superselection (charge) structure
- superselection structure of the (short-distance) scaling limit of the theory could be related to some kind of asymptotic morphism

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Superselection Theory 1/2

Data:

- $\mathscr{H}$ separable;
- $O \subset \mathbb{R}^4 \mapsto \mathscr{A}(O) \subset B(\mathscr{H})$ net of von Neumann algebras (generated by observables measurable in O) satisfying Haag duality
$$\mathscr{A}(O) = \mathscr{A}(O')';$$
- $\gamma \mapsto U(\gamma)$ unitary representation of $\mathscr{P}_+^\uparrow$ with positive energy, such that
$$U(\gamma)\mathscr{A}(O)U(\gamma)^* = \mathscr{A}(\gamma.O);$$
- $\Omega \in \mathscr{H}$ unique such that $U(x)\Omega = \Omega$ (vacuum).

Superselection sectors are meant to identify representations which describe "localized excitations of the vacuum" and are given by classes of localized endomorphisms:

$$\Delta(O) := \{\rho \in \mathrm{End}(\mathscr{A}) \,:\, \rho(A) = A \;\forall A \in \mathscr{A}(O')\}$$

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Superselection Theory 2/2

Theorem ([Doplicher-Roberts '90])

$\exists!\, O \mapsto \mathscr{F}(O)$ field net, $g \in G \mapsto V(g)$, G compact (global) gauge group, such that:

- $\mathscr{F}(O)^G = \mathscr{A}(O)$;
- $\forall \rho \in \Delta(O)\ \exists \psi_1, \dots, \psi_d \in \mathscr{F}(O)$ *orthogonal isometries* ($\psi_i^* \psi_j = \delta_{ij}$), $v_{[\rho]}$ *d-dimensional irrep of G, with*

$$\mathrm{Ad}\, V(g)(\psi_i) = \sum_{j=1}^{d} v_{[\rho]}(g)_{ij} \psi_j, \quad \rho(A) = \sum_{j=1}^{d} \psi_j A \psi_j^*;$$

- $\mathscr{F}$ *has normal commutation relations, defined by $k \in Z(G)$ with $k^2 = e$.*

$\mathscr{F}(\mathscr{A}) = \mathscr{F}$, $G(\mathscr{A}) = G$ are the canonical DR field net and gauge group of $\mathscr{A}$

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Scaling Algebras 1/2

On C*-algebra of bounded functions $\lambda \in \mathbb{R}_+^\times \mapsto \underline{F}_\lambda \in \mathscr{F}$ define:

$$\|\underline{F}_\lambda\| := \sup_\lambda \|\underline{F}_\lambda\|,$$

$$\underline{\alpha}_{(\Lambda,x)}(\underline{F})_\lambda := \mathrm{Ad}\, U(\Lambda, \lambda x)(\underline{F}_\lambda), \qquad (\Lambda, x) \in \mathscr{P}_+^\uparrow,$$

$$\underline{\beta}_g(\underline{F})_\lambda := \mathrm{Ad}\, V(g)(\underline{F}_\lambda), \qquad g \in G.$$

Definition

Local scaling algebra of O:

$$\underline{\mathfrak{F}}(O) := \left\{ \underline{F} : \underline{F}_\lambda \in \mathscr{F}(\lambda O), \lim_{\gamma \to e} \|\underline{\alpha}_\gamma(\underline{F}) - \underline{F}\| = 0, \lim_{g \to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0 \right\}$$

If $G = \{e\}$ then $\mathscr{F} = \mathscr{A}$ and we set $\underline{\mathfrak{A}} := \underline{\mathfrak{F}}$ (original scaling algebra of [Buchholz-Verch '95])

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Scaling Algebras 2/2

- Continuity condition w.r.t. translations $\iff$ $\underline{F}_\lambda$ has a "phase space occupation" independent of λ $\iff$ $\hbar$ not rescaled.
- Continuity condition w.r.t. G $\iff$ $\underline{F}_\lambda$ has a "charge transfer" independent of λ
- Typical elements

$$\underline{F}_\lambda = \int dx\, dg\, h(x,g) V(g) U(\lambda x) e^{i\phi_\lambda(f)} U(\lambda x)^* V(g)^*,$$

where $\phi_\lambda(x) = Z_\lambda \phi(\lambda x)$ is the usual renormalized field.
- We consider "all possible renormalization schemes" compatible with above requirements.

