COURSE MATHEMATICAL ANALYSIS 2 - BSC IN ENGINEERING SCIENCES
ACADEMIC YEAR 2017-18
EXERCISE CLASS OF 29/09/17

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1. Sequences

1.1. Compute, when it exists, the limit for $n \to +\infty$ of the following sequences:

(a) $a_n = \left(\dfrac{2-3n}{2n+1}\right)^3$;

(b) $a_n = \sqrt[n]{3^n + 7^n}$;

(c) $a_n = \displaystyle\sum_{k=1}^{n} \frac{9}{10^k}$;

(d) $a_n = n\left(\sqrt{1+\dfrac{1}{n}} - \sqrt{1-\dfrac{1}{n}}\right)$;

(e) $a_n = \dfrac{n \sin n}{n^2+1}$;

(f) $a_n = \log(n + \sqrt{1+n^2}) - \log n$;

(g) $a_n = \log(n - \sqrt{n^2-1}) + \log n$.

(h) $a_n = n \sin\left(1 - \cos\dfrac{5}{n}\right)$;

(i) $a_n = \left(\dfrac{n}{n-1}\right)^{n+1}$;

(j) $a_n = \dfrac{n! - (n+1)!}{n^2 e^n}$;

(k) $a_n = \left(\dfrac{n^2+4n+1}{(n+1)^2}\right)^{n+1}$;

(l) $a_n = \left[\log n - \dfrac{1}{2}\log(n^2+1)\right] \sin n$;

(m) $a_n = \dfrac{1-\cos\frac{3}{n}}{\sin\frac{3}{n^2}}$.

Solution. (a)-(g) Solved in class.

(h) The limit is an $\infty \cdot 0$ indeterminate form. There holds:

$$a_n = \frac{\sin\left(1-\cos\frac{5}{n}\right)}{1-\cos\frac{5}{n}} \cdot \frac{1-\cos\frac{5}{n}}{\left(\frac{5}{n}\right)^2} \cdot \left(\frac{5}{n}\right)^2 \cdot n.$$

Since $1-\cos\frac{5}{n} \to 0$ as $n \to +\infty$ and $\frac{\sin x}{x} \to 1$ as $x \to 0$, one obtains $\lim_{n\to+\infty} \frac{\sin\left(1-\cos\frac{5}{n}\right)}{1-\cos\frac{5}{n}} = 1$. Moreover, since $\frac{1-\cos x}{x^2} \to \frac{1}{2}$ as $x \to 0$, one has $\lim_{n\to+\infty} \frac{1-\cos\frac{5}{n}}{\left(\frac{5}{n}\right)^2} = \frac{1}{2}$. Finally, clearly $\lim_{n\to+\infty} \left(\frac{5}{n}\right)^2 \cdot n = 0$, and therefore $\lim_{n\to+\infty} a_n = 1 \cdot \frac{1}{2} \cdot 0 = 0$.

(i) It's a 1^∞ indeterminate form. One can write

$$a_n = \left(\frac{n-1+1}{n-1}\right)^{n+1} = \left[\left(1+\frac{1}{n-1}\right)^{n-1}\right]^{\frac{n+1}{n-1}}.$$

From this, since $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$, and clearly $\lim_{n\to+\infty} \frac{n+1}{n-1} = 1$, we obtain $\lim_{n\to+\infty} a_n = e^1 = e$.

(j) Since $(n+1)! = (n+1)n!$, we have

$$a_n = \frac{n! - (n+1)n!}{n^2 e^n} = -\frac{nn!}{n^2 e^n} = -\frac{(n-1)!}{e^n} = -\frac{(n-1)!}{e^{n-1}} \cdot \frac{1}{e},$$

and since obviously $\frac{(n-1)!}{e^{n-1}} \to +\infty$, we obtain $a_n \to -\infty$.

(k) Solved in class.

(l) The sequence in square brackets is an $\infty - \infty$ indeterminate form, while $\sin n$ has no limit as $n \to +\infty$. Using the properties of the logarithm we can write

$$a_n = \log\left(\frac{n}{\sqrt{n^2+1}}\right) \sin n,$$

and since $\frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$, we have $\log\left(\frac{n}{\sqrt{n^2+1}}\right) \to 0$, so that, using the fact that the sequence $\{\sin n\}$ is bounded, we obtain $a_n \to 0$.

(m) It's a $\frac{0}{0}$ indeterminate form. We have

$$a_n = 3 \cdot \frac{1-\cos\frac{3}{n}}{\left(\frac{3}{n}\right)^2} \cdot \frac{\frac{3}{n^2}}{\sin\frac{3}{n^2}},$$

so, arguing as in (h), we see that $a_n \to \frac{3}{2}$.

1.2. Verify, using the definition, the validity of the following limits:

(a) $\displaystyle\lim_{n\to+\infty} \frac{n^2+1}{n} = +\infty;$

(b) $\displaystyle\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = 0;$

(c) $\displaystyle\lim_{n\to+\infty} \log(n+1) - \log n = 0;$

(d) $\displaystyle\lim_{n\to+\infty} \frac{n^2+4}{2n^2+3} = \frac{1}{2};$

(e) $\displaystyle\lim_{n\to+\infty} \sqrt{n^2+n} - n = \frac{1}{2}.$

Solution. Solved in class

2. Series

2.1. Prove that the following series converge to the indicated sums:

(a) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} = \frac{1}{2};$

(b) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+2)} = \frac{3}{4};$

(c) $\displaystyle\sum_{n=0}^{+\infty} e^{-2n} = \frac{e^2}{e^2-1};$

(d) $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{\frac{n+1}{2}}} = \frac{1}{1+\sqrt{2}}.$

Solution. Solved in class