Scaling algebras for charge carrying quantum fields and superselection structure at short distances

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Dedicated to Detlev Buchholz, on the occasion of his 60th birthday.

ABSTRACT. We report on a recent work on the extension to the case of fields carrying superselection charges of the method of scaling algebras, which has been introduced earlier as a means for analysing the short-distance behaviour of quantum field theories in the setting of the algebraic approach. This generalization is used to study the relation between the superselection structures of the underlying theory and the one of its scaling limit, and, in particular, to propose a physically motivated criterion for the preservation of superselection charges in the scaling limit. This allows the formulation of an intrinsic notion of charge confinement as proposed by D. Buchholz.

1. Introduction

A great deal of information about the short distance properties of quantum field theory can be obtained through the use of renormalization group (RG) methods (consider, just to mention one example, the parton distributions in deep inelastic scattering processes). From a conceptual point of view, however, such applications are not completely satisfactory, as they always rely on a description of the theory in terms of (unobservable) fields, which, as is well known, is in general not unique (think for instance of Borchers classes, of bosonization in two dimensional models or of the more recent discoveries of dualities in supersymmetric $d = 4$ gauge theories).

In order to overcome these problems, a framework for a model-independent, canonical analysis of the short distance behaviour of QFT was developed in [**3**] in the context of the algebraic approach to QFT [**13**], in which a theory is defined only in terms of its algebras of local observables, and therefore the ambiguities inherent to the use of unobservable fields completely disappear. This has been accomplished through the introduction of the *scaling algebra*. We will give a summary of this work in section 2. A major result of this programme has been the formulation of an

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intrinsic confinement criterion [**2**], not relying on attaching a physical interpretation to unobservable fields (see section 4 below for some more details).

In this context, it is also interesting to study the short distance properties of superselection charges. As we will briefly recall in section 3, it has been one of the most relevant achievements of algebraic quantum field theory to show that the set of charges, as well as the unobservable fields carrying them, are completely encoded in the structure of the representations of the algebra of observables. It is therefore natural to try to analyse the short distance behaviour of charges through a generalization of the scaling algebra method to charge carrying fields. Such a generalization has been performed in [**8**], where it has also been used to formulate a notion of preservation of charges under the scaling (short distance) limit. We will describe the main results of this work in sections 4 and 5, the first one dealing with the case of charges which are localizable in bounded regions of Minkowski spacetime, while the second one treats the case of charges localizable in certain unbounded regions called spacelike cones.

2. An algebraic approach to the renormalization group

The basic principle of the algebraic approach to quantum field theory is that a theory is completely characterized by an assignment $O \to \mathscr{A}(O)$ of open double cones O in Minkowski space $\mathbb{R}^4$ (i.e. O is the non-void intersection of a forward and a backward light cone) to von Neumann algebras $\mathscr{A}(O)$ acting on a separable Hilbert space $\mathscr{H}$ (the *vacuum space*), the algebras $\mathscr{A}(O)$ having the physical interpretation of being generated by the observables of the system under consideration which can be measured with an experiment performed in the region O. Such a correspondence is subject to the following basic assumptions:

1. *isotony*: if $O_1 \subset O_2$ then $\mathscr{A}(O_1) \subset \mathscr{A}(O_2)$;
2. *locality*: if O_1 is spacelike separated from O_2, then $\mathscr{A}(O_1)$ and $\mathscr{A}(O_2)$ commute;
3. *covariance*: there is a unitary, strongly continuous representation U on $\mathscr{H}$ of the (proper, orthocronous) Poincarè group $\mathscr{P}_+^\uparrow$, such that , if $\alpha_{(\Lambda,a)} := \mathrm{Ad}\, U(\Lambda, a)$, $\alpha_{(\Lambda,a)}(\mathscr{A}(O)) = \mathscr{A}(\Lambda O + a)$;
4. *spectrum condition*: the spectrum of the representation of the translations group $a \in \mathbb{R}^4 \to U(\mathbb{1}, a)$ is contained in the closed forward light cone;
5. *existence of the vacuum*: there exists a unique (up to a phase) translation invariant unit vector $\Omega \in \mathscr{H}$, furthermore Ω is cyclic for $\bigcup_O \mathscr{A}(O)$.

Each assignment $O \to \mathscr{A}(O)$ satisfying the above assumptions will be called a *local, covariant net of observable algebras* on $\mathscr{H}$, and will also be denoted with $\mathscr{A}$ for brevity. By the same symbol we will indicate the C*-algebra which is the norm closure of $\bigcup_O \mathscr{A}(O)$, called the *quasi-local algebra* of the given net. We also define, for an arbitrary region $R \subset \mathbb{R}^4$, the C*-algebra $\mathscr{A}(R) \subset \mathscr{A}$ as the one generated by the algebras $\mathscr{A}(O)$ with $O \subset R$.

The following notation will be useful: for $R \subset \mathbb{R}^4$, R' is the set of all points which are spacelike to all points of R.

We will now describe the main results in [**3**], which give an adaptation to this framework of the renormalization group ideas.

The key observation in this analysis is that the conventional renormalization group transformations $(R_\lambda)_{\lambda>0}$ possess the characteristic feature of scaling localizations of operators by λ, and their 4-momentum transfer by λ^{-1}, so that the "phase space occupation" of orbits $\lambda \to R_\lambda(A)$ of (bounded) operators under RG transformation is independent of λ. This can be shown to be equivalent to the following uniform continuity property with respect to Poincaré transformations:

$$\lim_{(\Lambda,a)\to(\mathbb{1},0)} \sup_{\lambda>0} \|\alpha_{(\Lambda,\lambda a)}(R_\lambda(A)) - R_\lambda(A)\| = 0. \tag{2.1}$$

Of course this property does not fix the RG transformations $(R_\lambda)_{\lambda>0}$ uniquely, but we will see that it contains enough information as to enable us to perform the short distance scaling limit of a given theory.

