COURSE MATHEMATICAL ANALYSIS 2 - BSC IN ENGINEERING SCIENCES
ACADEMIC YEAR 2017-18
EXERCISE CLASS OF 13/10/17

G. MORSELLA

1. Improper integrals

1.1. Tell if the following improper integrals are convergent, and in the positive case compute them:

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-1/|x|}}{x^2}\,dx$; (b) $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{\sin^\alpha x}\,dx, \quad \alpha \in \mathbb{R}$;

(c) $\displaystyle\int_0^{+\infty} \frac{x^{1/4}+2}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$; (d) $\displaystyle\int_{-1}^{+\infty} \frac{\log(x^2(1+x))}{(1+|x|)^2}\,dx$.

Solution. (a)-(c) Solved in class.

(d) It is necessary to study the behavior of the integrand function for $x \to -1^+$, $x \to 0$ and $x \to +\infty$. One has

$$\frac{\log(x^2(1+x))}{(1+|x|)^2} = \frac{2\log|x| + \log(1+x)}{(1+|x|)^2} \sim \begin{cases} \log(1+x) & \text{for } x \to -1^+, \\ \log|x| & \text{for } x \to 0, \end{cases}$$

and being, for all $a > 0$,

$$\int_0^a \log t\,dt = \lim_{\varepsilon\to 0^+} \int_\varepsilon^a \log t\,dt = \lim_{\varepsilon\to 0^+} (a\log a - a - \varepsilon\log\varepsilon + \varepsilon) = a(\log a - 1),$$

it follows that $x \mapsto \log(1+x)$ is integrable in a (right) neighborhood of $x = -1$ and $x \mapsto \log|x|$ is integrable in a neighborhood of $x = 0$. Moreover, being

$$\lim_{x\to+\infty} \frac{\log(x^2(1+x))}{(1+|x|)^{1/2}} = 0,$$

one has, eventually for $x \to +\infty$,

$$\frac{\log(x^2(1+x))}{(1+|x|)^2} = \frac{\log(x^2(1+x))}{(1+|x|)^{1/2}} \cdot \frac{1}{(1+|x|)^{3/2}} \le \frac{1}{(1+|x|)^{3/2}} \sim \frac{1}{x^{3/2}},$$

and then the given function is integrable in $[-1, +\infty)$. In order to compute the integral, it is convenient to split it as

$$\int_{-1}^{+\infty} \frac{\log(x^2(1+x))}{(1+|x|)^2}\,dx = \int_{-1}^{0} \frac{2\log(-x) + \log(1+x)}{(1-x)^2}\,dx + \int_0^{+\infty} \frac{2\log x + \log(1+x)}{(1+x)^2}\,dx.$$

Then there holds, integrating by parts,

$$\int \frac{2\log(-x) + \log(1+x)}{(1-x)^2}\,dx = \frac{2\log(-x) + \log(1+x)}{1-x} - \int \frac{1}{1-x}\left(\frac{2}{x} + \frac{1}{1+x}\right) dx,$$

and one readily verifies that

$$\frac{1}{1-x}\cdot\frac{1}{x} = \frac{1}{x} + \frac{1}{1-x}, \qquad \frac{1}{1-x}\cdot\frac{1}{1+x} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right),$$

from which

$$\frac{1}{1-x}\left(\frac{2}{x} + \frac{1}{1+x}\right) = \frac{2}{x} + \frac{5}{2(1-x)} + \frac{1}{2(1+x)},$$

and consequently

$$\int_{-1}^{0}\frac{2\log(-x)+\log(1+x)}{(1-x)^2}\,dx=$$
$$=\lim_{\substack{\alpha\to-1^+\\ \varepsilon\to0^-}}\left[\frac{\log(x^2(1+x))}{1-x}-\log x^2+\frac{5}{2}\log|1-x|-\frac{1}{2}\log(1+x)\right]_{\alpha}^{\varepsilon}.$$

