COURSE MATHEMATICAL ANALYSIS 2 - BSC IN ENGINEERING SCIENCES
ACADEMIC YEAR 2017-18
EXERCISE CLASS OF 6/10/17

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1. Series

1.1. Tell if the following series with positive terms are convergent or divergent:

$$\text{(a)} \sum_{n=1}^{+\infty} \frac{n}{n^2+1}; \qquad \text{(b)} \sum_{n=1}^{+\infty} \sqrt[n]{n};$$

$$\text{(c)} \sum_{n=1}^{+\infty} \frac{3n^2+1}{n^4+n+1}; \qquad \text{(d)} \sum_{n=1}^{+\infty} \big(\sqrt[3]{n^3+1}-n\big);$$

$$\text{(e)} \sum_{n=1}^{+\infty} \frac{n}{2^n}; \qquad \text{(f)} \sum_{n=1}^{+\infty} \frac{n^n}{2^n n!};$$

$$\text{(g)} \sum_{n=1}^{+\infty} \frac{x^n}{n}, \quad x>0; \qquad \text{(h)} \sum_{n=1}^{+\infty} \frac{n^2}{n!};$$

Solution. Solved in class.

1.2. Study the convergence of the following series:

$$\text{(a)} \sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}; \qquad \text{(b)} \sum_{n=1}^{+\infty} \frac{n^2e^{1/n}-n^2-2}{n^3+\sqrt{n^2+3}};$$

$$\text{(c)} \sum_{n=1}^{+\infty} \left[(2n^2+1)\log\frac{3n^2+1}{3n^2+2}\right]^n; \qquad \text{(d)} \sum_{n=1}^{+\infty} \frac{n^{\sqrt{n}}}{n!};$$

$$\text{(e)} \sum_{n=1}^{+\infty} \left[\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^{1/\sin(1/n)}-e\right].$$

Solution. (a) Clearly the series has positive terms. Applying the ratio test one has

$$\frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} < 1 \qquad \text{for } n\to+\infty,$$

and then the series is convergent.

(b) Being, for $n\to+\infty$,

$$n^2(e^{1/n}-1)-2 = n\frac{e^{1/n}-1}{1/n}-2 \to +\infty,$$

the series has eventually positive terms. Moreover from the expansion $e^{1/n} = 1+\frac{1}{n}+o\left(\frac{1}{n}\right)$ one has, still for $n\to+\infty$,

$$\frac{n^2e^{1/n}-n^2-2}{n^3+\sqrt{n^2+3}} = \frac{n+o(n)}{n^3(1+\sqrt{1/n^4+3/n^6})} \sim 1/n^2,$$

and therefore the series convergences by the asymptotic comparison test.

(c)-(e) Solved in class.

1.3. For each of the following series, determine the set of $x \in \mathbb{R}$ for which it converges:

$$\text{(a)} \sum_{n=1}^{+\infty} x^{n!}; \qquad \text{(b)} \sum_{n=1}^{+\infty} \log\left(\frac{n+n^x}{n^x}\right);$$

$$\text{(c)} \sum_{n=1}^{+\infty} \frac{x^n}{1+|x|^n}; \qquad \text{(d)} \sum_{n=1}^{+\infty} \left(\frac{1+x}{n+x}\right)^n.$$

Solution. (a) Solved in class.

(b) Since $\frac{n+n^x}{n^x} > 1$ the series has positive terms. If $x \geq 1$ the generic terms has the asymptotic behaviour, for $n \to +\infty$,

$$\log\left(1+\frac{1}{n^{x-1}}\right) \sim \frac{1}{n^{x-1}},$$

and then the series converges for $x > 2$ and diverges for $x \in [1,2)$. If instead $x < 1$ one has

$$\log\left(\frac{n+n^x}{n^x}\right) = (1-x)\log n + \log\left(1+\frac{1}{n^{1-x}}\right) \to +\infty,$$

and then the series diverges. Summing up, the series converges if and only if $x > 2$.

(c) Solved in class.

(d) Being

$$\left|\frac{1+x}{n+x}\right| \to 0 \qquad \text{per } n \to +\infty,$$

the series converges (absolutely), by the root test, for all $x \in \mathbb{R}$.

1.4. Study the absolute and conditional convergence of the following series:

$$\text{(a)} \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{n}{n^2+1}; \qquad \text{(b)} \sum_{n=1}^{+\infty} (-1)^n \frac{n+\sin n}{n^2+9};$$

$$\text{(c)} \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{n^5}{(n+1)!}; \qquad \text{(d)} \sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}.$$

Solution. (a) The absolute value of the generic term has the asymptotic behaviour

$$\left|(-1)^{n-1}\frac{n}{n^2+1}\right| = \frac{n}{n^2+1} \sim \frac{1}{n},$$

and therefore the series is not absolutely convergent. But clearly $\frac{n}{n^2+1} \to 0$ and it is decreasing, since

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{1-x^2}{(x^2-1)^2} < 0 \Leftrightarrow |x| > 1,$$

and then the series is conditionally convergent by Leibniz's rule.

