# Infinitely many new curves of the Fučík spectrum

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Dedicated to the memory of our Fathers

ABSTRACT. - In this paper we present some results on the Fučík spectrum for the Laplace operator, that give new information on its structure. In particular, these results show that, if  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  with N > 1, then the Fučík spectrum has infinitely many curves asymptotic to the lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ , where  $\lambda_1$  denotes the first eigenvalue of the operator  $-\Delta$  in  $H_0^1(\Omega)$ . Notice that the situation is quite different in the case N = 1; in fact, in this case the Fučík spectrum may be obtained by direct computation and one can verify that it includes only two curves asymptotic to these lines.

RÉSUMÉ. - Nous présentons des résultats qui donnent nouvelles informations sur la structure du spectre de Fučík pour l'opérateur de Laplace. En particulier, ces résultats montrent que, si  $\Omega$  est un domaine borné de  $\mathbb{R}^N$  avec N > 1, alors le spectre de Fučík a un nombre infini de courbes qui ont comme asymptotes les droites  $\{\lambda_1\} \times \mathbb{R}$  et  $\mathbb{R} \times \{\lambda_1\}$ , où  $\lambda_1$  est la première valeur propre de l'operateur  $-\Delta$  in  $H_0^1(\Omega)$ . La situation est bien différente dans le cas N = 1; en effect, dans ce cas on peut vérifier qu'il y a seulement deux courbes dans le spectre de Fučík, qui ont ces droites comme asymptotes.

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### 1 Introduction

Let  $\Omega$  be a bounded connected domain of  $\mathbb{R}^N$  with  $N \geq 1$  and set  $u^+ = \max\{u, 0\}$ ,  $u^- = \max\{-u, 0\}$ . The Fučík spectrum of the Laplace operator  $-\Delta$  in  $H_0^1(\Omega)$  is defined as the set  $\Sigma$  of all the pairs  $(\alpha, \beta) \in \mathbb{R}^2$  such that the Dirichlet problem

$$\Delta u - \alpha u^{-} + \beta u^{+} = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial \Omega, \tag{1.1}$$

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has nontrivial solutions (i.e.  $u \in H_0^1(\Omega), u \neq 0$ ).

The Fučík spectrum arises, for example, in the study of problems of the type

 $\Delta u + g(x, u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial \Omega \tag{1.2}$ 

where g is a Carathéodory function in  $\Omega \times \mathbb{R}$  such that

$$\lim_{t \to -\infty} \frac{g(x,t)}{t} = \alpha, \qquad \lim_{t \to +\infty} \frac{g(x,t)}{t} = \beta \qquad \forall x \in \Omega$$
(1.3)

with  $\alpha$  and  $\beta$  in  $\mathbb{R}$ . These problems may lack of compactness in the sense that the well known Palais-Smale compactness condition fails if the pair  $(\alpha, \beta)$  belongs to the Fučík spectrum  $\Sigma$ .

After the pionering researches in [1, 8] and the first papers [12, 17], many works have been devoted to study these problems (see, for example, the references in [27–29]). In [27–29] we obtained new solutions of problems of this type using a method which does not require to know wether or not  $(\alpha, \beta) \in \Sigma$  and, in addition, may be useful to obtain new information on the structure of  $\Sigma$  (a similar method is used also in [9, 10]).

Let us denote by  $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$  the eigenvalues of  $-\Delta$  in  $H_0^1(\Omega)$ . It is clear that  $\Sigma$  contains all the pairs  $(\lambda_i, \lambda_i)$  (which are the only pairs  $(\alpha, \beta)$  in  $\Sigma$  such that  $\alpha = \beta$ ) and includes the lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ ; if  $\alpha \neq \lambda_1, \beta \neq \lambda_1$  and  $(\alpha, \beta) \in \Sigma$ , then  $\alpha > \lambda_1, \beta > \lambda_1$  and the eigenfunctions u corresponding to  $(\alpha, \beta)$  are sign changing functions; moreover,  $(\alpha, \beta) \in \Sigma$  if and only if  $(\beta, \alpha) \in \Sigma$  because a function u solves (1.1) if and only if -u solves (1.1) with  $(\beta, \alpha)$  in place of  $(\alpha, \beta)$ .

Several papers have been devoted to study the structure of  $\Sigma$  and its relation with existence and multiplicity of solutions for equations with asymmetric nonlinearities (see, for example, [2–4, 6, 7, 11–25, 33–37] etc. ...). In [12] it is shown that the two lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$  are isolated in  $\Sigma$ . Many results concern the curve in  $\Sigma$  emanating from each pair  $(\lambda_i, \lambda_i)$  (local existence and multiplicity, variational characterizations, local and global properties, etc. ...).

Combining these results, one can infer, in particular, that  $\Sigma$  contains a first curve which passes through  $(\lambda_2, \lambda_2)$  and extends to infinity. In [15] the authors prove directly the existence of such a first curve, show that it is asymptotic to the lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ , give a variational characterization of it and deduce that all the corresponding eigenfunctions have exactly two nodal regions (extending the well known nodal domain theorem of Courant).

In the case N = 1,  $\Sigma$  may be obtained by direct computation. It consists of curves emanating from the pairs  $(\lambda_i, \lambda_i) \forall i \in \mathbb{N}$ ; if *i* is an even positive integer, there exists only one curve while, if *i* is odd, there exist exactly two curves emanating from  $(\lambda_i, \lambda_i)$ . All these curves are smooth, unbounded and decreasing (i.e., on each curve,  $\alpha$  decreases as  $\beta$  increases); moreover, on each curve,  $\alpha$  tends to an eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$  as  $\beta \to$  $+\infty$ ; conversely, for every eigenvalue  $\lambda_i$  there exist exactly three curves asymptotic to the lines  $\{\lambda_i\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_i\}$ ; they pass, respectively, through the pairs  $(\lambda_{2i-1}, \lambda_{2i-1})$ ,  $(\lambda_{2i}, \lambda_{2i})$  and  $(\lambda_{2i+1}, \lambda_{2i+1})$ . In particular, if N = 1, there are only two nontrivial curves of  $\Sigma$ , asymptotic to  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ .

On the contrary, the situation is quite different in the case N > 1. In fact, using the method developed in [27–29], we can show that, if  $\Omega$  is a domain of  $\mathbb{R}^N$  with N > 1, there exist infinitely many curves of the Fučík spectrum  $\Sigma$ , asymptotic to the lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ . In the present paper we consider the case  $N \geq 3$ . The case N = 2, which requires more refined estimates, is considered in [32].

The main result of this paper may be stated as follows.

**Theorem 1.1** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$  with  $N \geq 3$ . Then, there exists a nondecreasing sequence  $(b_k)_k$  of positive numbers, having the following properties. For every positive integer k and for all  $\beta > b_k$  there exists  $\alpha_{k,\beta} > \lambda_1$  such that the pair  $(\alpha_{k,\beta},\beta)$  belongs to the Fučík spectrum  $\Sigma$ . Moreover, for every positive integer k,  $\alpha_{k,\beta}$  depends continuously on  $\beta$  in  $]b_k, +\infty[$ ,  $\alpha_{k,\beta} < \alpha_{k+1,\beta}$  for all  $\beta > b_{k+1}$  and  $\lim_{\beta \to +\infty} \alpha_{k,\beta} = \lambda_1.$ 

The proof follows directly from Theorem 2.1. It is clear that, if we replace  $(\alpha, \beta)$  and u by  $(\beta, \alpha)$  and -u, from Theorem 1.1 we obtain infinitely many curves of  $\Sigma$  asymptotic to the line  $\mathbb{R} \times \{\lambda_1\}$ .

Notice that, even for k = 1, Theorem 1.1 does not give the first curve of the Fučík spectrum (see for istance [15]) since, for all  $\beta > b_k$ , the pair  $(\alpha_{k,\beta},\beta)$  does not belong to the first curve (see also Remark 5.8 for more details).

The method we use for the proof is completely variational. For all  $\beta > 0$ , we consider the functional  $f_{\beta}$  defined by  $f_{\beta}(u) = \int_{\Omega} [|Du|^2 - \beta(u^+)^2] dx$ , constrained on the set  $S = \{u \in H_0^1(\Omega) : \int_{\Omega} (u^-)^2 dx = 1\}$ . For  $\beta > 0$  large enough, the eigenfunction u is obtained as a constrained critical point for  $f_{\beta}$  on S, while  $\alpha$  arises as the Lagrange multiplier with respect to the constraint S.

For every positive integer k, the eigenfunction  $u_{k,\beta}$  corresponding to the pair  $(\alpha_{k,\beta},\beta)$ , we obtain in this way, presents k bumps; for  $\beta > 0$  large enough, the set  $\{x \in \Omega : u_{k,\beta}(x) < 0\}$  is a connected open subset of  $\Omega$  while the set  $\{x \in \Omega : u_{k,\beta}(x) > 0\}$  has exactly k connected components. As  $\beta \to +\infty$ , the bumps concentrate near points. We describe the asymptotic behaviour of the concentration points and, in particular, we show that, if the distance between two concentration points tends to zero as  $\beta \to +\infty$ , then the approaching rate is less than the concentration rate, so that the bumps remain quite distinct; moreover, we describe the asymptotic profile of the rescaled bumps.

Finally, let us point out a natural question: where come from the curves given by Theorem 1.1? (they might come from bifurcations of the first curve of the Fučík spectrum, or from pairs  $(\lambda_i, \lambda_i)$  of higher eigenvalues, or may be they do not meet the line  $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha = \beta\}$ , etc...). It is a widely open problem which perhaps might produce interesting results (see also Remark 5.9 for a more detailed discussion of this

question). The paper is organized as follows. In Section 2 we state the main results which, in particular, imply and specify Theorem 1.1. In Section 3 we describe the variational framework and introduce a functional  $f_{\beta,\varepsilon}$ , converging to  $f_{\beta}$  as  $\varepsilon \to 0$ , which for all  $\varepsilon > 0$  presents more manageable variational properties with respect to  $f_{\beta}$ . In Section 4 we obtain some useful asymptotic estimates as  $\beta \to +\infty$ . Finally, in Section 5 we let  $\varepsilon \to 0$ , and prove the main results. Also we discuss some generalizations, forthcoming results on related questions, open problems, etc. ....

### 2 Statement of the main results

Let us denote by  $e_1$  the positive eigenfunction related to the first eigenvalue  $\lambda_1$ , normalized in  $L^2(\Omega)$ , i.e.  $e_1 \in H_0^1(\Omega)$ ,  $e_1 > 0$ ,  $\Delta e_1 + \lambda_1 e_1 = 0$  in  $\Omega$  and  $\int_{\Omega} e_1^2 dx = 1$  (since  $\Omega$  is a connected domain,  $e_1$  is unique and strictly positive in  $\Omega$ ). For every open subset A of  $\mathbb{R}^N$ , we denote by  $\lambda_1(A) \leq \lambda_2(A) \leq \lambda_3(A) \leq \ldots$  the eigenvalues of  $-\Delta$  in  $H_0^1(A)$ ; every function in  $H_0^1(A)$  is extended outside A by the value zero. The main results presented in this paper may be gathered in the following theorem (which contains and specifies Theorem 1.1).

**Theorem 2.1** Let  $\Omega$  be a bounded connected domain of  $\mathbb{R}^N$  with  $N \geq 3$ . Then, there exists a nondecreasing sequence  $(b_k)_k$  of positive numbers, having the following properties. For every positive integer k and for all  $\beta > b_k$ , there exist  $\alpha_{k,\beta} > \lambda_1$  and  $u_{k,\beta} \in H_0^1(\Omega)$ , with  $u_{k,\beta}^+ \neq 0$  and  $u_{k,\beta}^- \neq 0$ , such that (1.1), with  $\alpha = \alpha_{k,\beta}$  and  $u = u_{k,\beta}$ , is satisfied for all  $\beta > b_k$ . Moreover, for every positive integer k,  $\alpha_{k,\beta}$  depends continuously on  $\beta$  in  $]b_k, +\infty[$ ,  $\alpha_{k,\beta} < \alpha_{k+1,\beta} \forall \beta > b_{k+1}, \alpha_{k,\beta} \to \lambda_1$ , as  $\beta \to +\infty$ , while  $u_{k,\beta} \to -e_1$  in  $H_0^1(\Omega)$ .

In addition, there exist r > 0 and, for all  $k \ge 1$  and  $\beta > b_k$ , k points  $x_{1,\beta}, \ldots, x_{k,\beta}$  in  $\Omega$  such that

1) dist
$$(x_{i,\beta},\partial\Omega) > \frac{r}{\sqrt{\beta}} \quad \forall i \in \{1,\ldots,k\}, \ |x_{i,\beta} - x_{j,\beta}| > \frac{2r}{\sqrt{\beta}} \text{ for } i \neq j;$$

2) 
$$u_{k,\beta}(x) \leq 0 \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{k} B\left(x_{i,\beta}, \frac{r}{\sqrt{\beta}}\right) \text{ and } u_{k,\beta}^{+} \neq 0 \text{ in } B\left(x_{i,\beta}, \frac{r}{\sqrt{\beta}}\right) \quad \forall i \in \{1, \dots, k\};$$

3) 
$$\lim_{\beta \to +\infty} e_1(x_{i,\beta}) = \max_{\Omega} e_1 \ \forall i \in \{1, \dots, k\}, \ \lim_{\beta \to +\infty} \sqrt{\beta} |x_{i,\beta} - x_{j,\beta}| = \infty \ for \ i \neq j;$$

4) if 
$$\rho_{\beta} > 0 \ \forall \beta > b_k$$
,  $\lim_{\beta \to +\infty} \rho_{\beta} = 0 \ and \lim_{\beta \to +\infty} (\rho_{\beta} \sqrt{\beta}) = \infty$ , then  
$$\lim_{\beta \to +\infty} \sup \left\{ |u_{k,\beta}(x) + e_1(x)| : x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, \rho_{\beta}) \right\} = 0;$$

5) if,  $\forall k \in \mathbb{N}, \forall i \in \{1, \dots, k\}, \forall \beta > b_k \text{ and } \forall x \in \sqrt{\beta} (\Omega - x_{i,\beta}) \text{ we set } U_{i,k,\beta}(x) = \frac{1}{s_{i,k,\beta}} u_{k,\beta} \left(\frac{x}{\sqrt{\beta}} + x_{i,\beta}\right) \text{ where } s_{i,k,\beta} = \sup \left\{ u_{k,\beta}(x) : x \in B\left(x_{i,\beta}, \frac{r}{\sqrt{\beta}}\right) \right\}, \text{ then the rescaled function } U_{i,k,\beta} \text{ converges as } \beta \to +\infty \text{ to the radial solution } U \text{ of the problem}$ 

 $\Delta U + U^+ = 0 \quad in \ \mathbb{R}^N, \qquad U(0) = 1 \tag{2.1}$ 

and the convergence is uniform on the compact subsets of  $\mathbb{R}^N$ .

The proof will be given in Section 5. Let us point out that Theorem 2.1 holds true also for N = 2, but in this case the proof requires more refined estimates; moreover, the asymptotic behaviour of  $u_{k,\beta}$ , as  $\beta \to +\infty$ , is quite different in the cases N = 2and N > 2. In fact, if N = 2, we have  $\lim_{\beta \to +\infty} s_{i,k,\beta} = 0 \ \forall k \in \mathbb{N}, \ \forall i \in \{1, \ldots, k\}$  while, if N > 2,  $\lim_{\beta \to +\infty} s_{i,k,\beta} = c \ \forall k \in \mathbb{N}, \ \forall i \in \{1, \ldots, k\}$ , where c is a positive constant depending only on N and  $\sup_{\Omega} e_1$ . This different behaviour is strictly related to the fact that, if U is the radial solution of problem (2.1), then  $\inf_{\mathbb{R}^N} U = -\infty$  for N = 2, while  $\inf_{\mathbb{R}^N} U > -\infty$  for N > 2. The case N = 2 is presented in [32].

### 3 The variational framework

In order to prove Theorem 2.1, for every positive integer k we construct k-peaks eigenfunctions of the following type. For every  $\beta > 0$ , let us set  $r_{\beta} = \frac{3\bar{r}_1}{\sqrt{\beta}}$  where  $\bar{r}_1$  is the radius of the balls in  $\mathbb{R}^N$  for which the first eigenvalue of the Laplace operator is equal to 1, i.e.

$$\min\left\{\int_{B(0,\bar{r}_1)} |Du|^2 dx : u \in H^1_0(B(0,\bar{r}_1)), \int_{B(0,\bar{r}_1)} u^2 dx = 1\right\} = 1.$$

Let us consider the set

$$\Omega_{k,\beta} = \{ (x_1, \dots, x_k) \in \Omega^k : |x_i - x_j| \ge 2r_\beta \text{ if } i \ne j, \operatorname{dist}(x_i, \partial \Omega) \ge r_\beta \text{ for } i = 1, \dots, k \}.$$

It is clear that  $\Omega_{k,\beta} \neq \emptyset$  for  $\beta$  large enough and that, if  $(x_1, \ldots, x_k) \in \Omega_{k,\beta}$ , the balls  $B(x_1, r_\beta), \ldots, B(x_k, r_\beta)$  are pairwise disjoint and included in  $\Omega$ .