Scaling Limits

φ locally normal state on $\mathscr{F} \rightsquigarrow \underline{\varphi}_\lambda(\underline{F}) := \varphi(\underline{F}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{F}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([D'Antoni-M.-Verch '04])

- $\mathrm{SL}^{\mathscr{F}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota \in I}$ *is independent of* φ.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{F}}$ *with GNS representation* $\pi_{0,\iota}$. *Then* $\mathscr{F}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{F}}(O))''$ *is a field net in vacuum representation.*
- $\exists G_{0,\iota} = G/N_{0,\iota}$ *such that* $\mathscr{A}_{0,\iota} = \mathscr{F}_{0,\iota}^{G_{0,\iota}}$.

$O \mapsto \mathscr{F}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{F}$.
Physical interpretation: $\mathscr{F}_{0,\iota}$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

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Scaling Limits and Superselection Sectors 1/2

$\mathscr{F}(\mathscr{A})_{0,\iota}$ is not in general the canonical DR field net for $\mathscr{A}_{0,\iota}$

General situation:

$$\begin{tikzcd}
& \mathscr{A} \arrow[dl, "\mathrm{SL}"'] \arrow[dr, "\mathrm{DR}"] & \\
\mathscr{A}_{0,\iota} \arrow[dr, "\mathrm{DR}"'] & & \mathscr{F}(\mathscr{A}) \arrow[dl, "\mathrm{SL}"] \\
& \mathscr{F}(\mathscr{A}_{0,\iota}) \supseteq \mathscr{F}(\mathscr{A}_{0,\iota})^{H(\mathscr{A}_{0,\iota})} = \mathscr{F}(\mathscr{A})_{0,\iota} &
\end{tikzcd}$$

$H(\mathscr{A}_{0,\iota}) \subset G(\mathscr{A}_{0,\iota})$ normal subgroup such that

$$G(\mathscr{A})_{0,\iota} = G(\mathscr{A})/N(\mathscr{A})_{0,\iota} = G(\mathscr{A}_{0,\iota})/H(\mathscr{A}_{0,\iota}).$$

$\mathscr{F}(\mathscr{A}_{0,\iota}) \supsetneq \mathscr{F}(\mathscr{A})_{0,\iota} \implies \mathscr{A}$ has confined charges.

E.g. in the Schwinger model:

$\mathscr{F}(\mathscr{A}) = \mathscr{A} \implies \mathscr{F}(\mathscr{A})_{0,\iota} = \mathscr{A}_{0,\iota} \subsetneq \mathscr{F}(\mathscr{A}_{0,\iota})$ [Buchholz-Verch '97]

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Scaling Limits and Superselection Sectors 2/2

Which sectors *survive* the scaling limit?

Physical picture $\rightsquigarrow$ pointlike charges survive.

- $\psi_j(\lambda) \in \mathscr{F}(\lambda O)$ of class $[\rho] \implies \psi_j(\lambda)\Omega$ charge $[\rho]$ in λO.
- $[\rho]$ pointlike $\implies$ energy of $\psi_j(\lambda)\Omega \sim \lambda^{-1}$.

