RENORMALIZATION GROUP IN ALGEBRAIC QUANTUM FIELD THEORY: PAST, PRESENT AND FUTURE

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ABSTRACT. We present an overview of some recent results on the renormalization group analysis in the framework of local quantum physics. In particular, we emphasize the role of the theory of superselection sectors, through the magnifying glass provided by the scaling limit, that leads to intrinsic notions of preserved and confined sectors. In the final section we discuss some further results and open problems.

1. INTRODUCTION

The so called "scaling algebras" have been introduced some time ago [4] to analyze the short distance behaviour of quantum field theories in a model independent manner, as a counterpart to the renormalization group analysis in the conventional approach. The idea is to associate to a given theory, described by a C^*-algebra with a net of subalgebras [15, 21], an algebra of functions defined on $\mathbb{R}_+$ and taking values in the local algebras. These functions should be regarded as "orbits" of the renormalization group and they are characterized by requiring that they occupy the same value of the "phase space" at each scale. Taking the limit as $\lambda \to 0$ of the vacuum expectation values of these functions one can analyze the short distance properties of the given quantum field theory.

A general classification of the short distance behaviour of the given theory encoded in a new net of algebras has been obtained which corresponds to the one known in perturbation theory. Moreover an intrinsic notion of "charge confinement" has been formulated [3]. A version of such a notion requires a general study of the short distance property of charged fields and superselection sectors (see [17]) performed in [11], that has led to a notion of charge preservation. The confined sectors are then identified with those sectors of the scaling limit theory which do not arise as limits of preserved sectors. Examples exhibiting both preserved and non preserved sectors have been constructed [10].

On the other hand, it is interesting to stress the use of scaling algebras in proving properties of the underlying (scale $\lambda = 1$) theory under requirements on the scaling limit. The type of local von Neumann algebras, the non-existence of product states for tangent spacelike regions, the proper outerness of gauge group action are some examples of such results.

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Here is a brief outline of the content of this note: in the next section, after some preliminaries on the notion of scaling limit, we review the main ideas and results pertaining the notions of charge preservation and confinement. Then we give a glimpse of the interplay between the $\lambda \to 0$ and $\lambda = 1$ cases. In the final section we offer a panoramic view on some related directions for further studies and open problems.

2. A DIVE INTO THE SCALING LIMIT

2.1. **Scaling algebras and scaling limit for field nets.** We consider a Poincaré covariant, normal field net $O \to \mathfrak{F}(O)$ over double cones on Minkwoski space. By this, we mean that $O \to \mathfrak{F}(O)$ is an isotonous net of von Neumann algebras acting irreducibly on an Hilbert space $\mathscr{H}$, on which commuting unitary representations U of the universal cover of the Poincaré group $\tilde{\mathscr{P}}_+^\uparrow$ and V of a compact gauge group G act, such that, using the notations $\alpha_{(\Lambda,a)} := \mathrm{Ad}\, U(\Lambda, a)$, $\beta_g := \mathrm{Ad}\, V(g)$,

$$\alpha_{(\Lambda,a)}(\mathfrak{F}(O)) = \mathfrak{F}(\Lambda O + a), \qquad \beta_g(\mathfrak{F}(O)) = \mathfrak{F}(O).$$

Furthermore the translations $U(\mathbb{1}, \cdot)$ satisfy the spectrum condition, and the vacuum $\Omega \in \mathscr{H}$ is the unique translation invariant unit vector, and it is also gauge invariant. Finally there is a $k \in Z(G)$ with $k^2 = e$ which defines the Bose and Fermi parts of elements $F \in \mathfrak{F}(O)$ according to

$$F_\pm := \frac{1}{2}(F \pm \beta_k(F)).$$

The observable net associated to $\mathfrak{F}$ is defined as usual as the fixed points net under the action of G:

$$\mathfrak{A}(O) := \mathfrak{F}(O)^G = \{F \in \mathfrak{F}(O) : \beta_g(F) = F, \text{ for all } g \in G\}.$$