(b) Since $n + \sin n \geq n - 1 \geq 0$, one has

$$\left|(-1)^n \frac{n+\sin n}{n^2+9}\right| = \frac{n+\sin n}{n^2+9} = \frac{1}{n} \cdot \frac{1+\frac{\sin n}{n}}{1+\frac{9}{n^2}} \sim \frac{1}{n},$$

and then the series is not absolutely convergent. In order to study the conditional convergence, it is convenient to write

$$(-1)^n \frac{n+\sin n}{n^2+9} = (-1)^n \frac{n}{n^2+9} + (-1)^n \frac{\sin n}{n^2+9}.$$

Now the first term in the right hand side of the above equation is the generic term of a conditionally convergent series, similarly to exercise (a) before, while for the second term one has

$$\left|(-1)^n \frac{\sin n}{n^2+9}\right| = \frac{|\sin n|}{n^2+9} \leq \frac{1}{n^2},$$

and then it is the generic term of an absolutely convergent series by the comparison test. Thus the given series, being the sum of two convergent series, is conditionally convergent.

(c) Applying the ratio test to the absolute values series, one has

$$\left|(-1)^n \frac{(n+1)^5}{(n+2)!}\right| \cdot \left|\frac{(n+1)!}{(-1)^{n-1}n^5}\right| = \frac{1}{n+2}\left(1+\frac{1}{n}\right)^5 \to 0,$$

and then the series is absolutely convergent.

(d) One has

$$a_n := \left|\frac{(-1)^n}{\sqrt{n}+(-1)^n}\right| = \frac{1}{\sqrt{n}+(-1)^n} \sim \frac{1}{\sqrt{n}},$$

and then the series is not absolutely convergent. In order to study the conditional convergence, it is not possible to apply Leibniz's rule, since the sequence of the absolute values is not decreasing. Indeed, there holds

$$\begin{aligned} a_{2n} < a_{2n-1} &\Leftrightarrow \sqrt{2n}+1 > \sqrt{2n-1}-1 \Leftrightarrow 4\sqrt{2n} > -5 \\ a_{2n+1} > a_{2n} &\Leftrightarrow \sqrt{2n+1}-1 < \sqrt{2n}+1 \Leftrightarrow 4\sqrt{2n} > -3, \end{aligned}$$

and the last two inequalities are of course always satisfied. However, this shows that the sum of two successive terms of the given series is always negative. This suggest to consider, e.g., the even partial sums

$$\begin{aligned} s_{2n} &= \sum_{k=1}^{2n}(-1)^k a_k = \sum_{j=1}^{n}(-a_{2j-1}+a_{2j}) = \sum_{j=1}^{n}\left(\frac{1}{\sqrt{2j}+1} - \frac{1}{\sqrt{2j-1}-1}\right) \\ &= \sum_{j=1}^{n}\frac{\sqrt{2j-1}-\sqrt{2j}-2}{(\sqrt{2j-1}-1)(\sqrt{2j}+1)}, \end{aligned}$$

which are then the partial sums of series with negative terms, with asymptotic behaviour

$$\frac{\sqrt{2j-1}-\sqrt{2j}-2}{(\sqrt{2j-1}-1)(\sqrt{2j}+1)} = -\frac{1}{(\sqrt{2j-1}-1)(\sqrt{2j}+1)}\left(\frac{1}{\sqrt{2j-1}+\sqrt{2j}}+2\right) \sim -\frac{1}{j}.$$

This implies, by asymptotic comparison, that $s_{2n} \to -\infty$. Moreover $s_{2n+1} = s_{2n} - \frac{1}{\sqrt{2n+1}-1} \to -\infty$, and therefore the given series is divergent to $-\infty$, and in particular not conditionally convergent.

1.5. For each of the following series, determine the set of $x \in \mathbb{R}$ for which it converges absolutely, and that for which it converges conditionally:

(a) $\displaystyle\sum_{n=1}^{+\infty} n^2(4-x^2)^{\frac{n^2}{2n+5}}$; (b) $\displaystyle\sum_{n=1}^{+\infty}\frac{(1-2x)^n}{n-\log n}$;

(c) $\displaystyle\sum_{n=1}^{+\infty} n\sin\left(\frac{(-1)^n}{n^x}\right)$, $x \geq 0$; (d) $\displaystyle\sum_{n=1}^{+\infty}\frac{\arctan(x^n)}{n}$.

Solution. (a) Being

$$\left[n^3|4-x^2|^{\frac{n^2}{2n+5}}\right]^{1/n} = (n^3)^{1/n}|4-x^2|^{\frac{n}{2n+5}} \to |4-x^2|^{1/2} \qquad \text{per } n \to +\infty,$$

the series converges absolutely for $-1 < 4-x^2 < 1$, i.e., for $x \in (-\sqrt{5},-\sqrt{3})\cup(\sqrt{3},\sqrt{5})$, and does not converge (absolutely, nor conditionally) for $x \in (-\infty,-\sqrt{5})\cup(-\sqrt{3},\sqrt{3})\cup(\sqrt{5},+\infty)$. Finally the series reduces to $\sum_n n^3$, clearly not convergent, for $x = \pm\sqrt{3}$, and to $\sum_n n^3(-1)^{\frac{n^2}{2n+5}}$, again not convergent (the generic term is not infinitesimal), for $x = \pm\sqrt{5}$.