We say that a function  $u \in H_0^1(\Omega)$  belongs to  $E_{x_1,\dots,x_k}^{\beta}$  (i.e. it is a k-peaks function with respect to the balls  $B(x_1,r_{\beta}), \dots, B(x_k,r_{\beta})$ ) if  $u^+ = \sum_{i=1}^k u_i^+$  where, for all  $i \in \{1,\dots,k\}, u_i^+ \in H_0^1(\Omega), u_i^+ \not\equiv 0, u_i^+ \ge 0$  in  $\Omega, \|u_i^+\|_{L^2(\Omega)}^{-2} \int_{\Omega} x \cdot [u_i^+(x)]^2 dx = x_i$  and  $u_i^+(x) = 0 \ \forall x \in \Omega \setminus B(x_i,r_{\beta}).$ 

For all  $\beta > 0$  and  $\varepsilon > 0$ , let us consider the functional  $f_{\beta,\varepsilon} : H_0^1(\Omega) \to \mathbb{R}$  defined by

$$f_{\beta,\varepsilon}(u) = \int_{\Omega} |Du|^2 dx - 2 \int_{\Omega} G_{\beta,\varepsilon}(u) dx, \qquad (3.1)$$

where  $G_{\beta,\varepsilon}(t) = \int_0^t g_{\beta,\varepsilon}(\tau) d\tau \ \forall t \in \mathbb{R}$ , with  $g_{\beta,\varepsilon}(t) = 0 \ \forall t \leq \varepsilon$  and  $g_{\beta,\varepsilon}(t) = \beta(t-\varepsilon)$  $\forall t > \varepsilon.$ 

Now, our aim is to find k-peaks functions that are constrained critical points for the functional  $f_{\beta,\varepsilon}$  constrained on the set  $S = \{u \in H_0^1(\Omega) : \int_{\Omega} (u^-)^2 dx = 1\}.$ 

Let us consider the set  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon}$  consisting of all the functions  $u \in E_{x_1,\ldots,x_k}^{\beta}$  such that  $\|u^-\|_{L^2(\Omega)} = 1$  and  $f'_{\beta,\varepsilon}(u)[u_i^+] = 0$  for  $i = 1,\ldots,k$ .

One can easily verify that for all  $\varepsilon > 0$ , if a function  $u \in E_{x_1,\dots,x_k}^{\beta}$  is a critical point for  $f_{\beta,\varepsilon}$  constrained on S, then  $u \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}$  and, for every  $i \in \{1,\dots,k\}, f'_{\beta,\varepsilon}(u+tu_i^+)[u_i^+]$ is positive for  $t \in [-1,0[$  and negative for t > 0 (since  $\frac{1}{\tau}g_{\beta,\varepsilon}(\tau)$  is strictly increasing with respect to  $\tau$  in  $]\varepsilon, +\infty[$ ); so the function u is the unique maximum point for  $f_{\beta,\varepsilon}$ on the set  $\{u + tu_i^+ : t \in [-1, +\infty[\}$  (notice that, for  $\varepsilon = 0, f_{\beta,0}$  and  $M_{x_1,\dots,x_k}^{\beta,0}$  do not have the same properties; it is the reason for which we first introduce the parameter  $\varepsilon > 0$  and then let  $\varepsilon \to 0$ ).

**Proposition 3.1** Let k be a positive integer,  $\beta > 0$  large enough so that  $\Omega_{k,\beta} \neq \emptyset$ and consider a point  $(x_1, \ldots, x_k) \in \Omega_{k,\beta}$ . Then, for all  $\varepsilon > 0$ ,  $M_{x_1, \ldots, x_k}^{\beta, \varepsilon} \neq \emptyset$  and the minimum of the functional  $f_{\beta,\varepsilon}$  on the set  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon}$  is achieved.

<u>**Proof**</u> We have  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon} \neq \emptyset$  because of the choice of the radius  $r_{\beta}$ . In fact, taking into account that  $\sqrt{\beta} r_{\beta} > \bar{r}_{1}$ , one can find k+1 nonnegative functions  $v_{1}, \ldots, v_{k}, \bar{v}$  in  $H_{0}^{1}(\Omega)$ such that  $v_i = 0$  in  $\Omega \setminus B(x_i, r_\beta)$ ,  $\int_{\Omega} |Dv_i|^2 dx < \beta \int_{\Omega} v_i^2 dx$ ,  $\int_{\Omega} x \cdot v_i^2(x) dx = x_i \int_{\Omega} v_i^2 dx$ ,  $\int_{\Omega} \bar{v} v_i dx = 0$  for  $i = 1, \ldots, k$  and  $\int_{\Omega} \bar{v}^2 dx = 1$ . Thus, taking into account that

$$\lim_{t \to +\infty} \frac{1}{t^2} f_{\beta,\varepsilon}(tv_i) = \int_{\Omega} |Dv_i|^2 dx - \beta \int_{\Omega} v_i^2 dx < 0$$
(3.2)

and that (since  $\varepsilon > 0$ )

$$\lim_{t \to 0^+} \frac{1}{t^2} f_{\beta,\varepsilon}(tv_i) = \int_{\Omega} |Dv_i|^2 dx > 0,$$
(3.3)

we infer that for all  $\varepsilon > 0$  there exist k positive numbers  $t_{1,\varepsilon}, \ldots, t_{k,\varepsilon}$  such that the

function  $u = \sum_{i=1}^{k} t_{i,\varepsilon} v_i - \bar{v}$  belongs to  $M_{x_1,\dots,x_k}^{\beta,\varepsilon}$ . Notice that  $\inf\{f_{\beta,\varepsilon}(u) : u \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}\} \ge \lambda_1$  for all  $\beta > 0, \varepsilon > 0, (x_1,\dots,x_k) \in \Omega_{k,\beta}$ . In fact, if  $u \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}$ , we have  $f_{\beta,\varepsilon}(u) = f_{\beta,\varepsilon}(-u^-) + \sum_{i=1}^k f_{\beta,\varepsilon}(u_i^+)$ , where  $f_{\beta,\varepsilon}(-u^-) = \int_{\Omega} |Du^-|^2 dx \ge \lambda_1$  (since  $||u^-||_{L^2(\Omega)} = 1$ ) and  $f_{\beta,\varepsilon}(u_i^+) > 0$  for  $i = 1,\dots,k$  (because  $u \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}$  implies  $f_{\beta,\varepsilon}(u_i^+) = \max\{f_{\beta,\varepsilon}(tu_i^+) : t \ge 0\} > 0$  for  $\varepsilon > 0$ ).

Now, let us consider a minimizing sequence  $(u_n)_n$  for  $f_{\beta,\varepsilon}$  on  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon}$ . The same arguments as above show that (since  $\sup\{f_{\beta,\varepsilon}(u_n): n \in \mathbb{N}\} < +\infty$ ) we have

$$\lambda_1 \le \liminf_{n \to \infty} f_{\beta,\varepsilon}(-u_n^-) \le \limsup_{n \to \infty} f_{\beta,\varepsilon}(-u_n^-) < +\infty$$
(3.4)

and, for i = 1, ..., k,

$$0 \le \liminf_{n \to \infty} f_{\beta,\varepsilon}((u_n^+)_i) \le \limsup_{n \to \infty} f_{\beta,\varepsilon}((u_n^+)_i) < +\infty.$$
(3.5)

Notice that  $f_{\beta,\varepsilon}(-u_n^-) = \int_{\Omega} |Du_n^-|^2 dx$ , so (3.4) implies that the sequence  $(u_n^-)_n$  is bounded in  $H_0^1(\Omega)$ .

Now, let us prove that also the sequences  $[(u_n^+)_i]_n$  are bounded in  $H_0^1(\Omega) \,\forall i \in \{1, \ldots, k\}$ . Taking into account that  $f'_{\beta,\varepsilon}(u_n)[(u_n^+)_i] = 0 \,\forall n \in \mathbb{N}, \,\forall i \in \{1, \ldots, k\}$ , we have

$$\int_{\Omega} |D(u_n^+)_i|^2 dx = \int_{\Omega} g_{\beta,\varepsilon}((u_n^+)_i)(u_n^+)_i dx \le \beta \int_{\Omega} (u_n^+)_i^2 dx.$$
(3.6)

Thus, it suffices to prove that the sequences  $[(u_n^+)_i]_n$  are bounded in  $L^2(\Omega)$  for  $i = 1, \ldots, k$ . Arguing by contradiction, assume that (up to a subsequence)  $\lim_{n \to \infty} ||(u_n^+)_i||_{L^2(\Omega)}$   $= \infty$  for some  $i \in \{1, \ldots, k\}$  and set  $(\bar{u}_n)_i = \frac{(u_n^+)_i}{||(u_n^+)_i||_{L^2(\Omega)}}$ . Then, (3.6) implies  $\int_{\Omega} |D(\bar{u}_n)_i|^2$   $dx \leq \beta \, \forall n \in \mathbb{N}$ ; so (up to a subsequence)  $[(\bar{u}_n)_i]_n$  converges weakly in  $H_0^1(\Omega)$ , in  $L^2(\Omega)$  and a.e. in  $\Omega$  to a function  $\bar{u}_i \in H_0^1(\Omega)$ . It follows that  $\int_{\Omega} |D\bar{u}_i|^2 dx \leq \beta$ ,  $\int_{\Omega} \bar{u}_i^2 dx = 1, \, \bar{u}_i \geq 0$  in  $\Omega$  and  $\bar{u}_i = 0$  in  $\Omega \setminus B(x_i, r_\beta)$ . Moreover, one can verify by direct computation that the properties  $f'_{\beta,\varepsilon}(u_n)[(u_n^+)_i] = 0 \, \forall n \in \mathbb{N}$  and  $\lim_{n \to \infty} ||(u_n^+)_i||_{L^2(\Omega)} = \infty$ imply  $\lim_{n \to \infty} \int_{\Omega} |D(\bar{u}_n)_i|^2 dx = \beta$ . As a consequence,

$$\lim_{n \to \infty} f_{\beta,\varepsilon}'(t(\bar{u}_n)_i)[(\bar{u}_n)_i] = 2t\beta - 2\int_{\Omega} g_{\beta,\varepsilon}(t\bar{u}_i)(\bar{u}_i)dx \quad \forall t \ge 0.$$
(3.7)

Then, since  $\int_{\Omega} \bar{u}_i^2 dx = 1$ , we obtain for all  $\varepsilon > 0$ 

$$\liminf_{t \to +\infty} \left[ t\beta - \int_{\Omega} g_{\beta,\varepsilon}(t\bar{u}_i)\bar{u}_i dx \right] = \liminf_{t \to +\infty} \int_{\Omega} [\beta t\bar{u}_i - g_{\beta,\varepsilon}(t\bar{u}_i)]\bar{u}_i dx > 0.$$
(3.8)

Notice that, if we set  $t_{n,i} = ||(u_n^+)_i||_{L^2(\Omega)}$ , we have  $f'_{\beta,\varepsilon}(t(\bar{u}_n)_i)[(\bar{u}_n)_i] > 0 \ \forall t \in ]0, t_{n,i}[$ . Since  $\lim_{n \to \infty} t_{n,i} = +\infty$ , it follows

$$\liminf_{n \to \infty} f_{\beta,\varepsilon}((u_n^+)_i) = \liminf_{n \to \infty} \int_0^{t_{n,i}} f_{\beta,\varepsilon}'(t(\bar{u}_n)_i)[(\bar{u}_n)_i]dt$$
$$\geq 2 \int_0^\tau \left[ t\beta - \int_\Omega g_{\beta,\varepsilon}(t\bar{u}_i)\bar{u}_i dx \right] dt \qquad \forall \tau > 0.$$
(3.9)

Then, as  $\tau \to +\infty$ , from (3.8) we obtain  $\lim_{n\to\infty} f_{\beta,\varepsilon}((u_n^+)_i) = +\infty$ , in contradiction with (3.5). Thus, we can say that also the sequences  $[(u_n^+)_i]_n$  are bounded in  $H_0^1(\Omega)$  for  $i = 1, \ldots, k$ . As a consequence, there exist  $u^-, u_1^+, \ldots, u_k^+$  in  $H_0^1(\Omega)$  such that (up to a

subsequence)  $u_n^-$  converges as  $n \to \infty$  to  $u^-$  and  $(u_n^+)_i$  converges to  $u_i^+$ , for  $i = 1, \ldots, k$ , weakly in  $H_0^1(\Omega)$ , in  $L^2(\Omega)$  and a.e. in  $\Omega$ .

Now, let us prove that  $u_i^+ \neq 0 \ \forall i \in \{1, \ldots, k\}$ . Arguing by contradiction, assume that  $u_i^+ \equiv 0$  for some  $i \in \{1, \ldots, k\}$ . Then (because of the  $L^2(\Omega)$  convergence) from (3.6) we infer that  $\lim_{n\to\infty} \int_{\Omega} |D(u_n^+)_i|^2 dx = 0$ , which implies  $\lim_{n\to\infty} f_{\beta,\varepsilon}((u_n^+)_i) = 0$ . Therefore, we obtain a contradiction if we prove that

$$\inf\{f_{\beta,\varepsilon}(v_i^+) : v \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}\} > 0 \qquad \forall \varepsilon > 0.$$
(3.10)

Taking into account that  $f_{\beta,\varepsilon}(v_i^+) = \max\{f_{\beta,\varepsilon}(tv_i^+): t > 0\}$ , it is clear that it suffices to prove that there exist two positive constants  $\rho_{\beta,\varepsilon}$  and  $c_{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(v) \geq c_{\beta,\varepsilon}$  $\forall v \in S_i(\rho_{\beta,\varepsilon}), \text{ where }$ 

$$S_{i}(\rho_{\beta,\varepsilon}) = \left\{ v \in H_{0}^{1}(B(x_{i}, r_{\beta})) : v \ge 0 \text{ in } B(x_{i}, r_{\beta}), \int_{B(x_{i}, r_{\beta})} |Dv|^{2} dx = \rho_{\beta,\varepsilon}^{2} \right\}.$$
(3.11)

In order to prove the existence of  $c_{\beta,\varepsilon} > 0$  and  $\rho_{\beta,\varepsilon} > 0$  with these properties, let us consider the positive integer j such that

$$\lambda_{\tilde{j}}(B(x_i, 3\bar{r}_1)) \le 1 < \lambda_{\tilde{j}+1}(B(x_i, 3\bar{r}_1)).$$
(3.12)

Taking into account the choice of  $r_{\beta}$ , it follows that

$$\lambda_{\tilde{j}}(B(x_i, r_\beta)) \le \beta < \lambda_{\tilde{j}+1}(B(x_i, r_\beta)).$$
(3.13)

Now, let us denote by  $\Sigma^1_\beta$  and  $\Sigma^2_\beta$  the closed subspaces of  $H^1_0(B(x_i, r_\beta))$  spanned by the eigenfunctions of the Laplace operator  $-\Delta$  in  $H_0^1(B(x_i, r_\beta))$ , corresponding to eigenvalues  $\lambda_j(B(x_i, r_\beta))$  with, respectively,  $1 \le j \le \tilde{j}$  and  $j \ge \tilde{j} + 1$ .

For all  $\beta > 0$  and  $\varepsilon > 0$ , there exists  $\nu_{\beta,\varepsilon} > 0$  such that, if  $v \in \Sigma^1_\beta$  and  $\int_{B(x_i,r_\beta)} |Dv|^2 dx \le 1$ 

 $\nu_{\beta,\varepsilon}^2, \text{ then } |v(x)| \leq \varepsilon \ \forall x \in B(x_i, r_\beta).$ For all  $v \in H_0^1(B(x_i, r_\beta))$  such that  $\int_{B(x_i, r_\beta)} |Dv|^2 dx \leq \nu_{\beta,\varepsilon}^2, \text{ set } v = v_{1,\beta} + v_{2,\beta}, \text{ with}$  $v_{1,\beta} \in \Sigma^1_{\beta}$  and  $v_{2,\beta} \in \Sigma^2_{\beta}$ . Then, taking into account that  $\int_{B(x_i, r_{\beta})} |Dv_{1,\beta}|^2 dx \leq \nu^2_{\beta, \varepsilon}$  and as a consequence  $v_{1,\beta} \leq \varepsilon$ , we have

$$f_{\beta,\varepsilon}(v) = f_{\beta,\varepsilon}(v_{1,\beta} + v_{2,\beta}) = f_{\beta,\varepsilon}(v_{1,\beta} + v_{2,\beta}) - f_{\beta,\varepsilon}(v_{1,\beta}) + f_{\beta,\varepsilon}(v_{1,\beta})$$
(3.14)

where  $f_{\beta,\varepsilon}(v_{1,\beta}) = \int_{B(x_i,r_\beta)} |Dv_{1,\beta}|^2 dx$  and

$$f_{\beta,\varepsilon}(v_{1,\beta}+v_{2,\beta}) - f_{\beta,\varepsilon}(v_{1,\beta}) \geq f_{\beta,\varepsilon}'(v_{1,\beta})[v_{2,\beta}] + \int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 - \beta \int_{B(x_i,r_\beta)} v_{2,\beta}^2 dx$$
$$= \int_{B(x_i,r_\beta)} [|Dv_{2,\beta}|^2 - \beta v_{2,\beta}^2] dx$$
$$\geq \left(1 - \frac{\beta}{\lambda_{\tilde{j}+1}(B(x_i,r_\beta))}\right) \int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 dx \qquad (3.15)$$

because  $f'_{\beta,\varepsilon}(v_{1,\beta})[v_{2,\beta}] = 0$  and  $\int_{B(x_i,r_\beta)} |Dv_{2,\beta}|^2 dx \ge \lambda_{\tilde{i}+1}(B(x_i,r_\beta)) \int_{B(x_i,r_\beta)} v_{2,\beta}^2 dx$ . It follows that, for a suitable constant  $\tilde{c}_{\beta,\varepsilon} > 0$ , we have  $f_{\beta,\varepsilon}(v) \ge \tilde{c}_{\beta,\varepsilon} \int_{B(x_i,r_\beta)} |Dv|^2 dx$  $\forall v \in H_0^1(B(x_i,r_\beta))$  such that  $\int_{B(x_i,r_\beta)} |Dv|^2 dx \le \nu_{\beta,\varepsilon}^2$ . Therefore, it follows easily that there exist two constants  $\rho_{\beta,\varepsilon} \in ]0, \nu_{\beta,\varepsilon}[$  and  $c_{\beta,\varepsilon} > 0$  satisfying all the required properties.