The scaling algebra associated to $\mathfrak{F}$ is defined following the case of observable nets [4]. On the C*-algebra of bounded functions $\lambda \in (0, +\infty) \to \underline{F}_\lambda \in \mathfrak{F}$ with norm given by $\|\underline{F}\| = \sup_{\lambda>0} \|\underline{F}_\lambda\|$, we define automorphic actions of $\tilde{\mathscr{P}}_+^\uparrow$ and G as

$$\underline{\alpha}_{(\Lambda,a)}(\underline{F})_\lambda := \alpha_{(\Lambda,\lambda a)}(\underline{F}_\lambda), \qquad \underline{\beta}_g(\underline{F})_\lambda := \beta_g(\underline{F}_\lambda).$$

Definition 2.1. The *local scaling field algebra* associated to the double cone O is the C*-algebra $\underline{\mathfrak{F}}(O)$ of all bounded functions $\underline{F}$ such that $\underline{F}_\lambda \in \mathfrak{F}(\lambda O)$ and

$$\lim_{(\Lambda,a)\to(\mathbb{1},0)} \|\underline{\alpha}_{(\Lambda,a)}(\underline{F}) - \underline{F}\| = 0, \qquad \lim_{g\to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0.$$

As thoroughly discussed in [4, 11], the first condition in the last equation selects functions $\underline{F}$ such that $\underline{F}_\lambda$ has a phase space occupation independent of λ, while the second condition amounts to assuming that we have to do with dimensionless charges (which is not really restrictive in $d = 4$).

To a locally normal state φ on $\mathfrak{F}$ we associate the family of states $\{\underline{\varphi}_\lambda\}_{\lambda>0}$ on $\underline{\mathfrak{F}}$ defined by

$$\underline{\varphi}_\lambda(\underline{F}) := \varphi(\underline{F}_\lambda),$$

and consider the set of its weak* limit points as $\lambda \to 0$ (called *scaling limit states* of $\mathfrak{F}$), which is found to be independent of the chosen state φ.

Theorem 2.2. *Let $\underline{\omega}_0$ be a scaling limit state. Then $\underline{\omega}_0$ is a pure vacuum state for $\underline{\mathfrak{F}}$, so that if $(\pi_0, \mathscr{H}_0, \Omega_0)$ is the associated GNS representation,*

$$\mathfrak{F}_0(O) := \pi_0(\underline{\mathfrak{F}}(O))''$$

defines a covariant, normal field net with gauge group $G_0 := G/N$, where $N \subset G$ is a suitable closed normal subgroup. Furthermore if $\mathfrak{A}_0$ is the scaling limit of the observable net associated to $\underline{\omega}_0 \restriction \underline{\mathfrak{A}}$, then $\mathfrak{A}_0$ is isomorphic to the restriction of $\mathfrak{F}_0^{G_0}$ to the subspace $\overline{\pi_0(\underline{\mathfrak{A}})\Omega_0}$.

Each of the nets $\mathfrak{F}_0$ arising in this way is called a *scaling limit field net* of $\mathfrak{F}$.

2.2. **Short distance analysis of DHR sectors.** Even if we start, in the above construction, from an $\mathfrak{F}$ which is a canonical DR complete field net [12], we should expect that the scaling limit net $\mathfrak{F}_0$ will not be complete in general, the reason for this being the possible existence of confined charges [3]. If we assume that $\mathfrak{F}$ is the complete field net of its observable net $\mathfrak{A}$, and denote by $\mathfrak{F}^0$ the complete field net obtained by DR reconstruction from $\mathfrak{A}_0$, then from the above results and those in [12] we get that the situation is summarized in the following diagram:

$$\begin{tikzcd}
& \mathfrak{A} \arrow[dl, "\mathrm{SL}"'] \arrow[dr, "\mathrm{DR}"] & \\
\mathfrak{A}_0 \arrow[dr, "\mathrm{DR}"'] & & \mathfrak{F} \arrow[dl, "\mathrm{SL}"] \\
& \mathfrak{F}^0 \supseteq \mathfrak{F}_0 &
\end{tikzcd}$$