In order to show this, we consider the C*-algebra $B(\mathbb{R}_+, \mathscr{A})$ of all norm bounded functions $\underline{A}$ from the positive reals to the quasi-local algebra, with pointwise defined operations and with the uniform norm $\|\underline{A}\| := \sup_{\lambda>0} \|\underline{A}_\lambda\|$, and define on it an automorphic action $\underline{\alpha}$ of $\mathscr{P}_+^\uparrow$ by

$$\underline{\alpha}_{(\Lambda,a)}(\underline{A})_\lambda := \alpha_{(\Lambda,\lambda a)}(\underline{A}_\lambda), \qquad (\Lambda, a) \in \mathscr{P}_+^\uparrow. \tag{2.2}$$

Then, keeping in mind equation (2.1) above, we make the following

Definition 2.1. The *scaling algebra* associated to the double cone O is the C*-algebra $\underline{\mathfrak{A}}(O)$ of functions $\underline{A} \in B(\mathbb{R}_+, \mathscr{A})$ such that:

(1) $\underline{A}_\lambda \in \mathscr{A}(\lambda O)$ for each $\lambda > 0$;
(2) $\lim_{(\Lambda,a)\to(\mathbb{1},0)} \|\underline{\alpha}_{(\Lambda,a)}(\underline{A}) - \underline{A}\| = 0$;

The corresponding quasi-local algebra will be denoted by $\underline{\mathfrak{A}}$.

According to the above discussion, the conditions (1) and (2) in the previous definition implement the renormalization group phase space properties on the functions $\lambda \to \underline{A}_\lambda$ in the scaling algebras, so that $\underline{\mathfrak{A}}$ has to be thought of as comprising the orbits of observables in $\mathscr{A}$ under all possible choices of the RG transformations.

To any given locally normal state φ on $\mathscr{A}$ (i.e. φ is a state on $\mathscr{A}$ such that its restriction to each local algebra $\mathscr{A}(O)$ is a normal state of $\mathscr{A}(O)$), we associate a family of states $(\underline{\varphi}_\lambda)_{\lambda>0}$ on the scaling algebra $\underline{\mathfrak{A}}$ defined by

$$\underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda), \qquad \lambda > 0,\ \underline{A} \in \underline{\mathfrak{A}}, \tag{2.3}$$

and we denote by $\mathrm{SL}_{\mathscr{A}}(\varphi)$ the set of limit points, in the weak* topology on the dual of $\underline{\mathfrak{A}}$, of the net $(\underline{\varphi}_\lambda)_{\lambda>0}$, as $\lambda \to 0$, i.e. $\underline{\omega}_0 \in \mathrm{SL}_{\mathscr{A}}(\varphi)$ if there exists a subnet $(\lambda_\kappa)_\kappa \subset \mathbb{R}_+$, $\lambda_\kappa \to 0$, such that $\underline{\omega}_0(\underline{A}) = \lim_\kappa \underline{\varphi}_{\lambda_\kappa}(\underline{A})$ for each $\underline{A} \in \underline{\mathfrak{A}}$. $\mathrm{SL}_{\mathscr{A}}(\varphi)$ is called the set of *scaling limit states* of φ, and is always non-empty by general compactness arguments.

Theorem 2.2. *The set $\mathrm{SL}_{\mathscr{A}}(\varphi)$ is independent of φ. For $\underline{\omega}_0 \in \mathrm{SL}_{\mathscr{A}}$, let $(\pi_0, \mathscr{H}_0, \Omega_0)$ be the corresponding GNS representation of $\underline{\mathfrak{A}}$ and define, for each double cone O,*

$$\mathscr{A}_0(O) := \pi_0(\underline{\mathfrak{A}}(O))''. \tag{2.4}$$

Then there exists a representation U_0 of $\mathscr{P}_+^\uparrow$ with respect to which the net $O \to \mathscr{A}_0(O)$ is a local, Poincaré covariant net with Ω_0 as vacuum vector.[1]

[1]This result holds true also in $d = 2, 3$ spacetime dimensions, but for $d = 2$ the vacuum need not be a pure state of $\mathscr{A}_0$.

Each net $\mathscr{A}_0$ arising as in the previous theorem will be called a *scaling limit net* of $\mathscr{A}$, and, in view of their construction, it is natural to regard them as describing the short distance limit of the underlying theory $\mathscr{A}$.

In general, the nets $\mathscr{A}_0$ determined by the different states $\underline{\omega}_0 \in \mathrm{SL}_{\mathscr{A}}$ will be non-isomorphic to each other, and this is expected to be the case for theories that, in the conventional framework, don't possess an ultraviolet fixed point. In this case we say that $\mathscr{A}$ has a *degenerate scaling limit*. On the converse, it may happen that all the scaling limit nets are isomorphic to each other. There are then two possibilities. The first one is that $\mathscr{A}_0(O) = \mathbb{C}\mathbb{1}$ for all O, and this occurs for theories in which the algebras of small regions contain, apart from the multiples of the identity, only operators which have a 4-momentum transfer which diverges much faster than the inverse of the radius of the localization region, so that it is impossible to find RG orbits with the right phase space properties. If this is the case, we speak of a *classical scaling limit*. The second possibility is that not all the local algebras are trivial, implying that $\mathscr{A}_0$ is actually non-commutative and infinite dimensional. This is of course the more interesting case, as the short distance properties of $\mathscr{A}$ are described by a single non-trivial theory, and when it is realized we say that $\mathscr{A}$ has a *unique quantum scaling limit*. This happens for dilatation invariant theories (which comply with the Haag-Swieca compactness condition, see [**3**] for details), which are isomorphic to their own scaling limits, and for the theory of the free scalar field in $d = 3, 4$ spacetime dimension, whose scaling limit theories are all isomorphic to the theory of the massless free scalar field [**4**].

3. Superselection charges and reconstruction of fields

Superselection theory is the study of the structure of the set of (unitary equivalence classes of irreducible) representations of the quasi-local algebra of observables which describe, in some specified sense, localized excitations of the vacuum. These classes are called *superselection sectors* or *charges*. Here we will limit ourselves to a brief expository account of the results of superselection theory which we will be of relevance in the subsequent analysis of the short distance behaviour of sectors, referring the reader to [**16, 13**] or to the original papers cited below for further discussion and details.

In order to formulate a precise notion of "localized excitation" we follow [**6**], and state the criterion below (where $\restriction$ denotes restriction, and $\cong$ unitary equivalence).