Theorem ([D'Antoni-M.-Verch '04])

With $\psi_j(\lambda)$ as above and

$$(\underline{\alpha}_h \psi_j)_\lambda := \int_{\mathbb{R}^4} dx\, h(x) \mathrm{Ad}\, U(\lambda x)(\psi_j(\lambda)),$$

there exists

$$\psi_j^0 = \operatorname*{s^*\text{-}lim}_{h \to \delta} \pi_{0,\iota}(\underline{\alpha}_h \psi_j) \in \mathscr{F}_{0,\iota}(O)$$

and ψ_j^0 is a $G_{0,\iota}$-multilplet which implements a DHR sector of $\mathscr{A}_{0,\iota}$.

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Some Advertisement

Furhter results:

- Non-preserved sectors can actually appear, models constructed using certain generalized free fields [Lutz '97, D'Antoni-M. '07]
- Model independent understanding of pointlike field renormalization and scaling of OPE (substitute for coupling constant renormalization) [Bostelmann-D'Antoni-M. '09]
- Discussion of scaling limit of subsystems $\mathscr{B} \subset \mathscr{A} \subset \mathscr{F}(\mathscr{A}) = \mathscr{F}(\mathscr{B})$, in connection with quantum Noether theorem [Conti-M. '09]
- Relation with quantum Gromov-Hausdorff metric [Bostelmann-Guido-Suriano]
- Application to scaling of models with factorizing S-matrix [Bostelmann-Lechner-M. '11]

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Main Idea

We rephrase Doplicher's question more precisely:
Question: It is possible to associate to DHR morphisms ρ_0 of $\mathscr{A}_{0,\iota}$, some sort of asymptotic morphism of $\mathscr{A}$?
We try to use two remarks:

- Connes-Higson asymptotic morphisms (from $\mathfrak{A}$ to $\mathfrak{A}$) can be equivalently described as morphisms $\rho : \mathfrak{A} \to \mathfrak{A}_0$ where $\mathfrak{A}_0 := C_b((0,1],\mathfrak{A})/C_0((0,1],\mathfrak{A})$, using a (set-theoretic) section $s : \mathfrak{A}_0 \to C_b((0,1],\mathfrak{A})$
- Analogy: $\underline{\mathfrak{A}} \leftrightarrow C_b((0,1],\mathfrak{A})$, $\ker \pi_{0,\iota} \leftrightarrow C_0((0,1],\mathfrak{A})$, $\mathscr{A}_{0,\iota} \leftrightarrow \mathfrak{A}_0$

Problems:

1. How to relate morphisms of $\mathscr{A}_{0,\iota}$ with morphisms $\mathscr{A} \to \mathscr{A}_{0,\iota}$?
2. How to deal with weak closure in $\mathscr{A}_{0,\iota}(O) = \pi_{0,\iota}(\underline{\mathfrak{A}}(O))^-$?

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- Analogy: $\underline{\mathfrak{A}} \leftrightarrow C_b((0,1],\mathfrak{A})$, $\ker \pi_{0,\iota} \leftrightarrow C_0((0,1],\mathfrak{A})$, $\mathscr{A}_{0,\iota} \leftrightarrow \mathfrak{A}_0$

Problems:

1. How to relate morphisms of $\mathscr{A}_{0,\iota}$ with morphisms $\mathscr{A} \to \mathscr{A}_{0,\iota}$?
2. How to deal with weak closure in $\mathscr{A}_{0,\iota}(O) = \pi_{0,\iota}(\underline{\mathfrak{A}}(O))^-$?

Relating $\mathscr{A}$ and $\mathscr{A}_{0,\iota}$ 1/2

Recall: $\mathscr{A}$ satisfies the split property if for all $O_1 \Subset O_2$ (i.e. $\bar{O}_1 \subset O_2$) there exists $\mathscr{N}$ type I factor such that

$$\mathscr{A}(O_1) \subset \mathscr{N} \subset \mathscr{A}(O_2)$$

Equivalently $\mathscr{A}(O_1) \vee \mathscr{A}(O_2)' \simeq \mathscr{A}(O_1) \otimes \mathscr{A}(O_2)'$

Theorem

- *$\mathscr{A}$, $\mathscr{B}$ nets of type III_1 factors with split property*
- $O_1 \Subset O_2 \Subset \cdots \Subset O_n \Subset \cdots$ *with* $\bigcup_n O_n = \mathbb{R}^4$

There exists isomorphism $\phi : \mathscr{A} \to \mathscr{B}$ such that $\phi(\mathscr{A}(O_n)) = \mathscr{B}(O_n)$.