It is then natural to interpret the charges of $\mathfrak{A}_0$ described by $\mathfrak{F}^0$ but not by $\mathfrak{F}_0$ as confined charges of $\mathfrak{A}$. An example of such a situation is provided by the Schwinger model, discussed in [3, 5], where $\mathfrak{A} = \mathfrak{F}$ and therefore $\mathfrak{A}_0 = \mathfrak{F}_0$, but, since $\mathfrak{A}_0$ has nontrivial sectors, $\mathfrak{F}^0$ is striclty bigger than $\mathfrak{F}_0$.

Conversely, one could try to characterize the charges of $\mathfrak{A}$ which survive in the scaling limit as charges of $\mathfrak{A}_0$. As shown in the examples discussed in the next subsection, it must be expected that in general sectors may disappear in the scaling limit. A natural condition of preservation for a DHR sector ξ under the scaling limit is obtained by requiring that states carrying a charge ξ in a region of radius λ have energy at most λ^{-1}, i.e. constrained only by the Heisenberg uncertainty principle, meaning that the corresponding charge is "pointlike". As discussed in [11], this can be rephrased in the following, apparently more technical, condition, which involves the multiplets of isometries implementing the sector ξ of [12].

Definition 2.3. A sector ξ of $\mathfrak{A}$ is preserved in the scaling limit state $\underline{\omega}_0 = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ if, for each double cone O_1 and each $\lambda > 0$, there exist multiplets of class ξ, $\psi_j(\lambda) \in \mathfrak{F}(\lambda O_1)$, $j = 1, \ldots, d$, such that, for each $\varepsilon > 0$ and each double cone O containing the closure of O_1, there exist $\underline{F}_j, \underline{F}'_j \in \underline{\mathfrak{F}}(O)$ with

$$\limsup_\kappa \left(\|(\psi_j(\lambda_\kappa) - \underline{F}_{j\lambda_\kappa})\Omega\| + \|(\psi_j(\lambda_\kappa) - \underline{F}'_{j\lambda_\kappa})^*\Omega\| \right) < \varepsilon.$$

We introduce the notation

$$(\underline{\alpha}_h \psi_j)_\lambda := \int_{\tilde{\mathscr{P}}_+^\uparrow} d\Lambda\, da\, h(\Lambda, a) \alpha_{(\Lambda,\lambda a)}(\psi_j(\lambda)),$$

where $h \in L^1(\tilde{\mathscr{P}}_+^\uparrow)$ and $d\Lambda\, da$ is the Haar measure on $\tilde{\mathscr{P}}_+^\uparrow$.

Theorem 2.4. *Let ξ be a DHR sector preserved in $\underline{\omega}_0$, and $\psi_j(\lambda) \in \mathfrak{F}(\lambda O_1)$ be as in the above definition. Then, with $\mathfrak{F}_0$ the scaling limit net determined by $\underline{\omega}_0$, there exists*

$$\boldsymbol{\psi}_j := s^*\text{-}\lim_{h\to\delta} \pi_0(\underline{\alpha}_h \psi_j) \in \mathfrak{F}_0(O),$$

(limit in the strong* operator topology) for each $O \supset \overline{O}_1$. If the $\psi_j(\lambda)$ transform under the action of G like a multiplet according to the irreducible representation v_ξ, independent of λ, then v_ξ is trivial on N and the $\boldsymbol{\psi}_j$ transform under the action of $G_0 = G/N$ like a multiplet according to the irreducible representation $v_{\xi,0}(gN) = v_\xi(g)$, $g \in G$. Furthermore the equation

$$\boldsymbol{\rho}(\boldsymbol{a}) := \sum_{j=1}^d \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^*, \qquad \boldsymbol{a} \in \mathfrak{A}_0,$$

defines an irreducible, covariant endomorphism with finite statistics of $\mathfrak{A}_0$ localized in O.