Thus, we can say that  $u_i^+ \not\equiv 0 \ \forall i \in \{1, \ldots, k\}$ . Moreover, as a further consequence of the  $L^2(\Omega)$  convergence, we have

$$\|u_i^+\|_{L^2(\Omega)}^{-2} \int_{\Omega} [u_i^+(x)]^2 x \, dx = x_i \qquad \forall i \in \{1, \dots, k\}.$$
(3.16)

From the weak  $H_0^1(\Omega)$  convergence, it follows that  $f'_{\beta,\varepsilon}(u_i^+)[u_i^+] \leq 0 \ \forall i \in \{1,\ldots,k\}$ . Therefore, for all  $i \in \{1,\ldots,k\}$  there exists  $t_i \in ]0,1]$   $(t_i$  depends also on  $\beta$  and  $\varepsilon$ ) such that the function  $\tilde{u} = -u^- + \sum_{i=1}^k t_i u_i^+$  belongs to  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon}$ . Moreover, since  $f_{\beta,\varepsilon}(t_i(u_n^+)_i) \leq f_{\beta,\varepsilon}((u_n^+)_i) \ \forall n \in \mathbb{N}$ , we have

$$\liminf_{n \to \infty} f_{\beta,\varepsilon}(t_i(u_n^+)_i) \le \liminf_{n \to \infty} f_{\beta,\varepsilon}((u_n^+)_i) \quad \forall i \in \{i, \dots, k\}.$$
(3.17)

It follows that

$$f_{\beta,\varepsilon}(\tilde{u}) \leq \liminf_{n \to \infty} f_{\beta,\varepsilon} \left( -u_n^- + \sum_{i=1}^k t_i(u_n^+)_i \right)$$
  
$$\leq \liminf_{n \to \infty} f_{\beta,\varepsilon} \left( -u_n^- + \sum_{i=1}^k (u_n^+)_i \right) = \inf\{f_{\beta,\varepsilon}(u) : u \in M_{x_1,\dots,x_k}^{\beta,\varepsilon}\}. (3.18)$$

Thus, we can conclude that the minimum of  $f_{\beta,\varepsilon}$  on  $M_{x_1,\ldots,x_k}^{\beta,\varepsilon}$  is achieved and  $f_{\beta,\varepsilon}(\tilde{u}) = \min\{f_{\beta,\varepsilon}(u) : u \in M_{x_1,\ldots,x_k}^{\beta,\varepsilon}\}$ .

q.e.d.

Proposition 3.1 allows us to introduce the function  $\varphi_{k,\beta,\varepsilon}: \Omega_{k,\beta} \to \mathbb{R}$  defined by

$$\varphi_{k,\beta,\varepsilon}(x_1,\ldots,x_k) = \min_{M_{x_1,\ldots,x_k}^{\beta,\varepsilon}} f_{\beta,\varepsilon} \quad \forall (x_1,\ldots,x_k) \in \Omega_{k,\beta}, \ \forall k \in \mathbb{N}, \ \forall \beta > 0, \ \forall \varepsilon > 0.$$
(3.19)

**Proposition 3.2** For every positive integer k, for all  $\beta > 0$  and  $\varepsilon > 0$  (with  $\beta$  large enough so that  $\Omega_{k,\beta} \neq \emptyset$ ), there exists  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$  such that  $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$ .

<u>**Proof**</u> Let us consider a sequence  $(x_{1,n}, \ldots, x_{k,n})$  in  $\Omega_{k,\beta}$  such that

$$\lim_{n \to \infty} \varphi_{k,\beta,\varepsilon}(x_{1,n},\ldots,x_{k,n}) = \sup_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}.$$
(3.20)

Then, there exists  $(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$  such that, up to a subsequence,  $(x_{1,n},\ldots,x_{k,n}) \to (x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$  as  $n \to \infty$ .

By Proposition 3.1, there exists  $u_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \min \{f_{\beta,\varepsilon}(u) : u \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}\}$ . For every  $n \in \mathbb{N}$ , let us consider the function  $\hat{u}_n \in M_{x_{1,n},\dots,x_{k,n}}^{\beta,\varepsilon}$  such that  $(\hat{u}_n^+)_i(x) = (u_{k,\beta,\varepsilon}^+)_i(x + x_{i,\beta,\varepsilon} - x_{i,n})$  and  $\hat{u}_n^-$  is the minimizing function for the minimum

$$\min\left\{\int_{\Omega} |Dv|^2 dx : v \in H_0^1(\Omega), v \ge 0 \text{ in } \Omega, \int_{\Omega} v^2 dx = 1, \\ \int_{\Omega} v \, (\hat{u}_n^+)_i dx = 0 \text{ for } i = 1, \dots, k\right\}.$$
(3.21)

One can verify by standard arguments that  $\hat{u}_n \to u_{k,\beta,\varepsilon}$  in  $H_0^1(\Omega)$  and  $f_{\beta,\varepsilon}(\hat{u}_n) \to f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})$  as  $n \to \infty$ . Moreover, we have  $\min\{f_{\beta,\varepsilon}(u) : u \in M_{x_{1,n},\dots,x_{k,n}}^{\beta,\varepsilon}\} \leq f_{\beta,\varepsilon}(\hat{u}_n)$  because  $\hat{u}_n \in M_{x_{1,n},\dots,x_{k,n}}^{\beta,\varepsilon} \forall n \in \mathbb{N}$ . Thus, we obtain

$$\sup_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon} = \lim_{n \to \infty} \varphi_{k,\beta,\varepsilon}(x_{1,n}, \dots, x_{k,n})$$

$$\leq \lim_{n \to \infty} f_{\beta,\varepsilon}(\hat{u}_n) = f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon}, \dots, x_{k,\beta,\varepsilon}),$$
(3.22)

which implies  $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$ .

#### q.e.d.

### 4 Asymptotic estimates

In this section we describe the asymptotic behaviour as  $\beta \to +\infty$  of the mini-max function  $u_{k,\beta,\varepsilon}$  obtained in Section 3. Here we need some notion on the capacity. For every bounded domain A of  $\mathbb{R}^N$ , with  $N \geq 3$ , the capacity of A is defined by

$$\operatorname{cap} A = \min\left\{\int_{\mathbb{R}^N} |Du|^2 dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \ u \ge 1 \text{ a.e. in } A\right\}.$$
 (4.1)

It is well known that there exists a unique minimizing function  $u_A$ . Moreover, if  $A_1, \ldots, A_s$ , with s > 1, are pairwise disjoint bounded domains of  $\mathbb{R}^N$ , then we have

$$\operatorname{cap}\left(\bigcup_{i=1}^{s} A_{i}\right) < \sum_{i=1}^{s} \operatorname{cap}(A_{i}).$$

$$(4.2)$$

In fact, if we set  $\check{u}(x) = \max\{u_{A_i} : i = 1, \dots, s\}$ , we obtain

$$\operatorname{cap}\left(\bigcup_{i=1}^{s} A_{i}\right) \leq \int_{\mathbb{R}^{N}} |D\check{u}|^{2} dx < \sum_{i=1}^{s} \int_{\mathbb{R}^{N}} |Du_{A_{i}}|^{2} dx = \sum_{i=1}^{s} \operatorname{cap}(A_{i}).$$
(4.3)

**Proposition 4.1** For all positive integer k and for all sequences  $(\beta_n)_n$ ,  $(\varepsilon_n)_n$  of positive numbers, let us consider a sequence  $(x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n})$  of points in  $\Omega^k$  and a sequence of functions  $(u_{k,\beta_n,\varepsilon_n})_n$  in  $H_0^1(\Omega)$  such that  $(x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}) \in \Omega_{k,\beta_n}$ ,  $u_{k,\beta_n,\varepsilon_n} \in M_{x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}}^{\beta_n,\varepsilon_n}$  and  $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \min\{f_{\beta_n,\varepsilon_n}(u) : u \in M_{x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}}^{\beta_n,\varepsilon_n}\} \forall n \in \mathbb{N}$ . Moreover assume that, as  $n \to \infty$ ,  $\beta_n \to +\infty$  and  $\varepsilon_n \to \varepsilon$  such that  $0 \le \varepsilon < \frac{U(3\bar{r}_1)\max_{\Omega}e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}$ . Then,  $u_{k,\beta_n,\varepsilon_n} \to -e_1$  in  $H_0^1(\Omega)$  and

$$\limsup_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(u_{k, \beta_n, \varepsilon_n}) - \lambda_1] \le k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_{\Omega} e_1\right)^2, \quad (4.4)$$

where, for short, we denote by  $\operatorname{cap}(\bar{r}_1)$  the capacity of the balls of radius  $\bar{r}_1$  in  $\mathbb{R}^N$ . If we assume in addition that  $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n} \quad \forall n \in \mathbb{N}$ , we can say that

$$\lim_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(u_{k, \beta_n, \varepsilon_n}) - \lambda_1] = k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_{\Omega} e_1\right)^2,$$
(4.5)

$$\lim_{n \to \infty} e_1(x_{i,\beta_n,\varepsilon_n}) = \max_{\Omega} e_1 \qquad \forall i \in \{1,\dots,k\},$$
(4.6)

$$\lim_{n \to \infty} \sqrt{\beta_n} |x_{i,\beta_n,\varepsilon_n} - x_{j,\varepsilon_n,\beta_n}| = \infty \qquad \text{for } i \neq j;$$
(4.7)

moreover, if we set  $U_{\varepsilon}(x) = \varepsilon - \frac{(\varepsilon + \max_{\Omega} e_1)}{\lim_{|x| \to \infty} U(x)} U(x) \ \forall x \in \mathbb{R}^N, \ \forall \varepsilon > 0, \ as \ n \to \infty \ we \ have$ 

$$u_{k,\beta_n,\varepsilon_n}\left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\right) \longrightarrow U_{\varepsilon}(x) \qquad \forall x \in \mathbb{R}^N, \ \forall i \in \{1,\dots,k\},$$
(4.8)

and the convergence is uniform on the compact subsets of  $\mathbb{R}^N$ .

<u>Proof</u> In the proof, for short, let us write  $x_{i,n}$  and  $u_n$  instead of  $x_{i,\beta_n,\varepsilon_n}$  and  $u_{k,\beta_n,\varepsilon_n}$ . Taking into account that  $r_{\beta_n} \to 0$ , standard arguments show that  $u_n^- \to e_1$  in  $H_0^1(\Omega)$ . Notice that  $\sup\{U_{\varepsilon}(x) : |x| \ge 3\bar{r}_1\} < 0$  if and only if  $\varepsilon < \frac{U(3\bar{r}_1)\max_{\Omega}e_1}{\lim_{|x|\to\infty}U(x)-U(3\bar{r}_1)}$ , as one can verify by direct computation. Then, in order to prove (4.4), we can consider the sequence  $(\tilde{u}_n)_n$  in  $M_{x_{1,n},\ldots,x_{k,n}}^{\beta_n,\varepsilon_n}$  defined as follows. For  $i = 1,\ldots,k$ ,

$$(\tilde{u}_n^+)_i(x) = U_{\varepsilon_n}(\sqrt{\beta_n} (x - x_{i,n})] \qquad \forall x \in B\left(x_{i,n}, \frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}}\right), \tag{4.9}$$

where  $\rho_{\varepsilon_n}$  is the radius of  $\operatorname{supp}(U_{\varepsilon_n}^+)$  (which, for large *n*, is a ball strictly contained in  $B(0, 3\bar{r}_1)$  because of the assumptions on  $\varepsilon$ ) and  $\tilde{u}_n^-$  is the function in  $H_0^1(\Omega)$  such that

$$\int_{\Omega} (\tilde{u}_n^-)^2 dx = 1, \ \tilde{u}_n^-(x) = 0 \ \forall x \in \bigcup_{i=1}^k B\left(x_{i,n}, \frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}}\right), \tag{4.10}$$

$$\int_{\Omega} |D\tilde{u}_n^-|^2 dx = \min\left\{\int_{\Omega} |Du|^2 dx : u \in H_0^1(\Omega), \int_{\Omega} u^2 dx = 1, u \ge 0 \text{ in } \Omega, \right.$$

$$u = 0 \text{ in } \bigcup_{i=1}^{k} B\left(x_{i,n}, \frac{\rho_{\varepsilon_n}}{\sqrt{\beta_n}}\right) \right\}.$$
 (4.11)

It is clear that  $\tilde{u}_n \in M_{x_{1,n},\dots,x_{k,n}}^{\beta_n,\varepsilon_n}$  (since  $f'_{\beta_n,\varepsilon_n}(\tilde{u}_n)[(\tilde{u}_n^+)_i] = 0$  for  $i = 1,\dots,k$ , as one can easily verify taking into account the properties of U). It follows that

$$f_{\beta_{n},\varepsilon_{n}}(u_{n}) = \min\{f_{\beta_{n},\varepsilon_{n}}(u) : u \in M_{x_{1,n},\dots,x_{k,n}}^{\beta_{n},\varepsilon_{n}}\}$$
  
$$\leq f_{\beta_{n},\varepsilon_{n}}(\tilde{u}_{n}) = f_{\beta_{n},\varepsilon_{n}}(-\tilde{u}_{n}^{-}) + \sum_{i=1}^{k} f_{\beta_{n},\varepsilon_{n}}((\tilde{u}_{n}^{+})_{i}).$$
(4.12)

A direct computation shows that

$$\lim_{n \to \infty} \beta_n^{\frac{N-2}{2}} f_{\beta_n, \varepsilon_n}((\tilde{u}_n^+)_i) = \int_{B(0, \rho_{\varepsilon}) \setminus B(0, \bar{r}_1)} |DU_{\varepsilon}|^2 dx \quad \text{for } i = 1, \dots, k.$$
(4.13)

Moreover,  $\tilde{u}_n^- \to e_1$  in  $H_0^1(\Omega)$  and  $f_{\beta_n,\varepsilon_n}(-\tilde{u}_n^-) = \int_{\Omega} |D\tilde{u}_n^-|^2 dx \to \lambda_1$ . If we set  $\tilde{v}_n = -\tilde{u}_n^- + e_1$ , we obtain

$$f_{\beta_n,\varepsilon_n}(-\tilde{u}_n^-) = \lambda_1 + \int_{\Omega} |D\tilde{v}_n|^2 - 2\lambda_1 \int_{\Omega} e_1 \tilde{v}_n dx$$
(4.14)

and, after rescaling,

$$\beta_n^{\frac{N-2}{2}} [f_{\beta_n,\varepsilon_n}(-\tilde{u}_n^-) - \lambda_1] = \int_{\sqrt{\beta_n}\Omega} |D\tilde{V}_n|^2 dx - \frac{2\lambda_1}{\beta_n} \int_{\sqrt{\beta_n}\Omega} e_1\left(\frac{x}{\sqrt{\beta_n}}\right) \tilde{V}_n(x) dx, \quad (4.15)$$

where  $\tilde{V}_n(x) = \tilde{v}_n\left(\frac{x}{\sqrt{\beta_n}}\right) \quad \forall x \in \sqrt{\beta_n}\Omega.$ 

Clearly, there exist  $x_1, \ldots, x_k$  in  $\overline{\Omega}$  such that, up to a subsequence,  $x_{i,n} \to x_i$ , as  $n \to \infty$ , for  $i = 1, \ldots, k$ . Moreover, arguing as in [27–29], one can find h  $(h \leq k)$  pairwise disjoint subsets  $S_1, \ldots, S_h$  of  $\{1, \ldots, k\}$  such that  $\bigcup_{j=1}^h S_j = \{1, \ldots, k\}$  and  $\sqrt{\beta_n} |x_{i,n} - x_{j,n}| \to \infty$  if i and j belong to different subsets while it remains bounded if i and j both belong to the same subset (it is clear that in this case  $x_i = x_j$ ). In addition, if  $S_j$  (for  $j = 1, \ldots, h$ ) consists of  $k_j$  elements, these arguments allow us to say that there exist  $k_j$  pairwise disjoint balls in  $\mathbb{R}^N$ ,  $B(y_1^j, \rho_{\varepsilon}), \ldots, B(y_{k_j}^j, \rho_{\varepsilon})$ , such that

$$\lim_{n \to \infty} \int_{\sqrt{\beta_n}\Omega} \left[ |D\tilde{V}_n(x)|^2 - 2\frac{\lambda_1}{\beta_n} e_1\left(\frac{x}{\sqrt{\beta_n}}\right) \tilde{V}_n(x) \right] dx = \sum_{j=1}^h m_j^2 \exp\left(\bigcup_{i=1}^{k_j} B(y_i^j, \rho_\varepsilon)\right),\tag{4.16}$$

where  $m_j = e_1(x_i)$  for  $i \in S_j$  (it is clear that different choices of i in  $S_j$  give the same constant  $m_j$ ). Thus, from (4.12),(4.13) and (4.16) we obtain

$$\limsup_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n,\varepsilon_n}(u_n) - \lambda_1] = k \int_{B(0,\rho_{\varepsilon}) \setminus B(0,\bar{r}_1)} |DU_{\varepsilon}|^2 dx + \sum_{j=1}^h m_j^2 \operatorname{cap}\left(\bigcup_{i=1}^{k_j} B(y_i^j,\rho_{\varepsilon})\right).$$
(4.17)

Since  $m_j \leq \max_{\Omega} e_1$  for  $j = 1, \ldots, h$  and

$$\operatorname{cap}\left(\bigcup_{i=1}^{k_j} B(y_i^j, \rho_{\varepsilon})\right) \leq \sum_{i=1}^{k_j} \operatorname{cap} B(y_i^j, \rho_{\varepsilon}) = k_j \operatorname{cap}(B(0, \rho_{\varepsilon})), \quad (4.18)$$

it follows

$$\limsup_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(u_n) - \lambda_1]$$

$$\leq k \int_{B(0,\rho_{\varepsilon})\setminus B(0,\bar{r}_{1})} |DU_{\varepsilon}|^{2} dx + \left(\max_{\Omega} e_{1}\right)^{2} \operatorname{cap}(B(0,\rho_{\varepsilon})) \sum_{j=1}^{h} k_{j}$$
  
$$= k \int_{\mathbb{R}^{N}\setminus B(0,\bar{r}_{1})} |DU_{\varepsilon}|^{2} dx = k \operatorname{cap}(\bar{r}_{1}) \left(\varepsilon + \max_{\Omega} e_{1}\right)^{2}, \qquad (4.19)$$

that is (4.4). Let us point out that in (4.4) we have the strict inequality if  $m_j < \max_{\Omega} e_1$ or  $k_j > 1$  for some  $j \in \{1, \ldots, h\}$  (because of (4.2)).