Hypotheses above:

- satisfied in free field theory, and expected to hold quite generally
- conditions are known which imply them for $\mathscr{A}$, $\mathscr{A}_{0,\iota}$ (nuclearity)

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Relating $\mathscr{A}$ and $\mathscr{A}_{0,\iota}$ 2/2

Proof.

- If $\mathscr{A}(O_k) \subset \mathscr{N}_k \subset \mathscr{A}(O_{k+1})$ type I factor, [Doplicher '82] gives a commuting diagram

$$\begin{array}{ccccccccc}
\mathscr{N}_1 & \subset & \mathscr{N}_2 & \subset & \mathscr{N}_3 & \subset \dots & \mathscr{A} \\
\downarrow \phi_{\mathscr{A}} & & \downarrow & & \downarrow & & \downarrow \\
B(\mathscr{K}) & \underset{T\mapsto T\otimes \mathbb{1}}{\hookrightarrow} & B(\mathscr{K}^{\otimes 2}) & \hookrightarrow & B(\mathscr{K}^{\otimes 3}) & \hookrightarrow \dots & \mathscr{O}^0_{\mathscr{K}}
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- $B(\mathscr{K}^{\otimes k-1}) \otimes \mathbb{1} \subset \phi_{\mathscr{A}}(\mathscr{A}(O_k))$ III_1 factor
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- $\mathscr{A}(O_k)$, $\mathscr{B}(O_k)$ hyperfinite (by split) III_1 factors $\implies$ isomorphic
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A "Weak Closure" of the Scaling Algebra

To exploit analogy $\mathscr{A}_{0,\iota} \leftrightarrow \mathfrak{A}_0$: need to lift elements $A \in \mathscr{A}_{0,\iota}(O) \setminus \pi_{0,\iota}(\underline{\mathfrak{A}}(O))$ to some kind of functions $\lambda \mapsto A_\lambda \in \mathscr{A}$

Definition ([D'Antoni-M.-Verch '04])

$\lambda \in \mathbb{R}_+ \mapsto A_\lambda \in \mathscr{A}(\lambda O)$ norm bounded is asymptotically contained in $\mathscr{A}_{0,\iota}(\hat{O})$ if $\forall \varepsilon > 0$, $\exists \underline{A}, \underline{A}' \in \underline{\mathfrak{A}}(\hat{O})$ s.t.

$$\limsup_\kappa \left(\|(A_{\lambda_\kappa} - \underline{A}_{\lambda_\kappa})\Omega\| + \|(A^*_{\lambda_\kappa} - \underline{A}'_{\lambda_\kappa})\Omega\| \right) < \varepsilon$$

$\underline{\mathfrak{A}}^\bullet(O) := \{\lambda \mapsto A_\lambda \text{ asymptotically contained in } \mathscr{A}_{0,\iota}(\hat{O}) \text{ for all } \hat{O} \supset O\}$

Theorem

$O \mapsto \underline{\mathfrak{A}}^\bullet(O)$ is a net of C^*-algebras, $\underline{\mathfrak{A}}(O) \subset \underline{\mathfrak{A}}^\bullet(O)$

Question: is it a local net?

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Extending $\pi_{0,\iota}$ to $\underline{\mathfrak{A}}^\bullet$ 1/2

We need to extend $\pi_{0,\iota}$ to $\underline{\mathfrak{A}}^\bullet$

Definition ([Bostelmann-D'Anotni-M. '09])

$\mathscr{A}$ has convergent scaling limit if $\exists$ subalgebra $\hat{\underline{\mathfrak{A}}} \subset \underline{\mathfrak{A}}$ such that

- $\underline{A} \in \hat{\underline{\mathfrak{A}}} \implies \exists \lim_{\lambda\to 0} \omega(\underline{A}_\lambda)$;
- $\pi_{0,\iota}(\hat{\underline{\mathfrak{A}}}(O))'' = \mathscr{A}_{0,\iota}(O)$.