2.3. **Tensor products and examples of preserved and non-preserved sectors.** A natural question that arises when trying to compute the scaling limit in concrete models is the one about the permutability of the scaling limit and tensor product of nets operations. Some answer has been given in [10]. In order to formulate it, we introduce the operators $\Theta_{\beta,O} : \mathfrak{F}(O) \to \mathscr{H}$ defined by $\Theta_{\beta,O}(F) := e^{-\beta H} F\Omega$, where H is the Hamiltonian, and we say that a theory is asymptotically p-nuclear for some $p \in (0,1)$ if

$$\limsup_{\lambda\to 0} \|\Theta_{\lambda\beta,\lambda O}\|_p < \infty,$$

$\|\cdot\|_p$ denoting the nuclear p-norm. It follows from [18] that a theory generated by a finite multiplet of scalar free fields with any masses is asymptotically p-nuclear for every $p \in (0,1)$.

Theorem 2.5. *Let $p \in (0,1/6)$ and let $\mathfrak{F}^{(i)}$, $i = 1,2$, be covariant, normal and asymptotically p-nuclear field nets. Furthermore, let $\mathfrak{F}$ denote the $\mathbb{Z}_2$-graded tensor product of $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$ and, for each scaling limit state $\underline{\omega}_0$ of $\mathfrak{F}$, let $\mathfrak{F}_0$, $\mathfrak{F}_0^{(1)}$, $\mathfrak{F}_0^{(2)}$ be the scaling limit nets corresponding to $\underline{\omega}_0$, $\underline{\omega}_0 \upharpoonright \underline{\mathfrak{F}}^{(1)}$, $\underline{\omega}_0 \upharpoonright \underline{\mathfrak{F}}^{(2)}$, respectively. Then if $\mathfrak{F}_0^{(i)}$, $i = 1,2$ satisfy twisted Haag duality, $\mathfrak{F}_0$ is isomorphic to the $\mathbb{Z}_2$-graded tensor product of $\mathfrak{F}_0^{(1)}$ and $\mathfrak{F}_0^{(2)}$.*

Using this result, it is possible to construct models which exhibit preserved and non-preserved sectors, with basic building blocks appropriate (generalized) free fields. In particular, if ϕ_j, $j = 1,\dots,d$, is a multiplet of generalized free fields with mass measure $d\rho(m) = dm$, transforming under a d-dimensional representation of a compact group G, and if we define the algebra $\mathfrak{F}(O)$ as the one generated by the fields $\Box^{n(\mathrm{diam} O)}\phi_j(f)$ with $\mathrm{supp} f \subset O$, and $n(\lambda) \to +\infty$ as $\lambda \to 0$, it can be shown that the scaling limit of $\mathfrak{F}$ is trivial, $\mathfrak{F}_0(O) = \mathbb{C}\mathbb{1}$. On the other hand, the results of [5] can be generalized to show that the net generated by any multiplet of free fields of mass m has as unique scaling limit the net generated by the same multiplet of free fields of mass $m = 0$, and that all the corresponding DHR sectors are preserved. Performing suitable tensor products of such theories one gets the following result [10].

Theorem 2.6. *For any choice of (G, N) with G a compact Lie group and $N \subset G$ a closed normal subgroup, there exists an observable net $\mathfrak{A}$ which has DHR sectors labelled by classes of representations of G, and such that its scaling limit theory $\mathfrak{A}_0$, which is unique, has only DHR sectors labelled by classes of representations of G/N, so that the sectors of $\mathfrak{A}$ corresponding to representations of G trivial on N are preserved, while the remaining ones are not.*