Let us compute the two limits one by one. One has

$$\begin{aligned}\lim_{\varepsilon\to0^-}&\frac{\log(\varepsilon^2(1+\varepsilon))}{1-\varepsilon}-\log\varepsilon^2+\frac{5}{2}\log|1-\varepsilon|-\frac{1}{2}\log(1+\varepsilon)\\ &=\lim_{\varepsilon\to0^-}\frac{\log\varepsilon^2-(1-\varepsilon)\log\varepsilon^2}{1-\varepsilon}+\frac{\log(1+\varepsilon)}{1-\varepsilon}+\frac{5}{2}\log|1-\varepsilon|-\frac{1}{2}\log(1+\varepsilon)\\ &=\lim_{\varepsilon\to0^-}\frac{2\varepsilon\log\varepsilon}{1-\varepsilon}=0,\end{aligned}$$

and

$$\begin{aligned}\lim_{\alpha\to-1^+}&\frac{\log(\alpha^2(1+\alpha))}{1-\alpha}-\log\alpha^2+\frac{5}{2}\log|1-\alpha|-\frac{1}{2}\log(1+\alpha)\\ &=\lim_{\alpha\to-1^+}\frac{2\log(1+\alpha)-(1-\alpha)\log(1+\alpha)}{2(1-\alpha)}+\frac{\log(\alpha^2)}{1-\alpha}-\log\alpha^2+\frac{5}{2}\log|1-\alpha|\\ &=\lim_{\alpha\to-1^+}\frac{(1+\alpha)\log(1+\alpha)}{2(1-\alpha)}+\frac{5}{2}\log 2=\frac{5}{2}\log 2,\end{aligned}$$

from which

$$\int_{-1}^{0}\frac{2\log(-x)+\log(1+x)}{(1-x)^2}\,dx=-\frac{5}{2}\log 2.$$

With computations analogous to the ones above one finds then

$$\int_{0}^{+\infty}\frac{2\log x+\log(1+x)}{(1+x)^2}\,dx=\lim_{\substack{\varepsilon\to0^+\\ r\to+\infty}}\left[-\frac{\log(x^2(1+x))}{(1+x)}-\frac{1}{1+x}+2\log\frac{x}{1+x}\right]_{\varepsilon}^{r},$$

and it is clear that the $r\to+\infty$ limit vanishes, while, analogously to above,

$$\lim_{\varepsilon\to0^+}-\frac{\log(\varepsilon^2(1+\varepsilon))}{(1+\varepsilon)}-\frac{1}{1+\varepsilon}+2\log\frac{\varepsilon}{1+\varepsilon}=\lim_{\varepsilon\to0^+}\frac{2\varepsilon\log\varepsilon}{1+\varepsilon}-1=-1.$$

In conclusion

$$\int_{-1}^{+\infty}\frac{\log(x^2(1+x))}{(1+|x|)^2}\,dx=1-\frac{5}{2}\log 2.$$

1.2. Tell for which $\alpha\in\mathbb{R}$ the following improper integrals are convergent, and if applicable compute them for the indicated values of α:

(a) $\displaystyle\int_0^1\frac{\sqrt{x}}{|\log x|^{\alpha}}\,dx;\quad \alpha=-1;$ (b) $\displaystyle\int_0^{+\infty}(x^{\alpha-1}-x^{3\alpha/2})e^{-\alpha x^2/2};\ \alpha=2.$

Solution. (a) The integrand is infinitesimal for $x\to0$ for all α. Near 1 $|\log x|^{-\alpha}$ behaves as $|x-1|^{-\alpha}$, which is integrable for $\alpha<1$. When $\alpha=-1$,

$$\int_0^1\sqrt{x}|\log x|\,dx=\frac{2}{9}x^{3/2}(-3\log x+2)\Big|_0^1=\frac{4}{9}$$

(b) Solved in class.