Satisfied by free fields.

Theorem

$\mathscr{A}$ with convergent scaling limit. There exists a morphism $\pi^\bullet_{0,\iota} : \underline{\mathfrak{A}}^\bullet \to \mathscr{A}^d_{0,\iota}$ which extends $\pi_{0,\iota}$ and such that $\mathscr{A}_{0,\iota}(O) \subset \pi^\bullet_{0,\iota}(\underline{\mathfrak{A}}^\bullet(O))$ for all O

$\mathscr{A}^d_{0,\iota}(O) := \mathscr{A}_{0,\iota}(O')'$ dual net

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Extending $\pi_{0,\iota}$ to $\underline{\mathfrak{A}}^\bullet$ 2/2

Proof.

- definition of $\pi^\bullet_{0,\iota}$: by
  $\langle \pi_{0,\iota}(\underline{B})\Omega_{0,\iota}, \pi^\bullet_{0,\iota}(A)\pi_{0,\iota}(\underline{C})\Omega_{0,\iota}\rangle = \lim_\kappa \omega(\underline{B}^*_{\lambda_\kappa} A_{\lambda_\kappa} \underline{C}_{\lambda_\kappa})$
  for $A \in \underline{\mathfrak{A}}^\bullet$, $\underline{B}, \underline{C} \in \underline{\mathfrak{A}}$
- show: $\pi_{0,\iota}(\underline{\alpha}_h A) \xrightarrow{s^*} \pi^\bullet_{0,\iota}(A)$ as $h \to \delta$,
  $(\underline{\alpha}_h A)_\lambda := \int dx\, h(x)\alpha_{\lambda x}(A_\lambda) \in \underline{\mathfrak{A}}$
- multiplicativity of $\pi^\bullet_{0,\iota}$: approximate $A_\lambda B_\lambda \Omega$ by $(\underline{\alpha}_h A)_\lambda (\underline{\alpha}_h B)_\lambda \Omega$ and use multiplicativity of $\pi_{0,\iota}$
- $\mathscr{A}_{0,\iota}(O) \subset \pi^\bullet_{0,\iota}(\underline{\mathfrak{A}}^\bullet(O))$: given $A_0 \in \mathscr{A}_{0,\iota}(O)$ we can find $\underline{A}_n \in \hat{\underline{\mathfrak{A}}}(O)$ and $\lambda_n \searrow 0$ with $\pi_{0,\iota}(\underline{A}_n) \xrightarrow{s^*} A_0$ and $\|(\underline{A}_{n+1,\lambda} - \underline{A}_{n,\lambda})\Omega\| < 1/2^n$, $\lambda < \lambda_n$. With $A_\lambda := \underline{A}_{n,\lambda}$, $\lambda_{n+1} < \lambda < \lambda_n$, we have $A \in \underline{\mathfrak{A}}^\bullet(O)$, $\pi^\bullet_{0,\iota}(A) = A_0$

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Asymptotic Morphisms and Scaling Limit Morphisms 1/3

From now on, $\mathscr{A}$ has convergent scaling limit.
We introduce a weak notion of asymptotic morphism

Definition

An asymptotic morphism relative to $\underline{\omega}_{0,\iota}$ is a family of maps $\rho_\lambda : \mathscr{A} \to \mathscr{A}$, $\lambda > 0$, such that

$$\lim_\kappa \|[\rho_{\lambda_\kappa}(A)^* - \rho_{\lambda_\kappa}(A^*)]\Omega\| = 0$$

$$\lim_\kappa \|[\rho_{\lambda_\kappa}(A + \alpha B) - \rho_{\lambda_\kappa}(A) + \alpha\rho_{\lambda_\kappa}(B)]\Omega\| = 0$$

$$\lim_\kappa \|[\rho_{\lambda_\kappa}(AB) - \rho_{\lambda_\kappa}(A)\rho_{\lambda_\kappa}(B)]\Omega\| = 0$$