2.4. **Short distance analysis of BF sectors.** In [11] the analysis of Subsection 2.1 and 2.2 has been generalized to cover the case of BF sectors (cf. [15]), which is physically interesting in view of the fact that gauge charges, if not confined, should give rise to BF sectors. This requires a redefinition of the scaling algebra inspired by the physical picture that BF charges, at least in an asymptotically free theory, should become localized in bounded regions at small scales (see [11] for details). The outcome is that starting at scale $\lambda = 1$ from a field net $C \to \mathfrak{F}(C)$ indexed by spacelike cones, one ends up with a scaling limit field net $O \to \mathfrak{F}_0(O)$ indexed by double cones. Consequently, the notion of preservation of charges is formulated in such a way as to associate to a preserved BF sector at scale $\lambda = 1$ a DHR sector, corresponding to the same gauge group (quotient) representation, in the scaling limit.

Denoting by $\mathrm{DHR}(\mathfrak{A}_0)$ the set of DHR sectors of $\mathfrak{A}_0$ and by $\mathrm{BF}(\mathfrak{A})$ the set of BF sectors of $\mathfrak{A}$, the situation can be summarized by the following diagram,

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "\mathrm{SL}"] \arrow[d] & & \mathfrak{A}_0 \arrow[d] \\
\mathrm{BF}(\mathfrak{A}) & \mathrm{BF}_0(\mathfrak{A}) \arrow[l, hook'] \arrow[r, dashed] & \mathrm{DHR}(\mathfrak{A}_0)
\end{tikzcd}$$

where $\mathrm{BF}_0(\mathfrak{A})$ is the set of preserved BF sectors of $\mathfrak{A}$, and were the dashed arrow is the map associating to each preserved BF sector of $\mathfrak{A}$ the corresponding DHR sector of $\mathfrak{A}_0$ according to the previous discussion. This leads to the following criterion for confined sectors: a sector $\xi \in \mathrm{DHR}(\mathfrak{A}_0)$ is confined if it is not in the image of $\mathrm{BF}_0(\mathfrak{A})$ under this map.

3. ROBERTS VARIATIONS

We would like to present a sample of the results mentioned at the end of the introduction. They are variations of some old results obtained by Roberts in a dilation invariant theory [20] and are surely known to experts but we present them as they give the flavour of the involved reasoning.

The first result is the existence of "entanglement" among observables localized in spacelike separated regions with a common bounday point. We prove in fact that no locally normal state can factorize across such regions.

Proposition 3.1. *Assume $\mathfrak{F}$ has a nonclassical scaling limit $\underline{\omega}_0$ which satisfies the Reeh-Schlieder property, i.e. $\overline{\mathfrak{F}_0(O)\Omega_0} = \mathscr{H}_0$ for all double cones O. Then for every pair of* *spacelike tangent double cones O_1 and O_2 there is no locally normal state φ of $\mathfrak{F}$ such that*

$$\varphi(FG) = \varphi(F) \cdot \varphi(G), \qquad F \in \mathfrak{F}(O_1),\, G \in \mathfrak{F}(O_2)\,. \tag{3.1}$$

Proof. Without loss of generality, we can prove the claim for O_1, O_2 which both contain the origin in their spacelike boundary, since the theory is translation covariant. Assuming that the formula (3.1) holds one gets for the lifts of φ to $\underline{\mathfrak{F}}$

$$\underline{\varphi}_\lambda(\underline{FG}) = \varphi(\underline{F}_\lambda \underline{G}_\lambda) = \varphi(\underline{F}_\lambda) \cdot \varphi(\underline{G}_\lambda) = \underline{\varphi}_\lambda(\underline{F}) \cdot \underline{\varphi}_\lambda(\underline{F})$$

for all $0 < \lambda \le 1$ and all $\underline{F} \in \underline{\mathfrak{F}}(O_1)$, $\underline{G} \in \underline{\mathfrak{F}}(O_2)$. Taking a subnet $\lambda_\kappa \to 0$ with $\underline{\omega}_0 = \lim_\kappa \underline{\varphi}_{\lambda_\kappa}$ one obtains

$$\underline{\omega}_0(\underline{F}\,\underline{G}) = \underline{\omega}_0(\underline{F}) \cdot \underline{\omega}_0(\underline{G})\,.$$