1.3. Tell for which $\alpha\in\mathbb{R}$ the following improper integrals are convergent:

(a) $\displaystyle\int_0^{+\infty}\frac{\arctan(\alpha x)-\alpha x}{\sqrt{x^{\alpha}}(2+\log(1+e^{3x^{\alpha}}))}\,dx;$ (b) $\displaystyle\int_0^{+\infty}\sin(x^{\alpha})\,dx.$

Solution. (a) To start with, we observe that, indicated by f the integrand function, for $\alpha=0$ there holds $f=0$, which is then integrable. For $\alpha\neq0$ we must study the asymptotic behavior of f for $x \to 0$ and for $x \to \infty$. We start with the behavior for $x \to 0$, separating the two cases $\alpha > 0$ and $\alpha < 0$. For $\alpha > 0$, recalling that $\arctan t = t - \frac{t^3}{3} + o(t^4)$, one has

$$f(x) \sim -\frac{\frac{\alpha^3 x^3}{3}}{x^{\alpha/2}(2+\log 2)} = -\frac{\alpha^3}{3(2+\log 2)}\frac{1}{x^{\alpha/2-3}} \qquad \text{for } x \to 0$$

and then the given function is integrable, in a neighborhood of $x = 0$, for $0 < \alpha < 8$. For $\alpha < 0$ one has instead $e^{3x^\alpha} \to +\infty$ for $x \to 0$, and then $\log(1 + e^{3x^\alpha}) \sim 3x^\alpha$, from which

$$f(x) \sim -\frac{\frac{\alpha^3 x^3}{3}}{x^{\alpha/2}(3x^\alpha)} = -\frac{\alpha^3}{9}x^{3-3\alpha/2} \to 0 \qquad \text{for } x \to 0$$

and then f is integrable, in a neighborhood of $x = 0$, for all $\alpha < 0$. Summing up, f is integrable, in a neighborhood of $x = 0$, for all $\alpha < 8$. We now consider the asymptotic behavior for $x \to +\infty$. For $\alpha > 0$ one has again $\log(1 + e^{3x^\alpha}) \sim 3x^\alpha$ and then

$$f(x) \sim -\frac{\alpha x}{x^{\alpha/2}(3x^\alpha)} = -\frac{\alpha}{3}\frac{1}{x^{3/2\alpha-1}} \qquad \text{for } x \to +\infty,$$

and therefore the given function is integrable, for $x \to +\infty$, for $\alpha > 4/3$. Finally for $\alpha < 0$ there holds

$$f(x) \sim -\frac{\alpha x}{x^{\alpha/2}(2+\log 2)} = -\frac{\alpha}{2+\log 2}x^{1-\alpha/2} \to -\infty \qquad \text{for } x \to +\infty,$$

and then f is not integrable for $x \to +\infty$ if $\alpha < 0$. In conclusion, the given integral is convergent for $4/3 < \alpha < 8$.

(b) The integrand function is continuous and bounded in a right (open) neighborhood of $x = 0$, and therefore it is integrable there, for all $\alpha \in \mathbb{R}$. For what concerns integrablity at infinity, clearly if $\alpha < 0$ one has the asymptotic behavior $\sin(x^\alpha) \sim 1/x^{|\alpha|}$, from which the integrability for $\alpha < -1$, and the non integrability for $-1 \leq \alpha < 0$ both follow. Moreover clearly the function is constant, thus non integrable, if $\alpha = 0$. Let then $\alpha > 0$. Making the change of variable $t = x^\alpha$ one reduces to studying the convergence of the integral

$$\int_1^{+\infty} \frac{\sin t}{t^\beta}\, dt,$$

where $\beta = (\alpha - 1)/\alpha < 1$. If then $\beta \in (0, 1)$, which corresponds to $\alpha > 1$, one has, integrating by parts,

$$\int_1^r \frac{\sin t}{t^\beta}\, dt = -\frac{\cos r}{r^\beta} + \cos 1 - \beta \int_1^r \frac{\cos t}{t^{\beta+1}}\, dt$$