Definition 3.1 (DHR selection criterion). A representation π of the quasi-local algebra is a *DHR representation* if, for every double cone O,

$$\pi \restriction \mathscr{A}(O') \cong \iota \restriction \mathscr{A}(O'), \tag{3.1}$$

with ι the (defining) vacuum representation of $\mathscr{A}$.

Assuming that the net $\mathscr{A}$ satisfies *Haag duality*, $\mathscr{A}(O) = \mathscr{A}(O')'$, and using the above unitary equivalence to identify the Hilbert space of the representation π with the vacuum Hilbert space $\mathscr{H}$, it is possible to equivalently describe the set of DHR representations by the set $\Delta_{\mathrm{t}}^{\mathrm{DHR}}$ of localizable, transportable endomorphisms ρ of $\mathscr{A}$, such a ρ being *localizable* if for some double cone O, $\rho(A) = A$ for $A \in \mathscr{A}(O')$, and *transportable* if for each O_1 there exists $\rho_1 \cong \rho$ localized in O_1. By $\Delta_{\mathrm{t}}^{\mathrm{DHR}}(O)$ we denote the ρ's localizable in O. For $\rho, \sigma \in \Delta_{\mathrm{t}}^{\mathrm{DHR}}$ the set of their *(global) intertwiners* is the set $\mathscr{I}(\rho, \sigma)$ of the $T \in \mathscr{A}$ such that $T\rho(A) = \sigma(A)T$ for each $A \in \mathscr{A}$, while *local* intertwiners $T \in \mathscr{I}(\rho,\sigma)_O$ are defined by requiring that this holds only for $A \in \mathscr{A}(O)$.

In the following we will only consider *covariant* endomorphisms $\rho \in \Delta_{\mathrm{t}}^{\mathrm{DHR}}$, namely those for which there exists a unitary, strongly continuous representation U_ρ on $\mathscr{H}$ of the (universal covering of the) Poincaré group, which satisfies the spectrum condition and such that

$$U_\rho(\Lambda,a)\rho(A)U_\rho(\Lambda,a)^* = \rho(\alpha_{(\Lambda,a)}(A)), \qquad A \in \mathscr{A},\ (\Lambda,a) \in \mathscr{P}_+^\uparrow.$$

Also, we will restrict to *finite statistics* endomorphisms, where this means, essentially, that the charge described by the sector of which such an endomorphism is a representative has exchange statistics which is (a suitable generalization of) ordinary Bose or Fermi statistics (see [**6**] for details). We denote by $\Delta_{\mathrm{c}}^{\mathrm{DHR}}$ the set of localizable, transportable, covariant endomorphisms with finite statistics. We remark that, under fairly general additional assumptions on the net $\mathscr{A}$, $\Delta_{\mathrm{t}}^{\mathrm{DHR}} = \Delta_{\mathrm{c}}^{\mathrm{DHR}}$ [**10, 11**]. We denote by $\mathrm{DHR}(\mathscr{A})$ the set of (covariant, finite statistics) DHR sectors of $\mathscr{A}$, i.e. the set of equivalence classes of irreducible elements of $\Delta_{\mathrm{c}}^{\mathrm{DHR}}$.

In models, one usually starts from a net of *field algebras* $O \to \mathscr{F}(O)$, whose (self-adjoint) elements are not observables in general, on which one as an action of a compact symmetry group G (the global gauge group) by a unitary representation V which leaves the field algebras globally invariant, $V(g)\mathscr{F}(O)V(g)^* = \mathscr{F}(O)$, and the observables are then defined as the gauge invariant part of the fields

$$\mathscr{A}(O) := \mathscr{F}(O)^G := \{F \in \mathscr{F}(O) \,:\, \beta_g(F) = F,\, \forall g \in G\},$$

where $\beta_g := \mathrm{Ad}V(g)$. Superselection sectors of $\mathscr{A}$ then arise through the factorial decomposition of V, and are hence in 1-1 correspondence with irreducible representations of the gauge group G [**5**].

The most remarkable result of superselection theory is that all this structure is actually encoded in the set $\Delta_{\mathrm{c}}^{\mathrm{DHR}}$, i.e. one can canonically reconstruct the unobservable fields and the gauge group starting only from the observable net and its DHR representations [**9**].

Theorem 3.2 (Doplicher-Roberts reconstruction). *There exists a Poincaré covariant net $O \to \mathscr{F}(O)$ on a Hilbert space $\mathscr{H}_{\mathscr{F}}$ containing $\mathscr{H}$, and a unitary strongly continuous representation V of a compact group G on $\mathscr{H}$, leaving $\mathscr{F}(O)$ and Ω invariant and commuting with $U(\mathscr{P}_+^\uparrow)$, such that:*

(1) *there exists $k \in G$, $k^2 = 1_G$, such that $\mathscr{F}$ satisfies normal commutation relations with respect to the Bose-Fermi grading defined by k (i.e. $F_\pm := 1/2(F \pm \beta_k(F))$ are the Bose and Fermi parts of $F \in \mathscr{F}(O)$);*
(2) *$\mathscr{A}(O) = \mathscr{F}(O)^G \restriction \mathscr{H}$;*
(3) *for each irreducible $\rho \in \Delta_{\mathrm{c}}^{\mathrm{DHR}}(O)$ there exist $\psi_1,\dots,\psi_d \in \mathscr{F}(O)$, and an irreducible d-dimensional unitary representation v_ρ of G, whose equivalence class depends only on the class of ρ, such that*

$$\psi_i^*\psi_j = \delta_{ij}\mathbb{1}, \quad \sum_{j=1}^d \psi_j\psi_j^* = \mathbb{1}, \quad \beta_g(\psi_i) = \sum_{j=1}^d \psi_j v_\rho(g)_{ji}, \tag{3.2}$$

$$\rho(A) = \sum_{j=1}^d \psi_j A \psi_j^*, \qquad A \in \mathscr{A}; \tag{3.3}$$

(4) *$\mathscr{F}(O)$ is generated by $\mathscr{A}(O)$ and the multiplets $\psi_1,\dots,\psi_d$ as above.*

The pair $(\mathscr{F}, V)$ is unique up to unitary equivalence.