Now, let us prove that, if we assume in addition that  $f_{\beta_n,\varepsilon_n}(u_n) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n}$  $\forall n \in \mathbb{N}$ , then we have (4.5). In fact, in this case we can show that

$$\liminf_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(u_n) - \lambda_1] \ge k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_{\Omega} e_1\right)^2.$$
(4.20)

In order to prove (4.20), let us choose  $\bar{x} \in \Omega$  such that  $e_1(\bar{x}) = \max_{\Omega} e_1$  and a sequence  $(\bar{x}_{1,n}, \ldots, \bar{x}_{k,n})_n$  in  $\Omega^k$  such that  $(\bar{x}_{1,n}, \ldots, \bar{x}_{k,n}) \in \Omega_{k,\beta_n} \quad \forall n \in \mathbb{N}$ ,

$$\lim_{n \to \infty} |\bar{x}_{i,n} - \bar{x}| = 0 \qquad \forall i \in \{1, \dots, k\}$$

$$(4.21)$$

and

$$\lim_{n \to \infty} \sqrt{\beta_n} |\bar{x}_{i,n} - \bar{x}_{j,n}| = \infty \quad \text{if } i \neq j.$$
(4.22)

Taking into account Proposition 3.1, for every  $n \in \mathbb{N}$  there exists  $\bar{u}_n \in M_{\bar{x}_{1,n},\dots,\bar{x}_{k,n}}^{\beta_n,\varepsilon_n}$  such that  $f_{\beta_n,\varepsilon_n}(\bar{u}_n) = \varphi_{k,\beta_n,\varepsilon_n}(\bar{x}_{1,n},\dots,\bar{x}_{k,n})$ . Notice that

$$f_{\beta_n,\varepsilon_n}(\bar{u}_n) = f_{\beta_n,\varepsilon_n}(-\bar{u}_n^-) + \sum_{i=1}^k f_{\beta_n,\varepsilon_n}((\bar{u}_n^+)_i)$$
(4.23)

where  $f_{\beta_n,\varepsilon_n}((\bar{u}_n^+)_i) > 0 \ \forall n \in \mathbb{N}, \ \forall i \in \{1,\ldots,k\}$ . Moreover, since  $\lim_{n \to \infty} r_{\beta_n} = 0$ , we have  $\bar{u}_n^- \to e_1$  in  $H_0^1(\Omega)$  and  $f_{\beta_n,\varepsilon_n}(-\bar{u}_n^-) \to \lambda_1$  as  $n \to \infty$ . If we set  $\bar{w}_n = -\bar{u}_n^- + e_1$ , we obtain

$$f_{\beta_n,\varepsilon_n}(-\bar{u}_n^-) = \lambda_1 + \int_{\Omega} (|D\bar{w}_n|^2 - 2\lambda_1 e_1 \bar{w}_n) dx.$$

$$(4.24)$$

Hence, taking into account (4.4), it follows that

$$\limsup_{n \to \infty} \beta_n^{\frac{N-2}{2}} \int_{\Omega} (|D\bar{w}_n|^2 - 2\lambda_1 e_1 \bar{w}_n) dx < +\infty, \tag{4.25}$$

namely

$$\limsup_{n \to \infty} \int_{\sqrt{\beta_n} \Omega} \left[ |D\bar{W}_n(x)|^2 - \frac{2}{\beta_n} \lambda_1 e_1\left(\frac{x}{\sqrt{\beta_n}}\right) \bar{W}_n(x) \right] dx < +\infty, \tag{4.26}$$

where  $\bar{W}_n(x) = \bar{w}_n\left(\frac{x}{\sqrt{\beta_n}}\right)$ .

As a consequence, arguing as in [27–29], one can verify that, for i = 1, ..., k, there exists  $\overline{W}^i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that (up to a subsequence)  $\overline{W}_n(x + \sqrt{\beta_n} \overline{x}_{i,n}) \to \overline{W}^i(x)$ ; moreover, the convergence is uniform on the compact subsets of  $\mathbb{R}^N$  and

$$\sum_{i=1}^{k} \int_{\mathbb{R}^{N}} |D\bar{W}^{i}|^{2} dx \leq \liminf_{n \to \infty} \int_{\sqrt{\beta_{n}}\Omega} \left[ |D\bar{W}_{n}(x)|^{2} - \frac{2}{\beta_{n}} \lambda_{1} e_{1}\left(\frac{x}{\beta_{n}}\right) \bar{W}_{n}(x) \right] dx. \quad (4.27)$$

Now, we examine the asymptotic behaviour of the functions  $(\bar{u}_n^+)_i$  for  $i = 1, \ldots, k$ . Let us set  $\bar{V}_{i,n}(x) = c_{n,i}(\bar{u}_n^+)_i \left(\frac{x}{\sqrt{\beta_n}} + \bar{x}_{i,n}\right) \forall x \in \sqrt{\beta_n} \Omega$  where  $c_{n,i} = \beta_n^{-\frac{N}{2}} ||(\bar{u}_n^+)_i||_{L^2(\Omega)}^{-1}$ . Then,  $\bar{V}_{i,n} \in H_0^1(B(0, 3\bar{r}_1))$ ,  $\int_{B(0,3\bar{r}_1)} \bar{V}_{i,n}^2 dx = 1$  and  $\int_{B(0,\bar{r}_1)} |D\bar{V}_{i,n}|^2 dx < 1 \ \forall n \in \mathbb{N}$ ,  $\forall i \in \{1, \ldots, k\}$  (because  $f'_{\beta_n,\varepsilon_n}(\bar{u}_n)[(\bar{u}_n^+)_i] = 0$ ). Therefore, up to a subsequence,  $\bar{V}_{i,n}$ converges to a function  $\bar{V}_i \in H_0^1(B(0, 3\bar{r}_1))$  in  $L^2$ , weakly in  $H_0^1$  and a.e. in  $B(0, 3\bar{r}_1)$ . Thus, we have  $\int_{B(0,3\bar{r}_1)} \bar{V}_i^2 dx = 1$  and  $\int_{B(0,3\bar{r}_1)} |D\bar{V}_i|^2 dx \leq 1$  for  $i = 1, \ldots, k$ . As a consequence, we obtain

$$\int_{\mathbb{R}^N} |D\bar{W}^i|^2 dx \ge \operatorname{cap}(\bar{r}_1) \left(\max_{\Omega} e_1\right)^2 \quad \forall i \in \{1, \dots, k\}$$
(4.28)

because the balls of radius  $\bar{r}_1$  have the smallest capacity among the domains whose first eigenvalue is less than or equal to 1. Moreover, since only these balls have this property, in the case  $\varepsilon = \lim_{n \to \infty} \varepsilon_n = 0$ , (4.4) and (4.28) allow us to say that  $\bar{W}^i = \max_{\Omega} e_1 \left[ 1 + \frac{U^-}{\lim_{|x|\to\infty} U(x)} \right]$  and  $\bar{V}^i = cU^+ \ \forall i \in \{1,\ldots,k\}$ , where  $c = ||U^+||_{L^2}^{-1}$ . Furthermore, the minimality property of  $\bar{u}_n$  implies that  $\bar{u}_n \left( \frac{x}{\sqrt{\beta_n}} + x_{i,n} \right) \to U_0(x) =$  $\max_{\Omega} e_1 \left| \lim_{|x|\to\infty} U(x) \right|^{-1} U(x)$  uniformly on the compact subsets of  $\mathbb{R}^N$  (as one can verify arguing as in [27–29]). In the case  $\varepsilon > 0$ , arguing as in the proof of Proposition 3.1, one can verify that there exist k positive numbers  $\bar{t}_1, \ldots, \bar{t}_k$  such that

$$\bar{t}_i \int_{B(0,3\bar{r}_1)} |D\bar{V}_i|^2 dx = \int_{B(0,3\bar{r}_1)} g_{1,\varepsilon}(\bar{t}_i \bar{V}_i) \bar{V}_i dx \qquad \forall i \in \{1,\dots,k\}$$
(4.29)

and

$$\bar{t}_{i}^{2} \int_{B(0,3\bar{r}_{1})} |D\bar{V}_{i}|^{2} dx - 2 \int_{B(0,3\bar{r}_{1})} G_{1,\varepsilon}(\bar{t}_{i}\bar{V}_{i}) dx \le \liminf_{n \to \infty} \beta_{n}^{\frac{N-2}{2}} f_{\beta_{n},\varepsilon_{n}}((\bar{u}_{n}^{+})_{i}).$$
(4.30)

Thus, taking into account (4.27), we obtain

$$\liminf_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(\bar{u}_n) - \lambda_1] \ge \sum_{i=1}^k F_{\varepsilon}(\bar{W}^i + \bar{t}_i \bar{V}_i), \qquad (4.31)$$

where  $F_{\varepsilon}: \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$  is the functional defined by

$$F_{\varepsilon}(v) = \int_{\mathbb{R}^N} |Dv|^2 dx - 2 \int_{\mathbb{R}^N} \Gamma_{\varepsilon}(v) dx$$
(4.32)

with  $\Gamma_{\varepsilon}(t) = \int_{0}^{t} \gamma_{\varepsilon}(\tau) d\tau$ , where  $\gamma_{\varepsilon}(\tau) = \tau - (\varepsilon + \max_{\Omega} e_{1}) \quad \forall \tau \geq \varepsilon + \max_{\Omega} e_{1}$  and  $\gamma_{\varepsilon}(\tau) = 0 \quad \forall \tau \leq \varepsilon + \max_{\Omega} e_{1}$ .

Now, notice that

$$F_{\varepsilon}(W^{i} + \bar{t}_{i}\bar{V}_{i}) \ge F_{\varepsilon}\left(U_{\varepsilon} + \max_{\Omega} e_{1}\right) > 0 \quad \forall \varepsilon > 0, \ \forall i \in \{1, \dots, k\}$$

$$(4.33)$$

because  $F_{\varepsilon}(U_{\varepsilon} + \max_{\Omega} e_1)$  is the mountain pass level for the functional  $F_{\varepsilon}$  while  $F_{\varepsilon}(\bar{W}^i + \bar{t}_i \bar{V}_i)$  is the maximum of  $F_{\varepsilon}$  on the continuous path  $\Pi : [0, +\infty[ \to \mathcal{D}^{1,2}(\mathbb{R}^N)$  defined by  $\Pi(t) = t\bar{W}^i$  for  $t \in [0, 1], \Pi(t) = \bar{W}^i + (t-1)\bar{t}_i\bar{V}_i \ \forall t \in [1, +\infty[$ , which satisfies  $\Pi(0) = 0$ ,  $\lim_{t \to +\infty} \|\Pi(t)\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = +\infty, \ F_{\varepsilon}(\Pi(0)) = 0$  and  $\lim_{t \to +\infty} F_{\varepsilon}(\Pi(t)) = -\infty$ , as one can verify by direct computation.

Thus, we finally obtain (4.20) taking into account that  $f_{\beta_n,\varepsilon_n}(u_n) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n} \geq \varphi_{k,\beta_n,\varepsilon_n}(\bar{x}_{1,n},\ldots,\bar{x}_{k,n}) = f_{\beta_n,\varepsilon_n}(\bar{u}_n)$  and that  $F_{\varepsilon}(U_{\varepsilon} + \max_{\Omega} e_1) = (\varepsilon + \max_{\Omega} e_1)^2 \operatorname{cap}(\bar{r}_1)$ . Let us point out that, indeed, we must have  $\bar{W}^i + \bar{t}_i \bar{V}_i = U_{\varepsilon} + \max_{\Omega} e_1$  otherwise in (4.20) we have the strict inequality, in contradiction with (4.4). In fact, the radial function  $U_{\varepsilon} + \max_{\Omega} e_1$  is the unique mountain pass type critical point for  $F_{\varepsilon}$  (as one can show by radial symmetrization arguments) while  $\bar{W}^i + \bar{t}_i \bar{V}_i$  is the maximum point for  $F_{\varepsilon}$  on the continuous path II. Therefore, taking also into account the minimality properties of  $\bar{u}_n$ , it follows that  $\bar{u}_n\left(\frac{x}{\sqrt{\beta_n}} + \bar{x}_{i,n}\right) \to U_{\varepsilon}(x) \; \forall x \in \mathbb{R}^N, \; \forall i \in \{1,\ldots,k\}$  and the convergence is uniform on the compact subsets of  $\mathbb{R}^N$ .

Thus, we can say that (4.5) is satisfied and that (4.6), (4.7) hold otherwise in (4.4) we have the strict inequality; as a consequence, arguing as before for  $\bar{u}_n$ , we can say that also (4.8) must hold otherwise we have the strict inequality in (4.20), in contradiction with (4.5).

Finally, notice that  $u_n^+ \to 0$  in  $H_0^1(\Omega)$ , which implies  $u_n \to -e_1$  in  $H_0^1(\Omega)$ ; so the proof is complete.

q.e.d.

**Proposition 4.2** For all positive integer k, for  $\beta > 0$  such that  $\Omega_{k,\beta} \neq \emptyset$  and for all  $\varepsilon \in ]0, \overline{\varepsilon}]$  with  $0 < \overline{\varepsilon} < \frac{U(3\overline{r}_1) \max_{\Omega} e_1}{\lim_{|x| \to \infty} U(x) - U(3\overline{r}_1)}$ , let us choose  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon})$  in  $\Omega_{k,\beta}$  and  $u_{k,\beta,\varepsilon}$  in  $M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$ ,

$$\lim_{\beta \to +\infty} \inf\{e_1(x_{i,\beta,\varepsilon}) : i \in \{1,\dots,k\}, \ \varepsilon \in ]0,\overline{\varepsilon}]\} = \max_{\Omega} e_1, \tag{4.34}$$

$$\lim_{\beta \to +\infty} \sqrt{\beta} \inf\{ |x_{i,\beta,\varepsilon} - x_{j,\beta,\varepsilon}| : \varepsilon \in ]0, \overline{\varepsilon}] \} = +\infty \quad \text{for } i \neq j.$$
(4.35)

Then, there exists  $\bar{r} \in ]0, 3\bar{r}_1[$  and  $\bar{\beta}_k > 0$  such that

$$\sup\left\{u_{k,\beta,\varepsilon}(x) : x \in \Omega \setminus \bigcup_{i=1}^{k} B\left(x_{i,\beta,\varepsilon}, \frac{\bar{r}}{\sqrt{\beta}}\right), \ \varepsilon \in ]0, \bar{\varepsilon}], \ \beta \ge \bar{\beta}_{k}\right\} < 0.$$
(4.36)

<u>**Proof**</u> By the minimality of  $u_{k,\beta,\varepsilon}$ , we have only to check near the spheres  $\partial B(x_{i,\beta,\varepsilon}, r_{\beta})$ . Arguing as in the proof of Proposition 4.1, one can verify that

$$\lim_{\beta \to +\infty} \sup\left\{ \left| u_{k,\beta,\varepsilon} \left( \frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon} \right) - U_{\varepsilon}(x) \right| : x \in K, \ \varepsilon \in ]0,\overline{\varepsilon}] \right\} = 0 \quad \forall i \in \{1,\dots,k\}$$

$$(4.37)$$

for every compact subset K of  $\mathbb{R}^N$ .

Therefore, in order to complete the proof, it suffices to notice that there exists  $\bar{r} \in ]0, 3\bar{r}_1[$  such that

 $\sup\{U_{\varepsilon}(x) : |x| \ge \bar{r}, \ \varepsilon \in ]0, \bar{\varepsilon}]\} < 0, \tag{4.38}$ 

as one can easily verify taking into account the choice of  $\bar{\varepsilon}$ .

q.e.d.