and since $t \mapsto \frac{\cos t}{t^{\beta+1}}$ is absolutely integrable and $\lim_{r\to+\infty} \frac{\cos r}{r^\beta} = 0$, one sees that the $r \to +\infty$ limit of the left hand side exists, i.e., the considerd integral is convergent. If then $\beta = 0$, i.e. $\alpha = 1$, the considered integral is clearly not convergent. For $\beta \in (-1, 0)$ one has moreover, integrating two times by parts,

$$\int_1^r \frac{\sin t}{t^\beta}\, dt = \left[-\frac{\cos t}{t^\beta} - \beta\frac{\sin t}{t^{\beta+1}}\right]_1^r - \beta(1+\beta)\int_1^r \frac{\sin t}{t^{\beta+2}}\, dt.$$

Being then $\beta + 2 > 1$ the $r \to +\infty$ limit of the integral in the right hand side exists, while the one of the expression in square brackets does not exist since $\beta < 0$, and then the integral in the left hand side is not convergent. Analogously, integrating by parts a sufficient number of times, one verifies that the integral is not convergent for any $\beta < 0$, i.e., for $\alpha \in (0, 1)$. Suming up, the considered integral is convergent if and only if $|\alpha| > 1$. Observe that this gives examples of integrable functions which do not vanish at infinity.

2. Sequences of functions

2.1. Determine the intervals of pointwise and uniform convergence of the following sequences of functions:

(a) $f_n(x) := \dfrac{x}{1+n^2x^2}$; (b) $f_n(x) := \dfrac{nx}{1+n^2x^2}$;

(c) $f_n(x) := \sqrt{x^2 + \dfrac{1}{n^2}}$; (d) $f_n(x) := n \log\left(1 + \dfrac{x}{n}\right)$;

(e) $f_n(x) := \sin(nx)e^{-nx}$; (f) $f_n(x) := x^n(e^{x/n} - 1)$;

(g) $f_n(x) := x(x^{1/n} - 1)$; (h) $f_n(x) := \sqrt[n]{1 + nx^n}, \quad x \geq 0$.

Solution. (a) Clearly, if $x \neq 0$, $\frac{x}{1+n^2x^2} \to 0$ for $n \to +\infty$, and $f_n(0) = 0$ for all $n \in \mathbb{N}$. Then $\{f_n\}$ converges pointwise to the identically vanishing function on $\mathbb{R}$. In order to see if the convergence is also uniform, we have to determine $\|f_n\| = \sup_{x\in\mathbb{R}} |f_n(x)|$. There holds

$$\frac{d}{dx}\left(\frac{x}{1+n^2x^2}\right) = \frac{1-n^2x^2}{(1+n^2x^2)^2} = 0 \Leftrightarrow x = \pm\frac{1}{n},$$

from which one sees that f_n has a relative minimum for $x = -\frac{1}{n}$ and a relative maximum for $x = \frac{1}{n}$. Moreover $f_n(x) \to 0$ for $x \to \pm\infty$, and then $\|f_n\| = f_n(1/n) = \frac{1}{2n} \to 0$ for $n \to +\infty$, so the convergence is uniform on $\mathbb{R}$.

(b) - (e) Solved in class.

(f) Since $e^{x/n} - 1 \sim x/n$ for $n \to +\infty$ and $x \neq 0$, it is evident that

$$\lim_{n\to+\infty} x^n(e^{x/n} - 1) = \begin{cases} +\infty & x > 1, \\ 0 & -1 \leq x \leq 1, \\ \nexists & x < -1, \end{cases}$$

and then the sequence $\{f_n\}$ converges pointwise to the identically vanishing function in $[-1, 1]$. One has furthermore $|f_n(x)| \leq |e^{x/n} - 1|$ for all $x \in [-1, 1]$, and one readily sees that $x \mapsto |e^{x/n} - 1|$ is decreasing for $x \in [-1, 0]$ and increasing for $x \in [0, 1]$, and then its maximum is in $x = -1$ or in $x = 1$. Being then $e^{1/n} - 1 \geq 1 - e^{-1/n}$ (since $\cosh t \geq 1$ for all $t \in \mathbb{R}$), one concludes that

$$\sup_{x\in[-1,1]} |f_n(x)| \leq e^{1/n} - 1 \to 0 \qquad \text{for } n \to +\infty,$$

which implies the uniform convergence in $[-1, 1]$ of the given sequence.