Even restricting attention to purely massive theories, the DHR sectors do not exhibit the most general possible localization properties: as it is shown in [**1**], in general a sector in such a theory is localized in unbounded regions C called *spacelike cones*, of the form $C := a + \bigcup_{\lambda>0} \lambda O_{x,y}$, where $a \in \mathbb{R}^4$ and $O_{x,y} = (x+V_+) \cap (y+V_-)$ is a double cone spacelike to the origin such that $x^2 = y^2$. More precisely we have the following:

Theorem 3.3. *If π is an irreducible translation covariant representation of $\mathscr{A}$ in which the single particle states are separated from the continuum by a gap in the spectrum, there exists an irreducible vacuum representation π_{vac} of $\mathscr{A}$ such that*

$$\pi \restriction \mathscr{A}(C') \cong \pi_{\mathrm{vac}} \restriction \mathscr{A}(C') \tag{3.4}$$

for each spacelike cone C.

The analysis of the representations satisfying (3.4) can be developed in close parallel to the one for DHR representations. Also in this case it suffices to consider morphisms $\rho : \mathscr{A} \to B(\mathscr{H})$ which are localized in spacelike cones and transportable, and to introduce as before the set $\Delta_{\mathrm{c}}^{\mathrm{BF}}$ of such morphisms which are also covariant and with finite statistics [**1**], and the set $\mathrm{BF}(\mathscr{A})$ of equivalence classes of the irreducible ones. Finally, a version of the reconstruction theorem 3.2 holds in this case, too, the only essential difference being that one gets a field net $C \to \mathscr{F}(C)$ indexed by spacelike cones instead of double cones [**9**].

4. Short distance analysis of sectors: DHR case

In order to study the short distance behaviour of superselection charges, and to characterize their preservation under scaling limits, it is natural, in view of the results summarized in the previous section, to look at the short distance behaviour of the associated charge carrying fields, and therefore to generalize the construction of the scaling limit of the observable algebra, outlined in section 2, to the canonical Doplicher-Roberts field net discussed in the previous section. In the present and following sections we will report on the results of [**8**] in which these problems are addressed. We refer the interested reader to this paper for proofs and further discussion.

Let then $\mathscr{A}$ be a local, covariant observable net satisfying Haag duality, and $\mathscr{F}$ the associated Doplicher-Roberts field net determined by the set $\Delta_{\mathrm{c}}^{\mathrm{DHR}}$ of covariant, finite statistics localized morphisms of $\mathscr{A}$, and G the corresponding gauge group. The construction of the scaling algebra associated to $\mathscr{F}$ and of its scaling limits proceeds in close parallel to the discussion of section 2: on the C*-algebra $B(\mathbb{R}_+, \mathscr{F})$ of all norm bounded functions $\underline{F} : \mathbb{R}_+ \to \mathscr{F}$, equipped with the sup-norm $\|\underline{F}\| = \sup_{\lambda>0} \|\underline{F}_\lambda\|$, we define automorphic actions of $\mathscr{P}_+^\uparrow$ and G by

$$\underline{\alpha}_{(\Lambda,a)}(\underline{F})_\lambda := \alpha_{(\Lambda,\lambda a)}(\underline{F}_\lambda), \qquad \underline{\beta}_g(\underline{F})_\lambda := \beta_g(\underline{F}_\lambda), \qquad (\Lambda, a) \in \mathscr{P}_+^\uparrow,\ g \in G. \tag{4.1}$$

Definition 4.1. The *field scaling algebra* associated to the double cone O is the C*-algebra $\underline{\mathfrak{F}}(O)$ of functions $\underline{F} \in B(\mathbb{R}_+, \mathscr{F})$ such that:

(1) $\underline{F}_\lambda \in \mathscr{F}(\lambda O)$;
(2) $\lim_{(\Lambda,a)\to(\mathbb{1},0)} \|\underline{\alpha}_{(\Lambda,a)}(\underline{F}) - \underline{F}\| = 0$;
(3) $\lim_{g\to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0$.

The corresponding quasi-local algebra will be denoted by $\underline{\mathfrak{F}}$.

Remark 4.2. As in the case of observables, conditions (1) and (2) in the above definition are again dictated by the phase space properties of renormalization group orbits, which are of course the same for observables as for charged fields. For what concerns condition (3), it is easy to see that, in view of the definition of the lifted action $\underline{\beta}_g$ of the gauge group, eq. (4.1), in which the action of G is not rescaled, it selects orbits $\lambda \to \underline{F}_\lambda$ which essentially transfer the same charge at all scales. This means that we consider here only *dimensionless* charges, which however is not really restrictive in $d = 4$, where Noether currents are believed not to acquire anomalous dimension.

For a locally normal state φ on $\mathscr{F}$, we again define a net $(\underline{\varphi}_\lambda)_{\lambda>0}$ of states on $\underline{\mathfrak{F}}$ by $\underline{\varphi}_\lambda(\underline{F}) := \varphi(\underline{F}_\lambda)$, and the set $\mathrm{SL}_{\mathscr{F}}(\varphi)$ of scaling limit states of φ as the set of limit points of such a net.

Theorem 4.3. *The set* $\mathrm{SL}_{\mathscr{F}}(\varphi)$ *is independent of* φ*. For* $\underline{\omega}_0 \in \mathrm{SL}_{\mathscr{F}}$*, let* $(\pi_0, \mathscr{H}_0, \Omega_0)$ *be the corresponding GNS representation of* $\underline{\mathfrak{F}}$ *and define, for each double cone* O*,*

$$\mathscr{F}_0(O) := \pi_0(\underline{\mathfrak{F}}(O))''. \tag{4.2}$$

Then there exist representations U_0 *of* $\mathscr{P}_+^\uparrow$ *and* V_0 *of a suitable factor group* $G_0 = G/N_0$ *with respect to which* $O \to \mathscr{F}_0(O)$ *is a normal, Poincaré covariant field net with* Ω_0 *as vacuum vector, such that*

$$\mathscr{A}_0(O) := \pi_0(\underline{\mathfrak{A}}(O))'' = \mathscr{F}_0(O)^{G_0}. \tag{4.3}$$

Each of the nets $\mathscr{F}_0$ arising as in the above theorem will be referred to as a *scaling limit (field) net* of $\mathscr{F}$.