(ρ_λ) is called tame if furhermore

- $\rho^\bullet(A) : \lambda \in \mathbb{R}_+ \mapsto \rho_\lambda(A) \in \mathscr{A}$ belongs to $\underline{\mathfrak{A}}^\bullet$
- $A \in \mathscr{A} \mapsto \rho^\bullet(A) \in \underline{\mathfrak{A}}^\bullet$ is continuous
- $\pi^\bullet_{0,\iota}(\rho^\bullet(\mathscr{A}_{\text{loc}})) \subset (\mathscr{A}_{0,\iota})_{\text{loc}}$

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Asymptotic Morphisms and Scaling Limit Morphisms 2/3

Following Connes-Higson and our analogy, asymptotic moprhisms should correspond to morphisms $\rho : \mathscr{A} \to \mathscr{A}_{0,\iota}$
Some should correspond to isomorphisms $\phi : \mathscr{A} \to \mathscr{A}_{0,\iota}$ (exist quite generally, shown above)

Definition

An asymptotic isomorphism is an asymptotic morphism (ϕ_λ) such that

- $\phi^\bullet : \mathscr{A} \to \underline{\mathfrak{A}}^\bullet$ is injective
- $\exists$ continuous section $\bar{s} : \mathscr{A}_{0,\iota} \to \underline{\mathfrak{A}}^\bullet$ of $\pi^\bullet_{0,\iota}$ such that $\phi^\bullet(\mathscr{A}) = \bar{s}(\mathscr{A}_{0,\iota})$

Note that there always exists a continuous section $s : \mathscr{A}_{0,\iota} \to \underline{\mathfrak{A}}^\bullet$ of $\pi^\bullet_{0,\iota}$ [Bartle-Graves '50s]
$\mathrm{AMor}_\iota(\mathscr{A}) := \{(\rho_\lambda)$ asymptotic morphism relative to $\underline{\omega}_{0,\iota}\}$
$\mathrm{AIso}_\iota(\mathscr{A}) := \{(\phi_\lambda)$ asymptotic isomorphism relative to $\underline{\omega}_{0,\iota}\}$

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- $\exists$ continuous section $\bar{s} : \mathscr{A}_{0,\iota} \to \underline{\mathfrak{A}}^\bullet$ of $\pi^\bullet_{0,\iota}$ such that $\phi^\bullet(\mathscr{A}) = \bar{s}(\mathscr{A}_{0,\iota})$

Note that there always exists a continuous section $s : \mathscr{A}_{0,\iota} \to \underline{\mathfrak{A}}^\bullet$ of $\pi^\bullet_{0,\iota}$ [Bartle-Graves '50s]

$\mathrm{AMor}_\iota(\mathscr{A}) := \{(\rho_\lambda)$ asymptotic morphism relative to $\underline{\omega}_{0,\iota}\}$
$\mathrm{AIso}_\iota(\mathscr{A}) := \{(\phi_\lambda)$ asymptotic isomorphism relative to $\underline{\omega}_{0,\iota}\}$

Asymptotic Morphisms and Scaling Limit Morphisms2/3

Following Connes-Higson and our analogy, asymptotic moprhisms should correspond to morphisms $\rho : \mathscr{A} \to \mathscr{A}_{0,\iota}$
Some should correspond to isomorphisms $\phi : \mathscr{A} \to \mathscr{A}_{0,\iota}$ (exist quite generally, shown above)

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Asymptotic Morphisms and Scaling Limit Morphisms 3/3