(g) Clearly there holds, for all $x \in [0, +\infty)$, $\lim_{n\to+\infty} x(x^{1/n} - 1) = 0$, and then the sequence $\{f_n\}$ converges pointwise to the identically vanishing function in $[0, +\infty)$. Moreover since $\lim_{x\to+\infty} x(x^{1/n} - 1) = +\infty$, one has, for every unbounded interval $I \subset [0, +\infty)$,

$$\sup_{x\in I} |f_n(x)| = +\infty \not\to 0 \qquad \text{for } n \to +\infty,$$

and then $\{f_n\}$ does not converge uniformly to the vanishing function in I. Let now $J = [0, a]$. In order to find the supremum in J of $|f_n(x)|$ we compute

$$\frac{d}{dx}x(x^{1/n} - 1) = \left(\frac{1}{n} + 1\right)x^{1/n} - 1 = 0 \;\Leftrightarrow\; x = x_n := \left(1 + \frac{1}{n}\right)^{-n},$$

and from the sign of the derivative one sees that f_n is increasing in $(0, x_n)$ and decreasing in $(x_n, +\infty)$. There follows that,

$$\sup_{x\in J} |f_n(x)| = \max\{|f_n(x_n)|, |f_n(a)|\} \to 0 \qquad \text{per } n \to +\infty,$$

and then $\{f_n\}$ converges uniformly in J, and then in every bounded interval in $[0; +\infty)$.

(h) As one sees easily, there holds

$$\lim_{n\to+\infty} \sqrt[n]{1 + nx^n} = f(x) := \begin{cases} 1 & x \in [0, 1], \\ x & x > 1. \end{cases}$$

Being then

$$\sqrt[n]{1+nx^n}-1 \le \sqrt[n]{1+n}-1 \qquad x\in[0,1],$$
$$\sqrt[n]{1+nx^n}-x = x\left(\sqrt[n]{\frac{1}{x^n}+n}-1\right) \le x(\sqrt[n]{1+n}-1) \qquad x>1,$$

one has $|f_n(x)-f(x)| \le \max\{1,x\}(\sqrt[n]{1+n}-1)$ for all $x\ge 0$, and then

$$\sup_{x\in[0,a]} |f_n(x)-f(x)| \le \max\{1,a\}(\sqrt[n]{1+n}-1) \to 0 \qquad \text{per } n\to+\infty,$$

which implies the uniform convergence of $\{f_n\}$ to f in every interval $[0,a]$, $a>0$ (and then in every bounded interval in $[0,+\infty)$). For what concerns the unbounded intervals, being, for $n\to+\infty$,

$$|f_n(n)-f(n)| = \sqrt[n]{1+n^{n+1}}-n \ge n^{1+1/n}-n = n(e^{\frac{1}{n}\log n}-1) \sim \log n \to +\infty,$$

the given sequence does not converge uniformly in any unbounded interval.

2.2. Given the sequence of functions

$$f_n(x) := \frac{x^n}{n} e^{-x\sqrt{n^2+n}}, \qquad x\in\mathbb{R},$$

(a) determine the intervals of pointwise and uniform convergence;
(b) compute $\lim_{n\to+\infty}\int_0^2 f_n(x)\,dx$;
(c) study the convergence of the improper integral $\int_0^{+\infty} f_n(x)\,dx$ and, if applicable, compute $\lim_{n\to+\infty}\int_0^{+\infty} f_n(x)\,dx$.