We stress the fact that, in general, $\mathscr{F}_0$ is not a canonical Doplicher-Roberts net for $\mathscr{A}_0$. This is essentially due to the fact that $\mathscr{F}_0$ describes sectors of $\mathscr{A}_0$ which may be thought of as short distance remnants of sectors of $\mathscr{A}$, which are, in general, only a proper subset of the sectors of $\mathscr{A}_0$: apart from technical problems (e.g. $\mathscr{A}_0$ and $\mathscr{F}_0$ need not satisfy Haag duality), it follows from equation (4.3) and from the general theory of [**9**] that, denoting by $\mathscr{F}^{(0)}$ and $G^{(0)}$ the canonical field net and gauge group of $\mathscr{A}_0$, $\mathscr{F}_0 \subset \mathscr{F}^{(0)}$ and $G_0 = G^{(0)}/N^{(0)}$ for an appropriate normal subgroup $N^{(0)}$ of $G^{(0)}$, so that the sectors of $\mathscr{A}_0$ induced by $\mathscr{F}_0$ are only those which are associated to irreducible representations of $G^{(0)}$ which are trivial on $N^{(0)}$. The situation is suggestively summarized in the following diagram [**2**]:

$$\begin{array}{ccccc} & & \mathscr{A} & & \\ & \swarrow^{\mathrm{SL}} & & \searrow^{\mathrm{DR}} & \\ \mathscr{A}_0 & & & & \mathscr{F} \\ & \searrow_{\mathrm{DR}} & & \swarrow_{\mathrm{SL}} & \\ & \mathscr{F}^{(0)} & \supseteq & \mathscr{F}_0 & \end{array}$$

Charges of $\mathscr{A}_0$ have been named *ultracharges* of $\mathscr{A}$ in [**2**], because they can be naturally interpreted as the charges described by the underlying theory in the ultraviolet (short distance) limit. If the inclusion $\mathscr{F}_0 \subset \mathscr{F}^{(0)}$ is proper, the sectors induced by $\mathscr{F}^{(0)}$ but not by $\mathscr{F}_0$ will then be naturally considered as *confined* ultracharges of $\mathscr{A}_0$ because they are only visible in the scaling limit, but cannot be created by operations at finite scales. An example of this situation is given by the Schwinger model (QED_2 with massless fermions): it can be shown that the observable algebra $\mathscr{A}$ of this model is isomorphic to (a central extension of) the net of the massive free scalar field, and therefore has no sectors, which entails $\mathscr{F} = \mathscr{A}$, and $\mathscr{F}_0 = \mathscr{A}_0$. But, as shown in [**2, 4**], $\mathscr{A}_0$ has nontrivial (BF) sectors, so that $\mathscr{F}_0 \subsetneq \mathscr{F}^{(0)}$. We remark that this notion of confinement of ultracharges does not suffer from the ambiguities of the conventional confinement notion (which is based on giving a physical interpretation to unobservable objects like quark and gluon fields), as everything is canonically fixed once the *observables* of the theory are known, as the diagram above makes clear.

We address here a sort of converse problem, and try to characterize charges that are preserved under the scaling limit operation. We remark that on physical grounds it cannot be expected that all sectors of a theory are preserved, as it may happen that localizing a charge in a region of radius λ requires energy that grows too fast as $\lambda \to 0$ (think for instance of the composition of two charges which strongly repel each other), and, recalling the discussion about the phase space properties of renormalization group orbits, it is to be expected that such a charge will not be preserved. Indeed, following this idea, it is possible to construct a class of field nets $\mathscr{F}$ such that the sectors associated to representations which are non-trivial on some proper normal subgroup $N \subset G$ disappear in the scaling limit [**7**] (while the other sectors are preserved in the sense of definition 4.4 below).

We need therefore some condition selecting sectors which are preserved under the scaling limit. In order to find such a criterion, we note that it is natural to expect that "pointlike" objects will survive the scaling limit, simply because they are scale invariant. But, quantum mechanically, this means that the energy required to localize such an object in a region of radius λ is not more than λ^{-1}, i.e., since there is no "internal structure" that has to be "squeezed", the localization energy is only limited by Heisenberg principle.

In order to implement this idea, consider, for a sector ξ, the associated Doplicher-Roberts multiplets $\psi_j(\lambda) \in \mathscr{F}(\lambda O)$ at each scale λ (see theorem 3.2). Then the states $\psi_j(\lambda)\Omega$ can be thought of as roughly describing a charge ξ localized in λO. The condition of ξ being pointlike can then be expressed as

$$\lim_{\Delta \nearrow \mathbb{R}^4} \Big(\sup_{\lambda>0} \|[E(\lambda^{-1}\Delta) - \mathbb{1}]\psi_j(\lambda)\Omega\| + \sup_{\lambda>0} \|[E(\lambda^{-1}\Delta) - \mathbb{1}]\psi_j(\lambda)^*\Omega\| \Big) = 0, \tag{4.4}$$

(with Δ a compact subset of momentum space, and E the spectral measure of the translation), i.e. we require that $\psi_j(\lambda)\Omega$, $\psi_j(\lambda)^*\Omega$ have energy that scales essentially as λ^{-1}.

We give below a slightly more general definition of charge preservation, were $\omega := \langle \Omega, (\cdot)\Omega \rangle$ denotes the vacuum state of $\mathscr{F}$.

Definition 4.4. A sector $\xi \in \mathrm{DHR}(\mathscr{A})$ is *preserved in the scaling limit state* $\underline{\omega}_0 = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ if, for each double cone O_1 and each $\lambda > 0$, there exist multiplets of class ξ, $\psi_j(\lambda) \in \mathscr{F}(\lambda O_1)$, $j = 1, \dots, d$, such that, for each $\epsilon > 0$ and each double cone O containing the closure of O_1, there exist $\underline{F}_j, \underline{F}'_j \in \underline{\mathfrak{F}}(O)$ with

$$\limsup_\kappa \Big(\|(\psi_j(\lambda_\kappa) - \underline{F}_{j\lambda_\kappa})\Omega\| + \|(\psi_j(\lambda_\kappa) - \underline{F}'_{j\lambda_\kappa})^*\Omega\| \Big) < \epsilon. \tag{4.5}$$

It can be shown that if $\psi_j(\lambda)$ satisfies (4.4) the corresponding sector is preserved in all scaling limit states.