Isomorphism $\phi : \mathscr{A} \to \mathscr{A}_{0,\iota}$ used to lift morphisms $\rho_0 : \mathscr{A}_{0,\iota} \to \mathscr{A}_{0,\iota}$ to morphisms $\rho : \mathscr{A} \to \mathscr{A}_{0,\iota}$

$$\Downarrow$$

ρ_0 should correspond to pairs $\big((\rho_\lambda), (\phi_\lambda)\big) \in \mathrm{AMor}_\iota(\mathscr{A}) \times \mathrm{AIso}_\iota(\mathscr{A})$
There is a natural notion of asymptotic equivalence of pairs

Theorem

There is a bijective correspondence between asymptotic equivalence classes of pairs $\big((\rho_\lambda), (\phi_\lambda)\big) \in \mathrm{AMor}_\iota(\mathscr{A}) \times \mathrm{AIso}_\iota(\mathscr{A})$ and unitary equivalence classes of morphisms $\rho_0 : \mathscr{A}_{0,\iota} \to \mathscr{A}_{0,\iota}$, given by

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Asymptotic Morphisms and Preserved Sectors 1/2

Recall: $[\rho]$ preserved $\implies \exists \psi_j(\lambda) \in \mathscr{F}(\lambda O)$ such that

$$\rho_0(A) = \sum_{j=1}^{d} \psi_j^0 A (\psi_j^0)^*$$

is a DHR morphism of $\mathscr{A}_{0,\iota}$, where $\psi_j^0 = \text{s*-lim}_{h \to \delta} \pi_{0,\iota}(\underline{\alpha}_h \psi_j)$

Define:

$$\rho(\lambda)(A) := \sum_{j=1}^{d} \psi_j(\lambda) A \psi_j(\lambda)^*$$

scaled DHR morphisms of $\mathscr{A}$ localized in λO, $[\rho(\lambda)] = [\rho]$

Question: What is the relation, if any, between $\rho(\lambda)$ and asymptotic morphism associated to ρ_0?

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Asymptotic Morphisms and Preserved Sectors 2/2

Theorem

- $\rho(A)^\bullet : \lambda \in \mathbb{R}_+ \mapsto \rho(\lambda)(A_\lambda) \in \mathscr{A}$ *belongs to* $\underline{\mathfrak{A}}^\bullet$ *for all* $A \in \underline{\mathfrak{A}}^\bullet$
- *if* $\big((\rho_\lambda), (\phi_\lambda)\big) \in \mathrm{AMor}_\iota(\mathscr{A}) \times \mathrm{AIso}_\iota(\mathscr{A})$ *is a pair associated to* ρ_0, *then*
$$\pi^\bullet_{0,\iota}(\rho^\bullet(A)) = \pi^\bullet_{0,\iota}(\rho(\bar{s}\phi(A))^\bullet)$$
for all $A \in \mathscr{A}$

Morally: $\rho_\lambda(A) \sim \rho(\lambda)(\bar{s}\phi(A))_\lambda)$

Question: Can one characterize confinement/preservation of charges in terms of a relation as above between asymptotic morphism and families of scaled morphisms of $\mathscr{A}$?

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Outline

Conclusions and Outlook

Summary:

- There exists a bijection between (pairs of suitable) asymptotic morphisms of $\mathscr{A}$ and morphisms of the scaling algebra $\mathscr{A}_{0,\iota}$
- Asymptotic morphism associated to a preserved sector $[\rho]$ is asymptotically equivalent to a family of scaled morphisms of class $[\rho]$

(Some of the many) open questions:

- How can we characterize asymptotic morphisms associated to localized morphisms of $\mathscr{A}_{0,\iota}$?
- Is it possible to encode further structure of superselection structure of $\mathscr{A}_{0,\iota}$ (tensor structure, statistics etc.)?
- Can we describe confinement/preservation of charges through asymptotic morphisms?
- Can we use asymptotic morphisms to define (noncommutative geometric) invariants of $\mathscr{A}$ with an interesting physical interpretation?

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