Solution. (a) There holds

$$|f_n(x)| = \frac{1}{n}e^{n\log|x|-x\sqrt{n^2+n}} = \frac{1}{n}e^{n(\log|x|-x)}e^{-x(\sqrt{n^2+n}-n)} = \frac{1}{n}e^{n(\log|x|-x)}e^{-x\frac{n}{\sqrt{n^2+n}+n}}.$$

From the inequality $\log x < x$ for all $x>0$, and from the fact that $x\mapsto \log|x|$ is strictly decreasing for $x<0$, there follows that there exists a unique $\bar{x}\in\mathbb{R}$ such that $\log|x|\le x$ if and only if $x\ge\bar{x}$, and moreover $\bar{x}\in(-1,0)$, which implies

$$\lim_{n\to+\infty} e^{n(\log|x|-x)} = \begin{cases} 0 & \text{se } x>\bar{x}, \\ 1 & \text{se } x=\bar{x}, \\ +\infty & \text{se } x<\bar{x}. \end{cases}$$

Observing then that $\exp[-x\frac{n}{\sqrt{n^2+n}+n}] \to e^{-x/2}$ for $n\to+\infty$, for all $x\in\mathbb{R}$, and that $n\mapsto x^n$ takes alternatively positive and negative values for $x<\bar{x}<0$, one sees that

$$\lim_{n\to+\infty} f_n(x) = \begin{cases} 0 & \text{if } x\ge\bar{x}, \\ \nexists & \text{if } x<\bar{x}. \end{cases}$$

Concerning uniform convergence, there holds

$$\frac{d}{dx}\frac{x^n}{n}e^{-x\sqrt{n^2+n}} = \frac{x^{n-1}}{n}(n-x\sqrt{n^2+n})e^{-x\sqrt{n^2+n}},$$

and then defining $x_n := \frac{n}{\sqrt{n^2+n}}$ one has that for n even f_n is non negative and decreasing in $[\bar{x},0)$ and in $(x_n,+\infty)$, and increasing in $(0,x_n)$, while for n odd f_n is increasing in $[\bar{x},x_n)$ and decreasing in $(x_n,+\infty)$. Being moreover $\lim_{x\to+\infty} f_n(x)=0$, one has

$$\|f_n\|_\infty = \sup_{x\in[\bar{x},+\infty)} |f_n(x)| = \max\{|f_n(x_n)|, |f_n(\bar{x})|\},$$

and being furthermore

$$f_n(x_n) = \frac{1}{n}e^{-n\left(1-\log\frac{n}{\sqrt{n^2+n}}\right)} \to 0 \qquad \text{for } n\to+\infty,$$

one concludes that $\|f_n\|_\infty\to 0$ and $f_n\to 0$ uniformly in $[\bar{x},+\infty)$.

(b) In view of point (a) $f_n\to 0$ uniformly in $[0,2]\subset[\bar{x},+\infty)$, and then thanks to the theorem of passage to the limit under the integral

$$\lim_{n\to+\infty}\int_0^2 f_n(x)\,dx = \int_0^2 0\,dx = 0.$$

(c) Since the considered integral is extended to an unbounded interval, we can not apply the theorem of passage to the limit under the intergal (even if the convergence is uniform on the whole integration interval). However, we can observe that being $\sqrt{n^2+n} \geq n$, there holds, for $n \to +\infty$,

$$\begin{aligned} 0 \leq \int_0^{+\infty} \frac{x^n}{n} e^{-x\sqrt{n^2+n}}\, dx &\leq \int_0^{+\infty} \frac{x^n}{n} e^{-nx}\, dx \\ &= (t = nx) = \frac{1}{n^{n+2}} \int_0^{+\infty} t^n e^{-t}\, dt = \frac{n!}{n^{n+2}} \to 0. \end{aligned}$$