For a scaled multiplet $\psi_j(\lambda)$ as in the above definition, we introduce the notation

$$(\underline{\alpha}_h \psi_j)_\lambda := \int_{\mathscr{P}_+^\uparrow} d\Lambda\, da\, h(\Lambda, a) \alpha_{(\Lambda, \lambda a)}(\psi_j(\lambda)), \tag{4.6}$$

where $h \in L^1(\mathscr{P}_+^\uparrow)$ and $d\Lambda\, da$ is the Haar measure on $\mathscr{P}_+^\uparrow$, and we note that, while in general $\psi_j(\cdot)$ does not belong to $\underline{\mathfrak{F}}(O_1)$, there holds $\underline{\alpha}_h \psi_j \in \underline{\mathfrak{F}}(O)$ if h has compact support and O is sufficiently large. We will write $h \to \delta$ to denote limit with respect to the partial ordering on the set of non-negative, continuous, compactly supported functions on $\mathscr{P}_+^\uparrow$ with $\int_{\mathscr{P}_+^\uparrow} h = 1$, defined by $g \succeq h$ if $\operatorname{supp} g \subseteq \operatorname{supp} h$.

Theorem 4.5. *Let ξ be a DHR sector preserved in $\underline{\omega}_0$, and $\psi_j(\lambda) \in \mathscr{F}(\lambda O_1)$ be as above. Then, with $\mathscr{F}_0$ the scaling limit net determined by $\underline{\omega}_0$, there exists*

$$\boldsymbol{\psi}_j := s^*\text{-}\lim_{h \to \delta} \pi_0(\underline{\alpha}_h \psi_j) \in \mathscr{F}_0(O),$$

(limit in the strong operator topology) for each $O \supset \overline{O}_1$. If the $\psi_j(\lambda)$ are chosen to transform like a multiplet according to the irreducible representation v_ξ, independent of λ, under the action β of G (cf. (3.2)) – which is always possible –, then the $\boldsymbol{\psi}_j$ transform under the action $\beta^{(0)} = \operatorname{Ad} V_0$ of G_0 like a multiplet according to the irreducible representation $v_\xi^{(0)}$ of G_0 given by $v_\xi^{(0)}(gN_0) = v_\xi(g)$, $g \in G$. Furthermore the equation*

$$\boldsymbol{\rho}(\boldsymbol{a}) := \sum_{j=1}^d \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^*, \qquad \boldsymbol{a} \in \mathscr{A}_0,$$

defines an irreducible, covariant, finite statistics endomorphism of $\mathscr{A}_0$ localized in O.

In view of the above results, and in particular of the fact that the gauge group representation does not change in passing to the scaling limit, it is natural to regard the sector of $\mathscr{A}_0$ induced by the DHR endomorphism $\boldsymbol{\rho}$ as the scaling limit of the preserved sector ξ.

We mention that all the sectors in the theory of a G-multiplet of free scalar fields, with G a compact Lie group, are preserved in every scaling limit state [**7**].

As an application of this notion of preservation, we state the following generalization of a result of Roberts [**15**] on the equivalence of local and global intertwiners in dilation invariant theories.

Theorem 4.6. *If the von Neumann field algebras $\mathscr{F}(O)$ are factors, $\mathscr{F}(O) \cap \mathscr{F}(O)' = \mathbb{C}\mathbb{1}$, and if each sector of $\mathscr{A}$ is preserved in some scaling limit state, then there holds the equivalence of local and global intertwiners:*

$$\mathscr{I}(\rho, \sigma) = \mathscr{I}(\rho, \sigma)_O$$

Hence if all charges are well behaved in the ultraviolet limit, much of the superselection structure can be determined purely locally.

5. Short distance analysis of sectors: BF case

In view of possible applications to physically interesting models, the analysis of the above section, being restricted to DHR sectors, is too narrow: sectors in nonabelian gauge theories are generically expected to be of BF type (the localization cone being viewed as a fattened version of the gauge flux string attached to a gauge charge), so that, if colour in QCD is not confined, it should be a BF charge. But the scaling limit theory of QCD ought to be a free theory of "quarks" and "gluons" (asymptotic freedom), and therefore colour should become a DHR charge in the limit, which however should not be identifiable with the limit of a preserved DHR charge of the underlying theory. Thus, without generalizing the result of the previous section to BF sectors, we would not be able to identify all the DHR sectors of a scaling limit theory which are the short distance limit of sectors of the underlying theory.

The main problem in performing such a generalization lies in the fact that fields creating BF sectors are themselves localized in spacelike cones (see the end of section 3), and such regions are (essentially) invariant under scaling, so that it is not a priori clear how to implement the phase space requirements characteristic of renormalization group orbits in this case. But, as just remarked, it is natural to expect that BF sectors in asymptotically free theories should become DHR sectors in the scaling limit, as the gauge string attached to charges should become weaker and weaker at small scales. Correspondingly, the associated fields should become more and more localized in bounded regions. We will therefore restrict attention to BF sectors whose associated fields are asymptotically localized in double cones, in a sense which we will readily make precise, and this will allow us to generalize the scaling algebra construction and the notion of charge preservation.

We consider a local, Poincaré covariant observable net $\mathscr{A}$ and the corresponding normal, Poincaré covariant Doplicher-Roberts field net $C \to \mathscr{F}(C)$ determined by the set of Poincaré covariant, finite statistics BF sectors of $\mathscr{A}$.[2] We will also assume that $\mathscr{F}$ satisfies a condition of *geometric modular action* stating that the modular group of the right wedge $W = \{x \in \mathbb{R}^4 : x_1 > |x_0|\}$ coincides with the Lorentz boosts in the x_1 direction (see [**8**] for precise definitions), which is generically satisfied (e.g. if the theory is defined by Wightman fields), and which follows from an analogous condition for $\mathscr{A}$ [**12**]. We will then again define on the C*-algebra $B(\mathbb{R}_+, \mathscr{F})$ actions $\underline{\alpha}$ and $\underline{\beta}$ of $\mathscr{P}_+^\uparrow$ and G as in (4.1), and we will say that $\underline{F} \in B(\mathbb{R}_+, \mathscr{F})$ is *asymptotically localized* in the double cone O if

$$\limsup_{\lambda \to 0} \sup_{\substack{\underline{A} \in \underline{\mathfrak{A}}(O') \\ \|\underline{A}\| \leq 1}} \|[\underline{F}_\lambda, \underline{A}_\lambda]\| = 0. \tag{5.1}$$