**Remark 4.3** Let us point out that the strict inequality (4.36) given by Proposition 4.2 is important because the condition  $u \leq 0$  in  $\Omega \setminus \bigcup_{i=1}^{k} B(x_{i,\beta,\varepsilon}, r_{\beta})$  is an unilateral constraint that would give rise to a variational inequality if u = 0 somewhere in  $\Omega \setminus \bigcup_{i=1}^{k} B(x_{i,\beta,\varepsilon}, r_{\beta})$ . On the contrary, since (4.36) holds, u satisfies the equation  $\Delta u + \tilde{\alpha}u =$ 0 in  $\Omega \setminus \bigcup_{i=1}^{k} B(x_{i,\beta,\varepsilon}, r_{\beta})$  for a suitable Lagrange multiplier  $\tilde{\alpha} > 0$ , as we show in next lemma.

**Lemma 4.4** Let us consider  $k, \beta, \varepsilon, \overline{\varepsilon}, x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon}$  and  $u_{k,\beta,\varepsilon}$  satisfying the same assumptions as in Proposition 4.2. Let  $\overline{\beta}_k$  be the positive number given by Proposition 4.2. Then, for all  $\beta > \overline{\beta}_k$  and  $\varepsilon \in ]0, \overline{\varepsilon}]$ , there exist Lagrange multipliers  $\alpha_{k,\beta,\varepsilon} \in \mathbb{R}$  and  $\mu_{i,\beta,\varepsilon} \in \mathbb{R}^N$ , for  $i = 1, \ldots, k$ , such that

$$\frac{1}{2}f_{\beta,\varepsilon}'(u_{k,\beta,\varepsilon})[\psi] = \int_{\Omega} \{-\alpha_{k,\beta,\varepsilon}u_{k,\beta,\varepsilon}^{-} + \sum_{i=1}^{k} (u_{k,\beta,\varepsilon}^{+})_{i}[\mu_{i,\beta,\varepsilon} \cdot (x - x_{i,\beta,\varepsilon})]\}\psi dx \quad \forall \psi \in H_{0}^{1}(\Omega).$$

$$(4.39)$$

Moreover,  $\alpha_{k,\beta,\varepsilon} = \int_{\Omega} |Du_{k,\beta,\varepsilon}^-|^2 dx$ ,  $\lim_{\beta \to +\infty} \alpha_{k,\beta,\varepsilon} = \lambda_1 \ \forall k \in \mathbb{N}, \ \forall \varepsilon \in ]0, \overline{\varepsilon}]$  and

$$\lim_{\beta \to +\infty} \beta^{-\frac{3}{2}} \mu_{i,\beta,\varepsilon} = 0 \qquad \forall i \in \{1,\dots,k\}, \ \forall \varepsilon \in ]0,\bar{\varepsilon}].$$
(4.40)

**Proof** Unlike the case of the smooth constraints  $\int_{\Omega} (u^-)^2 dx = 1$  and  $\int_{\Omega} [u_i^+(x)]^2 x \, dx = x_i \int_{\Omega} (u_i^+)^2 dx$ , for which the Lagrange multipliers theorem applies, the constraints  $f'_{\beta,\varepsilon}(u)[u_i^+] = 0$ , for  $i = 1, \ldots, k$ , do not satisfy suitable regularity conditions. However, they are "natural constraints", in the sense that they do not give rise to Lagrange multipliers (while the multipliers  $\alpha_{k,\beta,\varepsilon}$  and  $\mu_{i,\beta,\varepsilon}$  come from the other constraints).

Notice that  $u_{k,\beta,\varepsilon}$  is the unique maximum point for  $f_{\beta,\varepsilon}$  on the set  $\{u_{k,\beta,\varepsilon} + \sum_{i=1}^{k} t_i(u_{k,\beta,\varepsilon}^+)_i$ :  $t_i \ge -1$  for  $i = 1, \ldots, k\}$ ; moreover,  $f'_{\beta,\varepsilon} \left(u_{k,\beta,\varepsilon} + \sum_{i=1}^{k} t_i(u_{k,\beta,\varepsilon}^+)_i\right) [(u_{k,\beta,\varepsilon}^+)_i]$  is positive for  $t_i \in [-1, 0[$  and negative for  $t_i > 0$ .

In order to prove (4.39), arguing by contradiction, we assume that (4.39) is not satisfied for any choice of the multipliers  $\alpha_{k,\beta,\varepsilon}$  in  $\mathbb{R}$  and  $\mu_{1,\beta,\varepsilon}, \ldots, \mu_{k,\beta,\varepsilon}$  in  $\mathbb{R}^N$ . Then, it follows by standard methods that there exists a continuous map  $\eta : ]-1, +\infty]^k \to H_0^1(\Omega)$ , such that  $\eta(t_1, \ldots, t_k) = u_{k,\beta,\varepsilon} + \sum_{i=1}^k t_i(u_{k,\beta,\varepsilon}^+)_i$  if  $(t_1, \ldots, t_k) \notin [-1/2, 1/2]^k$ ,  $\|\eta(t)^-\|_{L^2(\Omega)} =$  $1, \eta(t) \in E_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^\beta \quad \forall t \in [-1, +\infty[^k, f_{\beta,\varepsilon}(\eta(t)) < f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) \quad \forall t \in [-1, +\infty[^k]$ .

Therefore, applying Brouwer Theorem (see [5] and also [26]), we infer that there exists  $t \in [-1/2, 1/2]^k$  such that  $\eta(t) \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$ , which gives a contradiction because  $f_{\beta,\varepsilon}(\eta(t)) < f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})$  and

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}\}.$$
 (4.41)

Thus, we can conclude that there exist the multipliers  $\alpha_{k,\beta,\varepsilon}$  in  $\mathbb{R}$  and  $\mu_{i,\beta,\varepsilon}$  in  $\mathbb{R}^N$  satisfying (4.39).

Now, if in (4.39) we set  $\psi = u_{k,\beta,\varepsilon}^-$ , we obtain  $\alpha_{k,\beta,\varepsilon} = \int_{\Omega} |Du_{k,\beta,\varepsilon}^-|^2 dx$ ; then, since  $r_{\beta} \to 0$ , it follows that  $\lim_{\beta \to +\infty} \alpha_{k,\beta,\varepsilon} = \lambda_1 \ \forall k \in \mathbb{N}, \ \forall \varepsilon \in ]0, \overline{\varepsilon}].$ 

In order to prove (4.40), for every  $i \in \{1, \ldots, k\}$  we set  $\psi = \psi_{i,\beta,\varepsilon}(x) = \frac{1}{\beta} (u_{k,\beta,\varepsilon}^+)_i(x)$  $[\mu_{i,\beta,\varepsilon} \cdot (x - x_{i,\beta,\varepsilon})]$ . Then, after rescaling, we obtain

$$\frac{1}{2}\beta^{\frac{N-2}{2}}f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[\psi_{i,\beta,\varepsilon}]$$

$$= \beta^{-\frac{3}{2}} \int_{B(0,3\bar{r}_{1})} (u_{k,\beta,\varepsilon}^{+})_{i} \left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right) \psi_{i,\beta,\varepsilon} \left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right) (\mu_{i,\beta,\varepsilon} \cdot x) dx$$
$$= \int_{B(0,3\bar{r}_{1})} \left[ (u_{k,\beta,\varepsilon}^{+})_{i} \left(\frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon}\right) \right]^{2} \left(\frac{\mu_{i,\beta,\varepsilon}}{\beta^{\frac{3}{2}}} \cdot x\right)^{2} dx.$$
(4.42)

Arguing by contradiction, assume that (up to a subsequence)  $\lim_{\beta \to +\infty} \beta^{-\frac{3}{2}} |\mu_{i,\beta,\varepsilon}| > 0$ . So, taking into account the properties of the function  $U_{\varepsilon}$ , from Proposition 4.1 we infer that

$$\lim_{\beta \to +\infty} |\mu_{i,\beta,\varepsilon}|^{-1} \beta^{\frac{N+1}{2}} f_{\beta,\varepsilon}'(u_{k,\beta,\varepsilon})[\psi_{i,\beta,\varepsilon}] = 0 \quad \forall i \in \{1,\dots,k\}, \ \forall \varepsilon \in ]0,\bar{\varepsilon}]$$
(4.43)

while

$$\lim_{\beta \to +\infty} |\mu_{i,\beta,\varepsilon}|^{-1} \beta^{\frac{3}{2}} \int_{B(0,3\bar{r}_1)} \left[ (u_{k,\beta,\varepsilon}^+)_i \left( \frac{x}{\sqrt{\beta}} + x_{i,\beta,\varepsilon} \right) \right]^2 \left( \frac{\mu_{i,\beta,\varepsilon}}{\beta^{\frac{3}{2}}} \cdot x \right)^2 dx > 0.$$
(4.44)

Thus, we get a contradiction and (4.40) is proved.

q.e.d.

Lemma 4.5 Let us consider  $k, \beta_n, \varepsilon_n, (x_{1,\beta_n,\varepsilon_n}, \ldots, x_{k,\beta_n,\varepsilon_n}), u_{k,\beta_n,\varepsilon_n}$  satisfying the same assumptions as in Proposition 4.1. Moreover, for all  $n \in \mathbb{N}$ , let us consider  $(\hat{x}_{1,\beta_n,\varepsilon_n}, \ldots, \hat{x}_{k,\beta_n,\varepsilon_n})$  in  $\Omega_{k,\beta_n}, \hat{u}_{k,\beta_n,\varepsilon_n}$  in  $M^{\beta_n,\varepsilon_n}_{\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n}}$  and assume that  $f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(x_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}) = \max_{\Omega_{k,\beta_n}} \varphi_{k,\beta_n,\varepsilon_n}, f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n}) \forall n \in \mathbb{N},$ 

$$\lim_{n \to \infty} \sqrt{\beta_n} \left( \hat{x}_{i,\beta_n,\varepsilon_n} - x_{i,\beta_n,\varepsilon_n} \right) = 0 \qquad \forall i \in \{1,\dots,k\}.$$
(4.45)

Then, we have

$$\lim_{n \to \infty} \sup_{\Omega} |\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}| = 0.$$
(4.46)

If we assume in addition that  $(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n}) \neq (\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,x_{k,\beta_n,\varepsilon_n}) \quad \forall n \in \mathbb{N},$ then  $\sup_{\Omega} |\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}| > 0$  and the rescaled function  $Z_{i,n}$  defined by

$$Z_{i,n}(x) = \left(\sup_{\Omega} |\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}|\right)^{-1} (\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}) \left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\right)$$
$$\forall x \in \sqrt{\beta_n} \left(\Omega - x_{i,\beta_n,\varepsilon_n}\right), \ \forall i \in \{1,\dots,k\},$$
(4.47)

up to a subsequence, converges as  $n \to \infty$  to a function  $Z_i$  which is a weak solution of the equation

$$\Delta Z + a(x)Z = 0 \quad in \ \mathbb{R}^N \tag{4.48}$$

where a(x) = 1 if  $x \in B(0, \bar{r}_1)$  and a(x) = 0 otherwise; moreover, the convergence is uniform on the compact subsets of  $\mathbb{R}^N$ . Furthermore, there exists  $i \in \{1, \ldots, k\}$  such that  $Z_i \neq 0$ . **<u>Proof</u>** For short, in the proof let us write  $\hat{u}_n, u_n, \hat{x}_{i,n}, x_{i,n}$  instead of  $\hat{u}_{k,\beta_n,\varepsilon_n}, u_{k,\beta_n,\varepsilon_n}, \hat{x}_{i,\beta_n,\varepsilon_n}, x_{i,\beta_n,\varepsilon_n}$ .

From Proposition 4.1, we obtain

$$\limsup_{n \to \infty} \beta_n^{\frac{N-2}{2}} [f_{\beta_n, \varepsilon_n}(\hat{u}_n) - \lambda_1] \le k \operatorname{cap}(\bar{r}_1) \left(\varepsilon + \max_{\Omega} e_1\right)^2.$$
(4.49)

Moreover, the assumptions on  $\hat{x}_{i,n}$  and  $x_{i,n}$  imply

$$\lim_{n \to \infty} \sqrt{\beta_n} \left| \hat{x}_{i,n} - \hat{x}_{j,n} \right| = \infty \quad \text{for } i \neq j, \ i, j \in \{1, \dots, k\}.$$

$$(4.50)$$

Hence, arguing as in the proof of Proposition 4.1, one can show that  $\hat{u}_n\left(\frac{x}{\sqrt{\beta_n}} + \hat{x}_{i,n}\right) \rightarrow U_{\varepsilon}(x) \ \forall x \in \mathbb{R}^N, \forall i \in \{1, \dots, k\} \text{ and the convergence is uniform on the compact subsets of } \mathbb{R}^N$  (in fact, all the conditions we use in Proposition 4.1 to prove the similar property for  $u_n$ , are also satisfied by  $\hat{u}_n$ ).

It follows that

$$\lim_{n \to \infty} \sup_{B(x_{i,n}, 2r_{\beta_n})} |\hat{u}_n - u_n| = 0 \quad \text{for } i = 1, \dots, k;$$
(4.51)

moreover, taking into account the minimality properties of  $\hat{u}_n$  and  $u_n$ , standard arguments allow us to say that

$$\lim_{n \to \infty} \sup_{\Omega \setminus \bigcup_{i=1}^k B(x_{i,n}, 2r_{\beta_n})} |\hat{u}_n - u_n| = 0;$$

$$(4.52)$$

thus, (4.46) is proved. It is clear that  $\sup_{\Omega} |\hat{u}_n - u_n| > 0$  if  $x_{i,n} \neq \hat{x}_{i,n}$  for some  $i \in \{1, \ldots, k\}$ , otherwise we should have  $\hat{x}_{i,n} = x_{i,n} \forall i \in \{1, \ldots, k\}$ . Therefore, if  $(\hat{x}_{1,n}, \ldots, \hat{x}_{k,n}) \neq (x_{i,n}, \ldots, x_{k,n}) \forall n \in \mathbb{N}, Z_{i,n}$  is well defined and, up to a subsequence, it converges as  $n \to \infty$  to a function  $Z_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $\sup_{\mathbb{R}^N} |Z_i| \leq 1$ .

For short, in next formulas we write  $s_n$  instead of  $\sup_{\Omega} |\hat{u}_n - u_n|$ .

From Lemma 4.4, if we denote by  $\hat{\alpha}_{k,\beta_n,\varepsilon_n}$  and  $\hat{\mu}_{i,\beta_n,\varepsilon_n}$  the Lagrange multipliers corresponding to the function  $\hat{u}_n$ , we obtain

$$\frac{1}{2}f'_{\beta_n,\varepsilon_n}(\hat{u}_n)[\psi] - \frac{1}{2}f'_{\beta_n,\varepsilon_n}(u_n)[\psi]$$

$$= \int_{\Omega} D(\hat{u}_{n} - u_{n}) \cdot D\psi \, dx - \int_{\Omega} [g_{\beta_{n},\varepsilon_{n}}(\hat{u}_{n}) - g_{\beta_{n},\varepsilon_{n}}(u_{n})]\psi dx$$

$$= (\alpha_{k,\beta_{n},\varepsilon_{n}} - \hat{\alpha}_{k,\beta_{n},\varepsilon_{n}}) \int_{\Omega} \hat{u}_{n}^{-}\psi dx - \alpha_{k,\beta_{n},\varepsilon_{n}} \int_{\Omega} (\hat{u}_{n}^{-} - u_{n}^{-})\psi dx$$

$$+ \sum_{i=1}^{k} \int_{\Omega} [(\hat{u}_{n}^{+})_{i} - (u_{n}^{+})_{i}][\hat{\mu}_{i,\beta_{n},\varepsilon_{n}} \cdot (x - \hat{x}_{i,n})]\psi dx$$

$$+ \sum_{i=1}^{k} \int_{\Omega} (u_{n}^{+})_{i} [(\hat{\mu}_{i,\beta_{n},\varepsilon_{n}} - \mu_{i,\beta_{n},\varepsilon_{n}}) \cdot (x - \hat{x}_{i,n})]\psi dx$$

$$+ \sum_{i=1}^{k} \int_{\Omega} (u_{n}^{+})_{i} [\mu_{i,\beta_{n},\varepsilon_{n}} \cdot (x_{i,n} - \hat{x}_{i,n})]\psi dx \quad \forall \psi \in H_{0}^{1}(\Omega). \quad (4.53)$$

Taking into account the minimality properties of  $\hat{u}_n$  and  $u_n$ , since  $\alpha_{k,\beta_n,\varepsilon_n} = \int_{\Omega} |Du_n^-|^2 dx$  and  $\hat{\alpha}_{k,\beta_n,\varepsilon_n} = \int_{\Omega} |D\hat{u}_n^-|^2 dx$ , it follows that

$$\limsup_{n \to \infty} \frac{1}{s_n} |\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n}| < +\infty.$$
(4.54)

Moreover, since  $\hat{x}_{i,n} = \left[\int_{\Omega} (\hat{u}_n^+)_i^2 dx\right]^{-1} \int_{\Omega} [(\hat{u}_n^+)_i(x)]^2 x \, dx$  and  $x_{i,n} = \left[\int_{\Omega} (u_n^+)_i^2 dx\right]^{-1} \int_{\Omega} [(u_n^+)_i(x)]^2 x \, dx$ , it follows by direct computation that

$$\limsup_{n \to \infty} \frac{\sqrt{\beta_n}}{s_n} \left| \hat{x}_{i,n} - x_{i,n} \right| < +\infty.$$
(4.55)

From Lemma 4.4 we have also  $\lim_{n\to\infty} \beta_n^{-\frac{3}{2}} \hat{\mu}_{i,\beta_n,\varepsilon_n} = 0$  and  $\lim_{n\to\infty} \beta_n^{-\frac{3}{2}} \mu_{i,\beta_n,\varepsilon_n} = 0$ . Now, we can prove that

$$\lim_{n \to \infty} \beta_n^{-\frac{3}{2}} s_n^{-1} |\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}| = 0.$$
(4.56)

Arguing by contradiction, assume that (up to a subsequence) the limit (4.56) is positive for some  $i \in \{1, ..., k\}$ . Then, for n large enough, we can consider the function

$$\bar{Z}_{i,n} = \beta_n^{\frac{3}{2}} s_n \left| \hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n} \right|^{-1} Z_{i,n}$$
(4.57)

which, as  $Z_{i,n}$ , remains uniformly bounded as  $n \to \infty$ . Moreover, there exists  $\mu'_i \in \mathbb{R}^N$ ,  $|\mu'_i| = 1$ , such that, up to a subsequence,  $|\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}|^{-1}(\hat{\mu}_{i,\beta_n,\varepsilon_n} - \mu_{i,\beta_n,\varepsilon_n}) \to \mu'_i$  as  $n \to \infty$ . Hence, after rescaling in (4.53) we infer that (up to a subsequence)  $\bar{Z}_{i,n}$  converges as  $n \to \infty$  to a bounded function  $\bar{Z}_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , such that

$$\int_{\mathbb{R}^N} [D\bar{Z}_i \cdot D\psi - a(x)\bar{Z}_i\psi] dx = \int_{B(0,3\bar{r}_1)} U_{\varepsilon}^+(x)\psi(x)(x\cdot\mu_i') dx \quad \forall \psi \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$
(4.58)

Now, set  $\Psi = (DU \cdot \mu'_i)$ . Since this function satisfies the equation  $\Delta \Psi + a(x)\Psi = 0$  in  $\mathbb{R}^N$ , we obtain

$$\int_{\mathbb{R}^N} [D\bar{Z}_i \cdot D\Psi - a(x)\bar{Z}_i\Psi]dx = 0, \qquad (4.59)$$

while  $\int_{B(0,3\bar{r}_1)} U_{\varepsilon}^+(x) \Psi(x)(x \cdot \mu'_i) dx < 0.$ 

Thus, we have a contradiction and we can conclude that (4.56) holds.