The conclusion now follows from the fact that, $\underline{\omega}_0$ being nonclassical, there is $\hat{\underline{F}} \in \underline{\mathfrak{F}}(O_1)$ such that $\pi_0(\hat{\underline{F}}) \neq \underline{\omega}_0(\hat{\underline{F}})I$ and hence, with $\underline{F} := \hat{\underline{F}} - \underline{\omega}_0(\hat{\underline{F}})\underline{I}$, $\pi_0(\underline{F}) \neq 0$ but $\underline{\omega}_0(\underline{F}) = 0$. One would then deduce that $\underline{\omega}_0(\underline{F}\,\underline{G}) = 0$ for all $\underline{G} \in \underline{\mathfrak{F}}(O_2)$, which is impossible by the Reeh-Schlieder property. □

The next result shows that under some requirements in restriction to $\mathfrak{F}(O)$ a gauge automorphism can not be inner. (This does not rule out the possibility that it is however implemented by a unitary in $\mathfrak{F}(O_1)$ with $O_1 \gg O$, as in a theory satisfying the split property.)

Proposition 3.2. *Let $V \in \mathfrak{F}(O)$ be unitary and $VF = \beta_g(F)V$ for all $F \in \mathfrak{F}(O)$. Then we have $g \in N$ for any closed normal subrgroup $N \subset G$ associated to a scaling limit state $\underline{\omega}_0$ as in Theorem 2.2.*

Proof. We sketch the proof in the case where $\mathfrak{F}$ is Bosonic. The general case is similar but requires more work. Take a double cone O and by translational covariance assume that the origin is at the "corner" of O, so that $\lambda O \subset O$ for all $\lambda \le 1$. Also, let W be a wedge spacelike separated from O with the origin at the corner of W, too.

If φ is the locally normal state on $\mathfrak{F}$ given by $\varphi(F) = \omega(VFV^*)$ one has for $\underline{F} \in \underline{\mathfrak{F}}(O)$, $\underline{F}' \in \underline{\mathfrak{F}}(W)$,

$$\underline{\varphi}_\lambda(\underline{F}'\,\underline{F}) = \omega(V\underline{F}'_\lambda \underline{F}_\lambda V^*) = \omega(\underline{F}'_\lambda \beta_g(\underline{F}_\lambda)) = \underline{\omega}_\lambda(\underline{F}'\underline{\beta}_g(\underline{F})).$$

Let $\lambda_\kappa \to 0$ be a subnet with $\underline{\omega}_0 = \lim_\kappa \underline{\varphi}_{\lambda_\kappa}$. Then since $\lim_{\lambda\to 0} \|(\varphi - \omega) \restriction \mathfrak{F}(\lambda O)\| = 0$ [20], one also has $\underline{\omega}_0 = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ and therefore, from the above equation,

$$\underline{\omega}_0(\underline{F}'\,\underline{F}) = \underline{\omega}_0(\underline{F}'\,\underline{\beta}_g(\underline{F})), \qquad \underline{F} \in \underline{\mathfrak{F}}(O),\, \underline{F}' \in \underline{\mathfrak{F}}(W).$$

Thus, since by a standard Reeh-Schlieder argument Ω_0 is cyclic for $\mathfrak{F}_0(W)$, it readily follows that

$$\pi_0(\underline{F})\Omega_0 = V_0(g)\pi_0(\underline{F})\Omega_0$$

for all $\underline{F}$, that is $g \in N$. □

4. PERSPECTIVES

In this final section we shortly address some topics that are the subject of current research and should lead to significant developments in the area.