Definition 5.1. Let C be a spacelike cone and $O \subset C$ a double cone. The *scaling algebra of asymptotically localized fields* associated to C and O is the C*-algebra $\underline{\mathfrak{F}}(C, O)$ of functions $\underline{F} \in B(\mathbb{R}_+, \mathscr{F})$ such that:

(1) $\underline{F}_\lambda \in \mathscr{F}(\lambda C)$;
(2) $\underline{F}$ is asymptotically localized in O;
(3) $\lim_{(\Lambda,a)\to(\mathbb{1},0)} \|\underline{\alpha}_{(\Lambda,a)}(\underline{F}) - \underline{F}\| = 0$;
(4) $\lim_{g\to e} \|\underline{\beta}_g(\underline{F}) - \underline{F}\| = 0$.

By $\underline{\mathfrak{F}}$ we will denote the C*-algebra generated by all the algebras $\underline{\mathfrak{F}}(C, O)$.

Let φ be a normal state on $B(\mathscr{H}_{\mathscr{F}})$. We associate to it the net of states $(\underline{\varphi}_\lambda)_{\lambda>0}$ on $\underline{\mathfrak{F}}$ and the set $\mathrm{SL}_{\mathscr{F}}(\varphi)$ of scaling limit states as in the previous section.

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[2] We refer the reader to [**9**] for a precise formulation of the conditions on $\mathscr{A}$ under which such a field net exists and for a thorough discussion of its properties.

THEOREM 5.2. *If, for O containing the origin, $\mathscr{A}(O) = \bigvee_{O_0 \ni 0} \mathscr{A}(O \cap O_0')$,*[3] *then $\mathrm{SL}_{\mathscr{F}}(\varphi)$ is independent of φ. For $\underline{\omega}_0 \in \mathrm{SL}_{\mathscr{F}}(\omega)$, let $(\pi_0, \mathscr{H}_0^{\times}, \Omega_0)$ be the corresponding GNS representation of $\underline{\mathfrak{F}}$ and define, for each double cone O,*

$$\mathscr{F}_0(O) := \bigcap_{C \supset O} \pi_0(\underline{\mathfrak{F}}(C, O))''. \tag{5.2}$$

Then, identifying $\mathscr{F}_0$ with the net obtained by restricting it to the cyclic Hilbert space $\mathscr{H}_0 := \overline{\mathscr{F}_0 \Omega_0}$, there exist representations U_0 of $\mathscr{P}_+^{\uparrow}$ and V_0 of a suitable factor group $G_0 = G/N_0$ with respect to which $O \to \mathscr{F}_0(O)$ is a normal, Poincaré covariant field net with Ω_0 as vacuum vector.

Each net $\mathscr{F}_0$ defined in this way will be called a *scaling limit net of asymptotically localized fields* of $\mathscr{F}$. Therefore we obtain, from a net $C \to \mathscr{F}(C)$ of fields localized in spacelike cones, a family of nets $O \to \mathscr{F}_0(O)$ of fields localized in double cones in the scaling limit, as expected in asymptotically free theories.

The notion of preserved BF charge is now a straightforward generalization of definition 4.4, which is best formulated by introducing, for regions $R_1, R_2 \subset \mathbb{R}^4$, the notation $R_1 \Subset R_2$ to mean that there is a neighbourhood of the identity $\mathscr{N} \subset \mathscr{P}_+^{\uparrow}$, such that $\Lambda R_1 + a \subset R_2$ for each $(\Lambda, a) \in \mathscr{N}$.

DEFINITION 5.3. A sector $\xi \in \mathrm{BF}(\mathscr{A})$ is *preserved in the scaling limit state* $\underline{\omega}_0 = \lim_\kappa \underline{\omega}_{\lambda_\kappa}$ if, for each double cone O_1, and each spacelike cone $C_1 \supset O_1$ and $\lambda > 0$, there exist multiplets of class ξ, $\psi_j^{C_1}(\lambda) \in \mathscr{F}(\lambda C_1)$, $j = 1, \ldots, d$, which are asymptotically localized in O_1 and fulfill the following conditions:

(1) for each $C_1, \hat{C}_1 \supset O_1$,

$$\lim_\kappa \Big(\|(\psi_j^{C_1}(\lambda_\kappa) - \psi_j^{\hat{C}_1}(\lambda_\kappa))\Omega\| + \|(\psi_j^{C_1}(\lambda_\kappa) - \psi_j^{\hat{C}_1}(\lambda_\kappa))^*\Omega\| \Big) = 0; \tag{5.3}$$

(2) for each $\epsilon > 0$ and each pair $C \supset O$ such that $\overline{O}_1 \subset O$, $C_1 \Subset C$, there exist $\underline{F}_j, \underline{F}'_j \in \underline{\mathfrak{F}}(C, O)$ with

$$\limsup_\kappa \Big(\|(\psi_j^{C_1}(\lambda_\kappa) - \underline{F}_{j\lambda_\kappa})\Omega\| + \|(\psi_j^{C_1}(\lambda_\kappa) - \underline{F}'_{j\lambda_\kappa})^*\Omega\| \Big) < \epsilon. \tag{5.4}$$

Actually, the preservation notion given in [**8**] is slightly more general than the present one, which is however more simply stated.