Now, after rescaling, we can let  $n \to \infty$  in (4.53); so, it follows by usual arguments that (up to a subsequence)  $Z_{i,n}$  converges as  $n \to \infty$  to a solution  $Z_i$  of the equation (4.48) and that the convergence is uniform on the compact subsets of  $\mathbb{R}^N$ .

In order to prove that  $Z_i \not\equiv 0$  for some  $i \in \{1, \ldots, k\}$ , we argue by contradiction and assume that  $Z_i \equiv 0$  for  $i = 1, \ldots, k$ . In this case,  $Z_{i,n} \to 0$  as  $n \to \infty$ ,  $\forall i \in \{1, \ldots, k\}$ , uniformly on the compact subsets of  $\mathbb{R}^N$ ; moreover, if we set  $z_n = \frac{1}{s_n}(\hat{u}_n - u_n)$ , taking into account the minimality properties of  $\hat{u}_n$  and  $u_n$ , we can say that (up to a subsequence)  $(z_n)_n$  converges uniformly in  $\Omega$  to a function z. Now we prove that  $z \equiv 0$  in  $\Omega$ , so we have a contradiction because  $\sup_{\Omega} |z_n| = 1 \ \forall n \in \mathbb{N}$ . In order to prove that  $z \equiv 0$  in  $\Omega$ , notice that

$$\lim_{n \to \infty} \sup \left\{ |z_n(x)| : x \in \bigcup_{i=1}^k B(x_{i,n}, 2r_{\beta_n}) \right\} = 0;$$
(4.60)

moreover, for *n* large enough so that  $\bigcup_{i=1}^{k} \bar{B}(\hat{x}_{i,n}, r_{\beta_n}) \subset \bigcup_{i=1}^{k} B(x_{i,n}, 2r_{\beta_n})$ , the function  $z_n$  satisfies in  $\Omega \setminus \bigcup_{i=1}^{k} \bar{B}(x_{i,n}, 2r_{\beta_n})$  the equation  $\Delta z_n + \frac{1}{s_n} (\hat{\alpha}_{k,\beta_n,\varepsilon_n} \hat{u}_n - \alpha_{k,\beta_n,\varepsilon_n} u_n) = 0$ . Let us consider the function  $w_n \in H_0^1(\Omega)$ , such that  $w_n = z_n$  in  $\bigcup_{i=1}^{k} \bar{B}(x_{i,n}, 2r_{\beta_n})$  and  $\Delta w_n = 0$  in  $\Omega \setminus \bigcup_{i=1}^{k} \bar{B}(x_{i,n}, 2r_{\beta_n})$ . Since  $\lim_{n \to \infty} \sup\{|z_n(x)| : x \in \bigcup_{i=1}^{k} \bar{B}(x_{i,n}, 2r_{\beta_n})\} = 0$ , it follows that also  $\lim_{n \to \infty} \sup_{\Omega} |w_n| = 0$ . If we set  $\tilde{z}_n = z_n - w_n$ , we obtain

$$\Delta \tilde{z}_n + \alpha_{k,\beta_n,\varepsilon_n} \tilde{z}_n + \alpha_{k,\beta_n,\varepsilon_n} w_n + \frac{1}{s_n} (\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n}) \hat{u}_n = 0 \quad \text{in } \Omega \setminus \bigcup_{i=1}^k \bar{B}(x_{i,n}, 2r_{\beta_n}).$$

$$(4.61)$$

Taking into account that  $\lim_{n \to \infty} \alpha_{k,\beta_n,\varepsilon_n} = \lim_{n \to \infty} \hat{\alpha}_{k,\beta_n,\varepsilon_n} = \lambda_1$ , that  $\limsup_{n \to \infty} \frac{1}{s_n} |\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n}| < +\infty$  and that  $\hat{u}_n \to -e_1$  in  $H_0^1(\Omega)$ , it follows that, up to a subsequence,  $\alpha_{k,\beta_n,\varepsilon_n} w_n + \frac{1}{s_n} (\hat{\alpha}_{k,\beta_n,\varepsilon_n} - \alpha_{k,\beta_n,\varepsilon_n}) \hat{u}_n \to ce_1$  for a suitable constant  $c \in \mathbb{R}$ . Now, let us set  $\tilde{z}_{n,1} = e_1 \int_{\Omega} \tilde{z}_n e_1 dx$  and  $\tilde{z}_{n,2} = \tilde{z}_n - \tilde{z}_{n,1}$ . From (4.61) we obtain

$$\left(1 - \frac{\alpha_{k,\beta_n,\varepsilon_n}}{\lambda_2}\right) \|\tilde{z}_{n,2}\|_{H^1_0(\Omega)}^2 - c_n \|\tilde{z}_{n,2}\|_{H^1_0(\Omega)} \le 0$$
(4.62)

for a suitable sequence  $(c_n)_n$  in  $\mathbb{R}$  such that  $\lim_{n \to \infty} c_n = 0$ . Since  $\lim_{n \to \infty} \alpha_{k,\beta_n,\varepsilon_n} = \lambda_1 < \lambda_2$ , it follows that  $\lim_{n \to \infty} \|\tilde{z}_{n,2}\|_{H^1_0(\Omega)} = 0$ . Therefore, we can say that (up to a subsequence)  $(\tilde{z}_n)_n$  and  $(z_n)_n$  converge to the function  $z = \bar{c}e_1$  where  $\bar{c} = \lim_{n \to \infty} \int_{\Omega} z_n e_1 dx$ .

On the other hand,  $\lim_{n \to \infty} z_n(x_{i,n}) = 0 \quad \forall i \in \{1, \dots, k\}$ . Therefore, we have

$$0 = \lim_{n \to \infty} z_n(x_{i,n}) = \lim_{n \to \infty} z(x_{i,n}) = \bar{c} \lim_{n \to \infty} e_1(x_{i,n}) = \bar{c} \max_{\Omega} e_1,$$
(4.63)

which implies  $\bar{c} = 0$  because  $\max_{\Omega} e_1 \neq 0$ . It follows that  $z \equiv 0$  in  $\Omega$ , which gives a contradiction.

Thus, we can conclude that  $Z_i \neq 0$  for some  $i \in \{1, \ldots, k\}$  and the proof is complete. *q.e.d.* 

**Lemma 4.6** Let  $Z_1, \ldots, Z_k$  be the functions obtained in Lemma 4.5. Then, for every  $i \in \{1, \ldots, k\}$ , there exists  $\tau_i \in \mathbb{R}^N$  such that  $Z_i(x) = (DU(x) \cdot \tau_i) \ \forall x \in \mathbb{R}^N$ . Moreover, there exists  $i \in \{1, \ldots, k\}$  such that  $\tau_i \neq 0$ .

<u>Proof</u> Notice that the function U is nondegenerate in the sense that, if  $Z \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  is a weak solution of the equation (4.48), then there exists  $\tau \in \mathbb{R}^N$  such that  $Z(x) = (DU(x) \cdot \tau) \forall x \in \mathbb{R}^N$  (for the proof, see analogous results proved in [27–29]). Therefore, since the function  $Z_i$  satisfies the equation (4.48) for  $i = 1, \ldots, k$  as proved in Lemma 4.5, it follows that for every  $i \in \{1, \ldots, k\}$  there exists  $\tau_i \in \mathbb{R}^N$ , having the required property. Moreover,  $\tau_i \neq 0$  for some  $i \in \{1, \ldots, k\}$  because  $Z_i \not\equiv 0$  for some  $i \in \{1, \ldots, k\}$ , as we proved in Lemma 4.5.

**Proposition 4.7** For all positive integer k, for  $\beta > 0$  large enough so that  $\Omega_{k,\beta} \neq \emptyset$ and for all  $\varepsilon > 0$ , let us consider  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon})$  in  $\Omega_{k,\beta}$  and  $u_{k,\beta,\varepsilon}$  in  $M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$ such that

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}.$$
(4.64)

Then, there exists  $\tilde{\beta}_k > 0$  such that, for all  $\beta > \tilde{\beta}_k$  and  $\varepsilon \in \left]0, \frac{1}{2} \frac{U(3\bar{r}_1) \max_{\Omega} e_1}{\lim_{|x| \to \infty} U(x) - U(3\bar{r}_1)}\right[, u_{k,\beta,\varepsilon}$  is a constrained critical point for the functional  $f_{\beta,\varepsilon}$  constrained on the set  $S = \{u \in H_0^1(\Omega) : \|u^-\|_{L^2(\Omega)} = 1\}.$ 

<u>Proof</u> Clearly, it suffices to prove that the Lagrange multipliers  $\mu_{i,\beta,\varepsilon}$  given by Lemma 4.4 vanish for  $\beta$  large enough, namely that there exists  $\tilde{\beta}_k > 0$  such that  $\mu_{i,\beta,\varepsilon} = 0$   $\forall \beta > \tilde{\beta}_k, \forall i \in \{1, \ldots, k\}, \forall \varepsilon \in \left]0, \frac{1}{2} \frac{U(3\bar{r}_1) \max_{\Omega} e_1}{\lim_{|x| \to \infty} U(x) - U(3\bar{r}_1)}\right[.$ 

Arguing by contradiction, assume that there exist a sequence  $(\beta_n)_n$  of positive numbers and a sequence  $(\varepsilon_n)_n$  in  $\left]0, \frac{1}{2} \frac{U(3\bar{r}_1) \max_{\Omega} e_1}{\lim_{|x|\to\infty} U(x)-U(3\bar{r}_1)}\right[$  such that  $\lim_{n\to\infty} \beta_n = +\infty$  and  $(\mu_{1,\beta_n,\varepsilon_n},\ldots,\mu_{k,\beta_n,\varepsilon_n}) \neq 0 \ \forall n \in \mathbb{N}$ . Whitout any loss of generality, we can assume that

$$|\mu_{1,\beta_n,\varepsilon_n}| = \max\{|\mu_{i,\beta_n,\varepsilon_n}| : i = 1,\dots,k\} \quad \forall n \in \mathbb{N}.$$
(4.65)

Up to a subsequence,  $\varepsilon_n \to \tilde{\varepsilon}$  and  $\frac{\mu_{1,\beta_n,\varepsilon_n}}{|\mu_{1,\beta_n,\varepsilon_n}|} \to \tilde{\mu}$ , as  $n \to \infty$ , for suitable  $\tilde{\varepsilon} \in \left[0, \frac{1}{2} \frac{U(3\bar{r}_1) \max_{\Omega} e_1}{\lim_{|x|\to\infty} U(x)-U(3\bar{r}_1)}\right]$  and  $\tilde{\mu} \in \mathbb{R}^N$ , such that  $|\tilde{\mu}| = 1$ . Now, let us choose  $(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n})$  in  $\Omega_{k,\beta_n}$  and  $\hat{u}_{k,\beta_n,\varepsilon_n}$  in  $M_{\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n}}^{\beta_n,\varepsilon_n}$  such that  $f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n})$  and  $\hat{x}_{i,\beta_n,\varepsilon_n} = x_{i,\beta_n,\varepsilon_n}$  for  $i = 2,\ldots,k$  while  $\hat{x}_{1,\beta_n,\varepsilon_n} = x_{1,\beta_n,\varepsilon_n} + \frac{\delta_n}{\sqrt{\beta_n}}\tilde{\mu}$  with  $\delta_n > 0 \ \forall n \in \mathbb{N}$ ,  $\lim_{n\to\infty} \delta_n = 0$  and, in addition,

$$\lim_{n \to \infty} \delta_n \beta_n^{\frac{N+3}{2}} |\mu_{1,\beta_n,\varepsilon_n}|^{-1} = 0.$$
(4.66)

Notice that this choice of  $(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n})$  in  $\Omega_{k,\beta_n}$  is indeed possible because  $\lim_{n\to\infty}\sqrt{\beta_n}|x_{i,\beta_n,\varepsilon_n}-x_{1,\beta_n,\varepsilon_n}|=\infty$  for  $i\neq 1$ , as proved in Proposition 4.1. Moreover, we have  $\sup_{\Omega}|\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n}|>0$   $\forall n\in\mathbb{N}$  because  $\hat{u}_{k,\beta_n,\varepsilon_n}\neq u_{k,\beta_n,\varepsilon_n}$ , since  $\delta_n>0$ . For short, let us write  $s_n$  instead of  $\sup_{\Omega}|\hat{u}_{k,\beta_n,\varepsilon_n}-u_{k,\beta_n,\varepsilon_n}|$ . One can verify by direct computation that

$$f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) = f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}) + f'_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})[\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}] + R_n$$
(4.67)

where

$$R_n \ge -\beta_n \int_{\Omega} (\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n})^2 dx \ge -\beta_n |\Omega| s_n^2.$$
(4.68)

From Lemmas 4.5 and 4.6, we infer that there exist  $\tau_1, \ldots, \tau_k$  in  $\mathbb{R}^N$  such that (up to a subsequence) the rescaled function  $\frac{1}{s_n}(\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n})\left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\right)$  converges as  $n \to \infty$  to  $(DU(x) \cdot \tau_i)$ , for  $i = 1, \ldots, k$ , uniformly on the compact subsets of  $\mathbb{R}^N$ . We say that  $\tau_1 \neq 0$  and  $\tau_i = 0$  for  $i \neq 1$ . In fact, for  $i = 1, \ldots, k$ , we have

$$\int_{B(0,3\bar{r}_{1})} \left[ \hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) \right]^{2} x \, dx = \int_{B(0,3\bar{r}_{1})} \left[ u_{k,\beta_{n},\varepsilon_{n}}^{+} \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) \right]^{2} x \, dx \\
+ 2 \int_{B(0,3\bar{r}_{1})} u_{k,\beta_{n},\varepsilon_{n}}^{+} \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) \left( \hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} - u_{k,\beta_{n},\varepsilon_{n}}^{+} \right) \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) x \, dx + o(s_{n}). \tag{4.69}$$

Taking into account the choice of  $(\hat{x}_{1,\beta_n,\varepsilon_n},\ldots,\hat{x}_{k,\beta_n,\varepsilon_n})$ , if  $i \neq 1$ , for *n* large enough we obtain

$$\int_{B(0,3\bar{r}_1)} \left[ \hat{u}_{k,\beta_n,\varepsilon_n}^+ \left( \frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n} \right) \right]^2 x \, dx$$

$$= \int_{B(0,3\bar{r}_1)} \left[ u_{k,\beta_n,\varepsilon_n}^+ \left( \frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n} \right) \right]^2 x \, dx = 0.$$
(4.70)

Therefore, as  $n \to \infty$ , we get

$$\int_{B(0,3\bar{r}_1)} U^+_{\tilde{\varepsilon}}(x) [DU_{\tilde{\varepsilon}}(x) \cdot \tau_i] x \, dx = 0 \quad \text{for } i = 2, \dots, k.$$

$$(4.71)$$

It follows that

$$\int_{B(0,3\bar{r}_1)} U_{\tilde{\varepsilon}}^+(x) [DU_{\tilde{\varepsilon}}(x) \cdot \tau_i] x \, dx = \frac{1}{2} \int_{B(0,\rho_{\tilde{\varepsilon}})} [DU_{\tilde{\varepsilon}}^2(x) \cdot \tau_i] x \, dx = -\frac{\tau_i}{2} \int_{B(0,\rho_{\tilde{\varepsilon}})} U_{\tilde{\varepsilon}}^2(x) dx = 0,$$
(4.72)

where  $\rho_{\tilde{\varepsilon}}$  denotes the radius of supp  $U_{\tilde{\varepsilon}}^+$  (which is a ball). Therefore, we have  $\tau_i = 0$  for  $i = 2, \ldots, k$ .