4.1. Scaling limit of localized endomorphisms. In order to study the short distance properties of charges, it would be interesting to work out a more intrinsic mechanism for defining scaling limit of DHR endomorphisms without using multiplets. Among other virtues, this could be useful in applications to low-dimensional theories, where no canonical DHR type-construction is available.

Since technical problems arise, it could even be worth examining new objects that are endomorphisms only in a weaker asymptotic sense, as suggested e.g. by the results in [19], in which a C^*-category equivalent to the superselection category of the scaling limit theory is constructed in terms of asymptotic charge shifters. It is an intriguing possibility to adapt to this purpose some techniques that to date have been mainly worked out in the framework of KK-theory.

4.2. Field content of the scaling limit. The way Haag-Kastler nets are related to Wightman fields [23] has been the subject of several investigations. In recent times some interesting advance has been made in [1] where a phase space condition has been introduced that allows the construction of pointlike fields affiliated to a given net of local algebras. The action of renormalization on pointlike fields is currently under investigation [2]. An analysis of convergence of n-point functions and operator product expansions in the scaling limit are also expected.

4.3. Scaling limit of subsystems vs. subsystems of scaling limits. Motivated by the problem of characterizing in abstract way the local nets generated by fields with a clear physical interpretation, a classification theorem for subsystems has been obtained in [6]. In a parallel development, the functorial properties of the DR-constructions have been considered in [8]. On the other hand, some aspects of the DR-analysis of the scaling limit have been considered in [11]. Therefore it is quite natural to compare the available classification results for subsystems of the ambient theory and of its scaling limit. If $\mathscr{A} \subset \mathfrak{A}$ is a "reasonable" subsystem (such as the one generated by the local implementers of the spacetime translations), one can show that

$$\begin{array}{ccccc} \mathfrak{A}_0 & \subset & \mathfrak{F}_0 & \subset & \mathfrak{F}^0 \\ \cup & & \cup & & \cup \\ \mathscr{A}_0 & \subset & \mathscr{F}_0 & \subset & \mathscr{F}^0 \end{array},$$

with an obvious meaning of the symbols. However a more detailed analysis of the second and the third vertical inclusions requires some additional work. Making use of the results in [6] one obtains that indeed $\mathfrak{F}^0 = \mathscr{F}^0$ for a number of interesting situations, including the fixpoints of free field nets [9]. In this way, one expects to derive some relations between (the local nets generated by) the Noether charges (i.e., quantum invariants of motion) of the given theory and those of its scaling limit.

4.4. **Scale dependant gauge transformations.** The analysis of the charge content of the scaling limit has been based upon the experimental fact that 4D physical charges do not depend on the scale. However, there is some reason to believe that in lower dimensions such a dependancy could show up. An example in this direction is given by the Schwinger model, in which the charge of the "electron" is dimensionful. In a related fashion, there should be some construction explaining how the symmetry group of the original theory behaves under scaling. The mechanism of group contractions advocated by Inonu and Wigner [16] may play a role in this context.

4.5. **Low dimensional models.** Perhaps the most succesful example of an interacting theory is the $P(\varphi)_2$ model. It has been conjectured that the scaling limit of this model is the same as for the corresponding free theory [13]. However, to date there is no complete analysis of the scaling limit of the free theory, even if it has been shown in [5] that all scaling limit theories contain, as a subtheory, a central extension of the massless free field in $d = 2$, obtained as the scaling limit of the subalgebra generated by the (smoothed out) Weyl operators.

4.6. **A noncommutative geometry setup.** From a purely mathematical point of view, it has been recognized that, under fairly general phase space conditions on the underlying theory, the way the scaling limit is defined fits pretty well with Gromov's tangent cone construction [14]. This leads to the possibility to think of a scaling limit as a kind of tangent space of the given local net in a suitable noncommutative geometry framework. As the tangent cone construction enjoys rather good functoriality properties, this also sheds some light on the fact that in [10] nuclearity conditions were employed in order to show the permutability of the scaling limit with the operation of tensor product of theories.

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