(b) Applying the root test to the series of absolute values, one has to compute

$$\lim_{n\to+\infty}\left[\frac{|1-2x|^n}{n-\log n}\right]^{1/n} = \lim_{n\to+\infty}\frac{|1-2x|}{e^{\frac{1}{n}\log(n-\log n)}} = |1-2x|,$$

which follows from $0 \leq \frac{1}{n}\log(n-\log n) \leq \frac{1}{n}\log n \to 0$. Then the series converges absolutely for $|1-2x| < 1$, which is equivalent to $x \in (0,1)$, and does not converge (absolutely, nor conditionally) for $x \in (-\infty,0)\cup(1,+\infty)$. If $x = 0$ the series reduces to $\sum_n \frac{1}{n-\log n}$ which has positive terms and is not convergent by asymptotic comparison, and for $x = 1$ it reduces to $\sum_n \frac{(-1)^n}{n-\log n}$, which is easily seen to be conditionally convergent by Leibniz's test.

(c) Observing that sine is an odd function, one sees that the given series is

$$\sum_{n=1}^{\infty} n\sin\left(\frac{(-1)^n}{n^x}\right) = \sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{1}{n^x}\right),$$

and then it is an alternating series.

Being then, for $n \to +\infty$,

$$\left|(-1)^n n \sin\left(\frac{1}{n^x}\right)\right| = n \sin\left(\frac{1}{n^x}\right) \sim \frac{1}{n^{x-1}},$$

one also sees that the series converges absolutely if and only if $x > 2$, and does not converge for $x \in [0, 1]$ because the generic term is not infinitesimal. Then, if $x \in (1, 2]$, consider the function $g(t) := t \sin(1/t^x)$, $t \geq 1$. There holds

$$g'(t) = \sin\frac{1}{t^x} - \frac{x}{t^x}\cos\frac{1}{t^x} = \frac{1}{t^x}\cos\frac{1}{t^x}\left(t^x \tan\frac{1}{t^x} - x\right),$$

and since $t^x \tan(1/t^x) \to 1 < x$ for $t \to +\infty$ and $\cos(1/t^x) > 0$ for t sufficiently large, it follows that g is eventually decreasing for $t \to +\infty$. Then, by Leibniz's rule, the given series is conditionally convergent for $x \in (1, 2]$.

(d) Again, being arctan an odd function, and recalling the asymptotic behaviour $\arctan t \sim t$ for $t \to 0$, one has, for $n \to +\infty$,

$$\frac{|\arctan(x^n)|}{n} = \frac{\arctan(|x|^n)}{n} \sim \begin{cases} \frac{\pi}{2n} & \text{if } |x| > 1, \\ \frac{\pi}{4n} & \text{if } |x| = 1, \\ \frac{|x|^n}{n} & \text{if } |x| < 1. \end{cases}$$

From this, one sees that the series is not absolutely convergent if $|x| \geq 1$, and is absolutely convergent for $|x| < 1$ (the series with generic term $|x|^n/n$ is convergent for $|x| < 1$ by, e.g., the root test). Moreover this also shows that for $x \geq 1$ (in which case $|x| = x$) the series is divergent, and then not conditionally convergent. For $x = -1$, being $\arctan(-1)^n = (-1)^n \arctan 1 = (-1)^n\pi/4$, the series reduces to $\sum_n (-1)^n \frac{\pi}{4n}$ which is conditionally convergent by Leibniz's rule. Finally, for $x < -1$, writing $x = -|x|$ we have to consider the alternating series $\sum_n (-1)^n \frac{\arctan(|x|^n)}{n}$. There holds

$$\frac{d}{dt}\left(\frac{\arctan(|x|^t)}{t}\right) = \frac{1}{t^2}\left(\frac{t}{|x|^t} \cdot \frac{\log|x|}{1 + |x|^{-2t}} - \arctan(|x|^t)\right),$$

and since $|x| > 1$ the expression in brackets converges to $-\frac{\pi}{2}$ for $t \to +\infty$, which shows that $\frac{\arctan(|x|^n)}{n}$ is eventually decreasing, and the considered series is conditionally convergent by Leibniz's rule.

1.6. Study the absolute and conditional convergence of the following series:

$$\text{(a)} \sum_{n=1}^{+\infty} \frac{\sin^2 n}{n^x}, \quad x \in \mathbb{R}; \qquad \text{(b)} \sum_{n=1}^{+\infty} \frac{\sin n}{\sqrt{n} + \cos n}.$$

Solution. Solved in class.