On the contrary, if i = 1, for n large enough we have

$$\int_{B(0,3\bar{r}_1)} \left[ u_{k,\beta_n,\varepsilon_n}^+ \left( \frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n} \right) \right]^2 x \, dx = 0 \tag{4.73}$$

while

$$\int_{B(0,3\bar{r}_{1})} \left[ \hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) \right]^{2} x \, dx$$

$$= \delta_{n} \tilde{\mu} \int_{B(0,3\bar{r}_{1})} \left[ \hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} \left( \frac{x}{\sqrt{\beta_{n}}} + x_{i,\beta_{n},\varepsilon_{n}} \right) \right]^{2} dx.$$
(4.74)

So, as  $n \to \infty$ , we obtain

$$\lim_{n \to \infty} \frac{\delta_n}{s_n} \tilde{\mu} \int_{B(0,\rho_{\tilde{\varepsilon}})} U_{\tilde{\varepsilon}}^2 dx = 2 \int_{B(0,\rho_{\tilde{\varepsilon}})} U_{\tilde{\varepsilon}}(x) [DU_{\tilde{\varepsilon}}(x) \cdot \tau_1] x \, dx = -\tau_1 \int_{B(0,\rho_{\tilde{\varepsilon}})} U_{\tilde{\varepsilon}}^2 dx \quad (4.75)$$

where, taking into account Lemma 4.6,  $\tau_1 \neq 0$  because  $\tau_i = 0$  for  $i \neq 1$ . As a consequence,  $\lim_{n \to \infty} \frac{\delta_n}{s_n} = |\tau_1| > 0$  and  $\tau_1 = -|\tau_1|\tilde{\mu}$ .

From (4.67), (4.68) and Lemma 4.4, we obtain

$$f_{\beta_{n},\varepsilon_{n}}(\hat{u}_{k,\beta_{n},\varepsilon_{n}}) - f_{\beta_{n},\varepsilon_{n}}(u_{k,\beta_{n},\varepsilon_{n}})$$

$$\geq 2 \int_{\Omega} \left\{ -\alpha_{k,\beta_{n},\varepsilon_{n}} u_{k,\beta_{n},\varepsilon_{n}}^{-} + \sum_{i=1}^{k} (u_{k,\beta_{n},\varepsilon_{n}}^{+})_{i} [\mu_{i,\beta_{n},\varepsilon_{n}} \cdot (x - x_{i,\beta_{n},\varepsilon_{n}})] \right\} (\hat{u}_{k,\beta_{n},\varepsilon_{n}} - u_{k,\beta_{n},\varepsilon_{n}}) dx$$

$$- \beta_{n} |\Omega| s_{n}^{2}. \quad (4.76)$$

Notice that, since  $\int_{\Omega} (\hat{u}_{k,\beta_n,\varepsilon_n})^2 dx = 1$  and  $\int_{\Omega} (u_{k,\beta_n,\varepsilon_n})^2 dx = 1$ , we have

$$2\int_{\Omega} u_{k,\beta_n,\varepsilon_n}^- (\hat{u}_{k,\beta_n,\varepsilon_n}^- - u_{k,\beta_n,\varepsilon_n}^-) dx = -\int_{\Omega} (\hat{u}_{k,\beta_n,\varepsilon_n}^- - u_{k,\beta_n,\varepsilon_n}^-)^2 dx \qquad \forall n \in \mathbb{N}; \quad (4.77)$$

moreover,  $\hat{u}_{k,\beta_n,\varepsilon_n}^+ \leq [u_{k,\beta_n,\varepsilon_n} + s_n]^+$  and  $\sup_{\Omega}(u_{k,\beta_n,\varepsilon_n}^- [u_{k,\beta_n,\varepsilon_n} + s_n]^+) \leq s_n^2$ ; thus, we get

$$\int_{\Omega} u_{k,\beta_{n},\varepsilon_{n}}^{-} (\hat{u}_{k,\beta_{n},\varepsilon_{n}} - u_{k,\beta_{n},\varepsilon_{n}}) dx$$

$$\leq \int_{\Omega} u_{k,\beta_{n},\varepsilon_{n}}^{-} (\hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} - u_{k,\beta_{n},\varepsilon_{n}}^{+}) dx - \int_{\Omega} u_{k,\beta_{n},\varepsilon_{n}}^{-} (\hat{u}_{k,\beta_{n},\varepsilon_{n}}^{-} - u_{k,\beta_{n},\varepsilon_{n}}^{-}) dx$$

$$= \int_{\Omega} u_{k,\beta_{n},\varepsilon_{n}}^{-} \hat{u}_{k,\beta_{n},\varepsilon_{n}}^{+} dx + \frac{1}{2} \int_{\Omega} (\hat{u}_{k,\beta_{n},\varepsilon_{n}}^{-} - u_{k,\beta_{n},\varepsilon_{n}}^{-})^{2} dx$$

$$\leq \frac{3}{2} |\Omega| s_{n}^{2} \qquad \forall n \in \mathbb{N}.$$
(4.78)

Therefore, after rescaling, it follows

$$\frac{\beta_n^{\frac{N+1}{2}}}{|\mu_{1,\beta_n,\varepsilon_n}|s_n} [f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) - f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})] \\
\geq \sum_{i=1}^k \int_{B(0,3\bar{r}_1)} u_{k,\beta_n,\varepsilon_n}^+ \left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\right) \\
\frac{1}{s_n} (\hat{u}_{k,\beta_n,\varepsilon_n} - u_{k,\beta_n,\varepsilon_n}) \left(\frac{x}{\sqrt{\beta_n}} + x_{i,\beta_n,\varepsilon_n}\right) \left(\frac{\mu_{i,\beta_n,\varepsilon_n}}{|\mu_{1,\beta_n,\varepsilon_n}|} \cdot x\right) dx - \frac{2s_n \beta_n^{\frac{N+3}{2}}}{|\mu_{1,\beta_n,\varepsilon_n}|} |\Omega| \quad (4.79)$$

for *n* large enough. Then, as  $n \to \infty$ , we obtain

$$\liminf_{n \to \infty} \frac{\beta_n^{\frac{N+1}{2}}}{s_n |\mu_{1,\beta_n,\varepsilon_n}|} [f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) - f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n})] \\
\geq \int_{B(0,3\bar{r}_1)} U_{\tilde{\varepsilon}}^+(x) [DU(x) \cdot \tau_1](\tilde{\mu} \cdot x) = -|\tau_1| \int_{B(0,3\bar{r}_1)} U_{\varepsilon}^+(x) [DU(x) \cdot \tilde{\mu}](\tilde{\mu} \cdot x) dx > 0, \tag{4.80}$$

which is a contradiction because

$$f_{\beta_n,\varepsilon_n}(\hat{u}_{k,\beta_n,\varepsilon_n}) = \varphi_{k,\beta_n,\varepsilon_n}(\hat{x}_{1,\beta_n,\varepsilon_n},\dots,\hat{x}_{k,\beta_n,\varepsilon_n}) \le \max_{\Omega_{k,\beta_n}}\varphi_{k,\beta_n,\varepsilon_n} = f_{\beta_n,\varepsilon_n}(u_{k,\beta_n,\varepsilon_n}).$$
(4.81)

Thus, the proof is complete.

### 5 Proof of the main results and final remarks

In this section we study the behaviour as  $\varepsilon \to 0$  of the function  $u_{k,\beta,\varepsilon}$  obtained by minimax methods in Sections 3 and 4. In particular, our aim is to show that for all  $\beta > \tilde{\beta}_k$ (see Proposition 4.7)  $\alpha_{k,\beta,\varepsilon} \to \alpha_{k,\beta}$ ,  $u_{k,\beta,\varepsilon} \to u_{k,\beta}$  as  $\varepsilon \to 0$  (up to a subsequence) for suitable  $\alpha_{k,\beta} \in \mathbb{R}$ ,  $u_{k,\beta} \in H_0^1(\Omega)$  and that  $u_{k,\beta}$  is an eigenfunction for the Fučík spectrum, corresponding to the pair  $(\alpha_{k,\beta}, \beta)$ , namely  $u_{k,\beta}$  solves the problem

$$\Delta u - \alpha_{k,\beta} u^- + \beta u^+ = 0 \quad \text{in } \Omega, \ u \in H^1_0(\Omega), \ u \neq 0 \text{ in } \Omega.$$
(5.1)

**Lemma 5.1** For all  $\beta > 0$  and  $\varepsilon > 0$ , let us consider a point  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon})$  in  $\Omega_{k,\beta}$ and a function  $u_{k,\beta,\varepsilon}$  in  $M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$ . Moreover, assume that (up to a subsequence)  $(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$  converges as  $\varepsilon \to 0$  to a point  $(x_{1,\beta},\ldots,x_{k,\beta})$ .

Then, up to a subsequence,  $-u_{k,\beta,\varepsilon}^- + \sum_{i=1}^k (u_{k,\beta,\varepsilon}^+)_i ||(u_{k,\beta,\varepsilon}^+)_i||_{L^2(\Omega)}^{-1}$  converges in  $H_0^1(\Omega)$ , as  $\varepsilon \to 0$ , to a function  $\bar{u}_k \in E_{x_{1,\beta},\dots,x_{k,\beta}}^\beta$ ; moreover,

$$\int_{\Omega} |D(\bar{u}_{k,\beta}^+)_i|^2 dx = \beta \qquad \forall i \in \{1,\dots,k\}$$
(5.2)

and

$$\int_{\Omega} |D\bar{u}_{k,\beta}^{-}|^{2} dx = \min\left\{\int_{\Omega} |Du^{-}|^{2} dx : u \in E_{x_{1,\beta},\dots,x_{k,\beta}}^{\beta}, \|u^{-}\|_{L^{2}(\Omega)} = 1, \\ \int_{\Omega} |Du_{i}^{+}|^{2} dx = \beta, \int_{\Omega} (u_{i}^{+})^{2} dx = 1 \text{ for } i = 1,\dots,k\right\}.$$
(5.3)

**<u>Proof</u>** Notice that, since  $u_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$ , we have

$$\int_{\Omega} |D(u_{k,\beta,\varepsilon}^+)_i|^2 dx = \int_{\Omega} g_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})(u_{k,\beta,\varepsilon}^+)_i dx < \beta \int_{\Omega} (u_{k,\beta,\varepsilon}^+)_i^2 dx$$
$$\forall \varepsilon > 0, \ \forall i \in \{1,\dots,k\}.$$
(5.4)

Let us set  $(\bar{u}_{k,\beta,\varepsilon}^+)_i = \|(u_{k,\beta,\varepsilon}^+)_i\|_{L^2(\Omega)}^{-1}(u_{k,\beta,\varepsilon}^+)_i$ . Then, we have

$$\int_{\Omega} |D(\bar{u}_{k,\beta,\varepsilon}^+)_i|^2 dx < \beta \qquad \forall \varepsilon > 0, \ \forall i \in \{1,\dots,k\}.$$
(5.5)

It follows that, up to a subsequence,  $(\bar{u}_{k,\beta,\varepsilon}^+)_i$  converges as  $\varepsilon \to 0$  to a function  $(\bar{u}_{k,\beta}^+)_i$ in  $L^2(\Omega)$ , weakly in  $H_0^1(\Omega)$  and a.e. in  $\Omega$ . Moreover, since

$$\limsup_{\varepsilon \to 0} \|u_{k,\beta,\varepsilon}^-\|_{H^1_0(\Omega)} < +\infty, \tag{5.6}$$

also  $u_{k,\beta,\varepsilon}^-$  converges as  $\varepsilon \to 0$  to a function  $u_{k,\beta}^-$  in  $L^2(\Omega)$ , weakly in  $H_0^1(\Omega)$  and a.e. in  $\Omega$ . As a consequence, the function  $\bar{u}_{k,\beta} = -u_{k,\beta}^- + \sum_{i=1}^k (\bar{u}_{k,\beta}^+)_i$  belongs to  $E_{x_{1,\beta},\dots,x_{k,\beta}}^\beta$ and  $\|\bar{u}_{k,\beta}^-\|_{L^2(\Omega)} = 1$ . Notice that, indeed,  $(\bar{u}_{k,\beta,\varepsilon}^+)_i \to (\bar{u}_{k,\beta}^+)_i \, \forall i \in \{1,\dots,k\}$  and  $u_{k,\beta,\varepsilon}^- \to u_{k,\beta}^-$  strongly in  $H_0^1(\Omega)$  as  $\varepsilon \to 0$ . In fact, we have

$$\int_{\Omega} |D(\bar{u}_{k,\beta}^+)_i|^2 dx = \beta \qquad \forall i \in \{1,\dots,k\}$$
(5.7)

and

$$\lim_{\varepsilon \to 0} \int_{\Omega} |Du_{k,\beta,\varepsilon}^{-}|^2 dx = \int_{\Omega} |Du_{k,\beta}^{-}|^2 dx.$$
(5.8)

For the proof, we argue by contradiction and assume that

$$\int_{\Omega} |Du_{k,\beta}^{-}|^{2} dx < \liminf_{\varepsilon \to 0} \int_{\Omega} |Du_{k,\beta,\varepsilon}^{-}|^{2} dx$$
(5.9)

or

$$\int_{\Omega} |D(\bar{u}_{k,\beta}^+)_i|^2 dx < \beta \qquad \text{for some } i \in \{1, \dots, k\}.$$
(5.10)

In this case, by slight modifications of the supports of  $u_{k,\beta}^-$  and  $(\bar{u}_{k,\beta}^+)_i$ , one can construct a function  $\tilde{u}_{k,\beta} \in E_{x_{1,\beta},\dots,x_{k,\beta}}^\beta$  such that  $\|\tilde{u}_{k,\beta}^-\|_{L^2(\Omega)} = 1$ ,  $\|(\tilde{u}_{k,\beta}^+)_i\|_{L^2(\Omega)} = 1$ ,

$$\int_{\Omega} |D\tilde{u}_{k,\beta}^{-}|^{2} dx < \liminf_{\varepsilon \to 0} \int_{\Omega} |Du_{k,\beta,\varepsilon}^{-}|^{2} dx$$
(5.11)

and

$$\int_{\Omega} |D(\tilde{u}_{k,\beta}^+)_i|^2 dx < \beta \qquad \text{for some } i \in \{1,\dots,k\}.$$
(5.12)

Without any loss of generality, we can assume that (5.12) is satisfied for i = 1. Then, for all  $\varepsilon > 0$ , let us consider the function  $\tilde{u}_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  such that  $(\tilde{u}_{k,\beta,\varepsilon}^+)_i = (u_{k,\beta,\varepsilon}^+)_i$  for  $i = 2, \dots, k$ ,  $(\tilde{u}_{k,\beta,\varepsilon}^+)_1(x) = t_\varepsilon(\tilde{u}_{k,\beta}^+)_1(x - x_{1,\beta,\varepsilon} + x_{1,\beta}) \quad \forall x \in \Omega$ , where  $t_\varepsilon$  is the positive number such that  $f'_{\beta,\varepsilon}(t_\varepsilon(\tilde{u}_{k,\beta}^+)_1)[(\tilde{u}_{k,\beta}^+)_1] = 0$  and  $\tilde{u}_{k,\beta,\varepsilon}^-$  is the nonnegative function in  $H_0^1(\Omega)$  such that  $\tilde{u}_{k,\beta,\varepsilon}^-(x) = 0 \quad \forall x \in \bigcup_{i=1}^k \operatorname{supp}(\tilde{u}_{k,\beta,\varepsilon}^+)_i, \|\tilde{u}_{k,\beta,\varepsilon}^-\|_{L^2(\Omega)} = 1$  and

$$\int_{\Omega} |D\tilde{u}_{k,\beta,\varepsilon}^{-}|^{2} dx = \min\left\{\int_{\Omega} |Du|^{2} dx : u \in H_{0}^{1}(\Omega), u \ge 0 \text{ in } \Omega, \\ u(x) = 0 \ \forall x \in \bigcup_{i=1}^{k} \operatorname{supp}(\tilde{u}_{k,\beta,\varepsilon}^{+})_{i}, \|u\|_{L^{2}(\Omega)} = 1\right\}.$$
(5.13)

Then, we have

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) - f_{\beta,\varepsilon}(\tilde{u}_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}((u_{k,\beta,\varepsilon}^+)_1) - f_{\beta,\varepsilon}((\tilde{u}_{k,\beta,\varepsilon}^+)_1) + f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}^-) - f_{\beta,\varepsilon}(\tilde{u}_{k,\beta,\varepsilon}^-),$$
(5.14)  
where  $f_{\beta,\varepsilon}((u_{k,\beta,\varepsilon}^+)_1) \ge 0 \ \forall \varepsilon > 0, \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}((\tilde{u}_{k,\beta,\varepsilon}^+)_1) = 0 \ \text{and} \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}^-) > \int_{\Omega} |D\tilde{u}_{k,\beta}^-|^2$ 

$$\begin{split} dx &\geq \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(\tilde{u}_{k,\beta,\varepsilon}^{-}).\\ \text{It follows that } f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) > f_{\beta,\varepsilon}(\tilde{u}_{k,\beta,\varepsilon}) \text{ for } \varepsilon > 0 \text{ small enough, which is a contradiction}\\ \text{because } \tilde{u}_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon} \text{ and } f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}\}.\\ \text{Thus, we can conclude that } u_{k,\beta,\varepsilon}^{-} \to u_{k,\beta}^{-} \text{ in } H_{0}^{1}(\Omega) \text{ as } \varepsilon \to 0 \text{ and } \int_{\Omega} |D(\bar{u}_{k,\beta}^{+})_{i}|^{2} dx = \beta\\ \forall i \in \{1,\dots,k\}. \end{split}$$

In a similar way we can prove (5.3). Arguing again by contradiction, assume that there exists  $\bar{u} \in E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}}$  such that  $\|\bar{u}^{-}\|_{L^{2}(\Omega)} = 1$ ,  $\int_{\Omega} |D\bar{u}^{+}_{i}|^{2} dx = \beta$ ,  $\|\bar{u}^{+}_{i}\|_{L^{2}(\Omega)} = 1$  $\forall i \in \{1,\dots,k\}$  and  $\int_{\Omega} |D\bar{u}^{-}|^{2} dx < \int_{\Omega} |Du^{-}_{k,\beta}|^{2} dx$ .