THEOREM 5.4. *Let ξ be a BF sector preserved in $\underline{\omega}_0$, and $\psi_j^{C_1}(\lambda) \in \mathscr{F}(\lambda C_1)$, asymptotically localized in O_1, be as above. Then, with $\mathscr{F}_0$ determined by $\underline{\omega}_0$, there exists*

$$\boldsymbol{\psi}_j := s^*\text{-}\lim_{h \to \delta} \pi_0(\underline{\alpha}_h \psi_j^{C_1})$$

(limit in the strong operator topology). $\boldsymbol{\psi}_j$ is independent of C_1 and belongs to $\mathscr{F}_0(O)$ for each $O \supset \overline{O}_1$. If the $\psi_j^{C_1}(\lambda)$ are chosen to transform like a multiplet according to the irreducible representation v_ξ, independent of λ, under the action β of G, then the $\boldsymbol{\psi}_j$ transform under the action $\beta^{(0)} = \mathrm{Ad}\, V_0$ of G_0 like a multiplet according to the irreducible representation $v_\xi^{(0)}$ of G_0. Furthermore, if the scaling limit vacuum Hilbert space $\mathscr{H}_0^{\mathrm{vac}} := \overline{\mathscr{A}_0 \Omega_0}$ is separable, the state*

$$\omega_\xi(\boldsymbol{a}) := \sum_{j=1}^d \langle \Omega_0, \boldsymbol{\psi}_j \boldsymbol{a} \boldsymbol{\psi}_j^* \Omega_0 \rangle, \qquad \boldsymbol{a} \in \mathscr{A}_0,$$

[3]The validity of this condition has been tested in free field models.

defines a GNS representation π_ξ *such that for each* $x \in \mathbb{R}^4$,

$$\pi_\xi \upharpoonright \mathscr{A}_0(O' + x) \cong \pi_0^{\mathrm{vac}} \upharpoonright \mathscr{A}_0(O' + x),$$

i.e., π_ξ *has the DHR property for the class of all translates of* O*.*

We obtain then that preserved BF sectors give rise, in the above sense, to some kind of DHR charges of the scaling limit theories, even if with somewhat weaker properties with respect to the full-fledged DHR sectors obtained in the previous section. The remaining open problem is to decide if $\mathscr{F}_0(O)^{G_0} = \mathscr{A}_0(O)$, and if $\mathscr{F}_0$ is irreducible. If these two conditions hold, we get a genuine DHR sector in the scaling limit. We refer the reader to [**8**] for further discussion of these points.

6. Conclusions and outlook

The discussion of the previous sections can be summarized in the following diagram:

$$\begin{array}{ccccc}
\mathscr{A} & & \xrightarrow{\quad\mathrm{SL}\quad} & & \mathscr{A}_0 \\
\downarrow & & & & \downarrow \\
\mathrm{BF}(\mathscr{A}) & \hookleftarrow & \mathrm{BF}_0(\mathscr{A}) & \dashrightarrow & \mathrm{DHR}(\mathscr{A}_0)
\end{array}$$

where $\mathrm{BF}_0(\mathscr{A})$ is the set of preserved BF sectors of $\mathscr{A}$, and were the dashed arrow is the map associating to each preserved BF sector the corresponding DHR sector according to theorem 5.4 (and following remarks). This leads to a natural notion of confinement of ultracharges: a sector $\xi \in \mathrm{DHR}(\mathscr{A}_0)$ is confined if it is not in the image of $\mathrm{BF}_0(\mathscr{A})$ under this map.

Future developments of the study exposed here will include a closer analysis of the problems mentioned at the end of the previous section, as well as the study of specific models, in particular in order to test the condition of charge preservation for BF sectors. Also, it would be desirable to develop a short distance analysis of sectors not relying on the Doplicher-Roberts theorem 3.2, so as to encompass sectors with braid group statistics. Some work in this direction has been reported on in [**14**].

References

[1] Detlev Buchholz, Klaus Fredenhagen, "Locality and the structure of particle states", *Commun. Math. Phys.* **84** (1982), 1–54.

[2] Detlev Buchholz, "Quarks, gluons, colour: Facts or fiction?", *Nucl. Phys.* **B469** (1996), 333–356.

[3] Detlev Buchholz, Rainer Verch, "Scaling algebras and renormalization group in algebraic quantum field theory", *Rev. Math. Phys.* **7** (1995), 1195–1239.

[4] Detlev Buchholz, Rainer Verch, "Scaling algebras and renormalization group in algebraic quantum field theory. II: Instructive examples", *Rev. Math. Phys.* **10** (1998), 775–800.

[5] Sergio Doplicher, Rudolf Haag, John E. Roberts, "Fields, observables and gauge transformations I", *Commun. Math. Phys.* **13** (1969), 1–23.

[6] Sergio Doplicher, Rudolf Haag, John E. Roberts, "Local observables and particle statistics I", *Commun. Math. Phys.* **23** (1971), 199–230.

[7] Claudio D'Antoni, Gerardo Morsella, "Scaling algebras and superselection sectors: study of a class of models", *Rev. Math. Phys.* **18** (2006), 565–594.

[8] Claudio D'Antoni, Gerardo Morsella, Rainer Verch, "Scaling algebras for charged fields and short-distance analysis for localizable and topological charges", *Ann. Henri Poincaré* **5** (2004), 809-870.

[9] Sergio Doplicher, John E. Roberts, “Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics”, *Commun. Math. Phys.* **131** (1990), 51–107.
[10] Klaus Fredenhagen, “On the existence of anti-particles”, *Commun. Math. Phys.* **79** (1981), 141–151.
[11] Daniele Guido, Roberto Longo, “Relativistic invariance and charge conjugation in quantum field theory”, *Commun. Math. Phys.* **148** (1992), 521–551.
[12] Daniele Guido, Roberto Longo, “An algebraic spin and statistics theorem”, *Commun. Math. Phys.* **172** (1995), 517–533.
[13] Rudolf Haag, *Local quantum physics*, IInd ed., Springer, 1996.
[14] Gerardo Morsella, “The structure of charges in the ultraviolet and an intrinsic notion of confinement”, *Operator Algebras and Mathematical Physics* (Constanta, 2001) (J. M. Combes, J. Cuntz, G. Elliot, G. Nenciu, H. Siedentop, S. Stratila, eds.), Theta Foundation, 2003, pp. 315–324.
[15] John E. Roberts, “Some applications of dilatation invariance to structural questions in the theory of local observables”, *Commun. Math. Phys.* **37** (1974), 273–286.
[16] John E. Roberts, *Lectures on algebraic quantum field theory*, “The algebraic theory of superselection sectors. Introduction and recent results”, (Palermo, 1989) (Daniel Kastler, ed.), World Scientific, 1990, pp. 1–112.

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