In this case, by slight modifications of the supports of  $\bar{u}^-$  and  $\bar{u}_i^+$  for  $i = 1, \ldots, k$ , one can find  $\check{u}_{k,\beta} \in E^{\beta}_{x_{1,\beta},\ldots,x_{k,\beta}}$  such that  $\int_{\Omega} |D(\check{u}^+_{k,\beta})_i|^2 dx < \beta \int_{\Omega} (\check{u}^+_{k,\beta})_i^2 dx \ \forall i \in \{1,\ldots,k\},$  $\|\check{u}^-_{k,\beta}\|_{L^2(\Omega)} = 1$  and  $\int_{\Omega} |D\check{u}^-_{k,\beta}|^2 dx < \int_{\Omega} |Du^-_{k,\beta}|^2 dx.$ 

It follows that there exist k positive numbers  $\check{t}_{1,\varepsilon},\ldots,\check{t}_{k,\varepsilon}$  such that  $f'_{\beta,\varepsilon}(\check{t}_{i,\varepsilon}(\check{u}^+_{k,\beta})_i)$   $[(\check{u}^+_{k,\beta})_i] = 0 \ \forall i \in \{1,\ldots,k\}$  and we can consider the function  $\check{u}_{k,\beta,\varepsilon}$  in  $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}$ defined in the following way: for  $i = 1,\ldots,k$ ,  $(\check{u}^+_{k,\beta,\varepsilon})_i(x) = (\check{u}^+_{k,\beta})_i(x - x_{i,\beta,\varepsilon} + x_{i,\beta})$  $\forall x \in \Omega \text{ and } \check{u}^-_{k,\beta,\varepsilon}$  is the nonnegative function in  $H^1_0(\Omega)$  such that  $\check{u}^-_{k,\beta,\varepsilon}(x) = 0 \ \forall x \in \cup_{i=1}^k \operatorname{supp}(u^+_{k,\beta,\varepsilon})_i, \|\check{u}^-_{k,\beta,\varepsilon}\|_{L^2(\Omega)} = 1$  and

$$\int_{\Omega} |D\check{u}_{k,\beta,\varepsilon}^{-}|^{2} dx = \min\left\{\int_{\Omega} |Du|^{2} dx : u \in H_{0}^{1}(\Omega), u \ge 0 \text{ in } \Omega, \\ u(x) = 0 \ \forall x \in \bigcup_{i=1}^{k} \operatorname{supp}(\check{u}_{k,\beta,\varepsilon}^{+})_{i}, \|u\|_{L^{2}(\Omega)} = 1\right\}.$$
(5.15)

Then, by direct computation, we obtain

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) - f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon}) - f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon}) + \sum_{i=1}^{k} f_{\beta,\varepsilon}((u_{k,\beta,\varepsilon}^{+})_{i}) - \sum_{i=1}^{k} f_{\beta,\varepsilon}((\check{u}_{k,\beta,\varepsilon}^{+})_{i}), \quad (5.16)$$

where  $f_{\beta,\varepsilon}((u_{k,\beta,\varepsilon}^+)_i) \ge 0 \ \forall \varepsilon > 0, \ \forall i \in \{1,\ldots,k\}, \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}((\check{u}_{k,\beta,\varepsilon}^+)_i) = 0 \ \forall i \in \{1,\ldots,k\}$ and

$$\lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}^{-}) = \int_{\Omega} |Du_{k,\beta}^{-}|^2 dx > \int_{\Omega} |D\check{u}_{k,\beta}^{-}|^2 dx \ge \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon}^{-}).$$
(5.17)

It follows that  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) > f_{\beta,\varepsilon}(\check{u}_{k,\beta,\varepsilon})$  for  $\varepsilon > 0$  small enough; so we have again a contradiction because  $\check{u}_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  and  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \min\{f_{\beta,\varepsilon}(u) : u \in M_{x_{1,\beta,\varepsilon},\dots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}\}$ . q.e.d.

Now, notice that we can consider the function  $\varphi_{k,\beta} : \Omega_{k,\beta} \to \mathbb{R}$  such that, for all  $(x_1, \ldots, x_k) \in \Omega_{k,\beta}$ 

$$\varphi_{k,\beta}(x_1,\ldots,x_k) = \min\left\{ \int_{\Omega} |Du^-|^2 dx : u \in E^{\beta}_{x_1,\ldots,x_k}, \|u^-\|_{L^2(\Omega)} = 1, \\ \int_{\Omega} |Du^+_i|^2 dx = \beta, \|u^+_i\|_{L^2(\Omega)} = 1 \text{ for } i = 1,\ldots,k \right\}.$$
 (5.18)

In fact, this minimum exists as we can infer from Proposition 3.1 and Lemma 5.1 (where we choose  $(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) = (x_1,\ldots,x_k) \ \forall \beta > 0, \ \forall \varepsilon > 0$ ).

**Lemma 5.2** If in Lemma 5.1 we assume in addition that  $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon} \quad \forall \varepsilon > 0$ , then  $\varphi_{k,\beta}(x_{1,\beta},\ldots,x_{k,\beta}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta}$ .

<u>**Proof**</u> Arguing by contradiction, assume that there exists  $(y_{1,\beta}, \ldots, y_{k,\beta}) \in \Omega_{k,\beta}$  such that  $\varphi_{k,\beta}(x_{1,\beta}, \ldots, x_{k,\beta}) < \varphi_{k,\beta}(y_{1,\beta}, \ldots, y_{k,\beta})$ .

Taking into account Lemma 5.1, we have  $\int_{\Omega} |D\bar{u}_{k,\beta}^-|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\ldots,x_{k,\beta})$ . Then, slight modifications of the supports of  $\bar{u}_{k,\beta}^-$  and  $(\bar{u}_{k,\beta}^+)_i$ , for  $i = 1,\ldots,k$ , allow us to construct a function  $v_{k,\beta} \in E_{x_{1,\beta},\ldots,x_{k,\beta}}^{\beta}$  such that  $\|v_{k,\beta}^-\|_{L^2(\Omega)} = 1$ ,

$$\int_{\Omega} |Dv_{k,\beta}^{-}|^2 dx < \varphi_{k,\beta}(y_{1,\beta},\dots,y_{k,\beta})$$
(5.19)

and

$$\int_{\Omega} |D(v_{k,\beta}^{+})_{i}|^{2} dx < \beta \int_{\Omega} (v_{k,\beta}^{+})_{i}^{2} dx \qquad \forall i \in \{1,\dots,k\},$$
(5.20)

which implies the existence of k positive numbers  $t_{1,\varepsilon}, \ldots, t_{k,\varepsilon}$  such that  $f'_{\beta,\varepsilon}(t_{i,\varepsilon}(v_{k,\beta}^+)_i)$  $[(v_{k,\beta}^+)_i] = 0 \ \forall i \in \{1,\ldots,k\}$ . Let us consider the function  $v_{k,\beta,\varepsilon}$  in  $M^{\beta,\varepsilon}_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}$  such that  $(v_{k,\beta,\varepsilon}^+)_i(x) = (v_{k,\beta}^+)_i(x - x_{i,\beta,\varepsilon} + x_{i,\beta}) \ \forall x \in \Omega, \ \forall i \in \{1,\ldots,k\}, \ \forall \varepsilon > 0 \ \text{and} \ v_{k,\beta,\varepsilon}^-$  is the nonnegative function in  $H^1_0(\Omega)$  such that  $v_{k,\beta,\varepsilon}^-(x) = 0 \ \forall x \in \bigcup_{i=1}^k \operatorname{supp}(v_{k,\beta,\varepsilon}^+)_i, \ \|v_{k,\beta,\varepsilon}^-\|_{L^2(\Omega)} = 1 \ \text{and}$ 

$$\int_{\Omega} |Dv_{k,\beta,\varepsilon}^{-}|^{2} dx = \min\left\{\int_{\Omega} |Dv|^{2} dx : v \in H_{0}^{1}(\Omega), v \ge 0 \text{ in } \Omega, \\ v(x) = 0 \ \forall x \in \bigcup_{i=1}^{k} \operatorname{supp}(v_{k,\beta,\varepsilon}^{+})_{i}, \ \|v\|_{L^{2}(\Omega)} = 1\right\}.$$
(5.21)

Moreover, let us consider a function  $w_{k,\beta,\varepsilon}$  in  $M_{y_{1,\beta},\ldots,y_{k,\beta}}^{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}) =$  $\varphi_{k,\beta,\varepsilon}(y_{1,\beta},\ldots,y_{k,\beta}) \ \forall \varepsilon > 0.$ Then, since  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$  and  $v_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$ , we obtain

$$f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) \le f_{\beta,\varepsilon}(v_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}(v_{k,\beta,\varepsilon}) + \sum_{i=1}^{k} f_{\beta,\varepsilon}((v_{k,\beta,\varepsilon}^{+})_{i}) \qquad \forall \varepsilon > 0$$
(5.22)

where  $\lim_{\varepsilon \to 0} f_{\beta,\varepsilon}((v_{k,\beta,\varepsilon}^+)_i = 0 \quad \forall i \in \{1,\ldots,k\} \text{ and } \lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(v_{k,\beta,\varepsilon}^-) = \int_{\Omega} |Dv_{k,\beta}^-|^2 dx < 0$  $\varphi_{k,\beta}(y_{1,\beta},\ldots,y_{k,\beta}).$ Moreover, we have

$$f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}) = f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}^{-}) + \sum_{i=1}^{k} f_{\beta,\varepsilon}((w_{k,\beta,\varepsilon}^{+})_{i})$$
(5.23)

where  $f_{\beta,\varepsilon}((w_{k,\beta,\varepsilon}^+)_i) \ge 0 \ \forall \varepsilon > 0$  and, by Lemma 5.1,  $\lim_{\varepsilon \to 0} f_{\beta,\varepsilon}(w_{k,\beta,\varepsilon}^-) = \varphi_{k,\beta}(y_{1,\beta},\ldots,$  $y_{k,\beta}$ ). It follows that, for  $\varepsilon > 0$  small enough,  $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) < \varphi_{k,\beta,\varepsilon}(y_{1,\beta},\ldots,y_{k,\beta,\varepsilon})$  $y_{k,\beta}$  which is a contradiction because  $\varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$ .

q.e.d.

**Proposition 5.3** Let us consider  $(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon})$  in  $\Omega_{k,\beta}$  and  $u_{k,\beta,\varepsilon}$  in  $M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$ , satisfying the same assumptions as in Proposition 4.7.

Then, up to a subsequence,  $(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) \to (x_{1,\beta},\ldots,x_{k,\beta})$  as  $\varepsilon \to 0$  and  $u_{k,\beta,\varepsilon}$ converges in  $H_0^1(\Omega)$  to a function  $u_{k,\beta} \in E_{x_{1,\beta},\ldots,x_{k,\beta}}^\beta$ , for all  $\beta > \tilde{\beta}_k$  (where  $\tilde{\beta}_k$  is the number obtained in Proposition 4.7). Moreover, for all  $\beta > \tilde{\beta}_k$ ,  $u_{k,\beta}$  solves the equation

$$\Delta u - \alpha_{k,\beta} u^- + \beta u^+ = 0 \qquad \text{in } \Omega, \qquad (5.24)$$

where  $\alpha_{k,\beta} = \int_{\Omega} |Du_{k,\beta}^-|^2 dx$ .

**<u>Proof</u>** As we proved in Proposition 4.7, for all  $\beta > \tilde{\beta}_k$  and  $\varepsilon \in \left[0, \frac{1}{2} \frac{U(3\bar{r}_1) \max_{\Omega} e_1}{\lim_{|x| \to \infty} U(x) - U(3\bar{r}_1)}\right]$  $u_{k,\beta,\varepsilon}$  is a weak solution of the equation

$$\Delta u - \alpha_{k,\beta,\varepsilon} u^- + g_{\beta,\varepsilon}(u) = 0 \qquad \text{in } \Omega, \tag{5.25}$$

where  $\alpha_{k,\beta,\varepsilon} = \int_{\Omega} |Du_{k,\beta,\varepsilon}|^2 dx.$ 

Moreover, by Lemma 5.1,  $-u_{k,\beta,\varepsilon}^- + \sum_{i=1}^k (u_{k,\beta,\varepsilon}^+)_i ||(u_{k,\beta,\varepsilon}^+)_i||_{L^2(\Omega)}^{-1}$  converges in  $H_0^1(\Omega)$ , as  $\varepsilon \to 0$ , to a function  $\bar{u}_{k,\beta} \in E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}}$ . Let us prove that

$$\liminf_{\varepsilon \to 0} \|(u_{k,\beta,\varepsilon}^+)_i\|_{L^2(\Omega)} > 0 \qquad \forall \beta > \tilde{\beta}_k, \ \forall i \in \{1,\dots,k\}.$$
(5.26)

Arguing by contradiction, assume that, up to a subsequence,  $\lim_{\varepsilon \to 0} ||(u_{k,\beta,\varepsilon}^+)_i||_{L^2(\Omega)} = 0$ for suitable  $\beta > \tilde{\beta}_k$  and  $i \in \{1, \dots, k\}$ . In this case, we have  $(u_{k,\beta,\varepsilon}^+)_i \to 0$  in  $H_0^1(\Omega)$  as  $\varepsilon \to 0$  (because  $f'_{\beta,\varepsilon}(u_{k,\beta,\varepsilon})[(u_{k,\beta,\varepsilon}^+)_i] = 0$  $\forall \varepsilon > 0$ ). Therefore, if we let  $\varepsilon \to 0$ , from (5.25) we obtain

$$\int_{B(x_{i,\beta},r_{\beta})} [D\bar{u}_{k,\beta}^{-} \cdot D\psi - \bar{\alpha}_{k,\beta}\bar{u}_{k,\beta}^{-}\psi]dx = 0 \qquad \forall \psi \in H_0^1(B(x_{i,\beta},r_{\beta}))$$
(5.27)

where  $\bar{\alpha}_{k,\beta} = \int_{\Omega} |D\bar{u}_{k,\beta}^-|^2 dx$ . Thus, we have a contradiction because  $D\bar{u}_{k,\beta} \neq 0$  on  $B(x_{i,\beta}, r_{\beta}) \cap \partial(\operatorname{supp} \bar{u}_{k,\beta}^{-}).$ 

Now, let us prove that

$$\limsup_{\varepsilon \to 0} \| (u_{k,\beta,\varepsilon}^+)_i \|_{L^2(\Omega)} < +\infty \qquad \forall \beta \ge \tilde{\beta}_k, \ \forall i \in \{1,\dots,k\}.$$
(5.28)

Arguing again by contradiction, assume that, up to a subsequence,  $\lim_{\varepsilon \to 0} \|(u_{k,\beta,\varepsilon}^+)_i\|_{L^2(\Omega)}$  $= +\infty$  for suitable  $\beta > \tilde{\beta}_k$  and  $i \in \{1, \ldots, k\}$ . Then, as  $\varepsilon \to 0$ , from (5.25) we obtain

$$\int_{\Omega} [D(\bar{u}_{k,\beta}^+)_i \cdot D\psi - \beta(\bar{u}_{k,\beta}^+)_i \psi] dx = 0 \qquad \forall \psi \in H^1_0(B(x_{i,\beta}, r_\beta)).$$
(5.29)

Thus, we still have a contradiction because  $D\bar{u}_{k,\beta} \neq 0$  on  $\partial(\operatorname{supp}(\bar{u}_{k,\beta}^+)_i)$ .

Therefore, we can say that for all  $\beta > \tilde{\beta}_k$  (up to a subsequence)  $u_{k,\beta,\varepsilon}$  converges in  $H_0^1(\Omega)$ , as  $\varepsilon \to 0$ , to a function  $u_{k,\beta} \in E_{x_{1,\beta},\dots,x_{k,\beta}}^{\beta}$ . Moreover, if we let  $\varepsilon \to 0$  in (5.25), we infer that, for all  $\beta > \beta_k$ ,  $u_{k,\beta}$  is a weak solution of the equation

$$\Delta u - \alpha_{k,\beta} u^- + \beta u^+ = 0 \quad \text{in } \Omega \tag{5.30}$$

with  $\alpha_{k,\beta} = \int_{\Omega} |Du_{k,\beta}^{-}|^2 dx$ . So the proof is complete.

q.e.d.

**Proposition 5.4** For all  $\beta > \tilde{\beta}_k$ , let  $u_{k,\beta} \in E^{\beta}_{x_{1,\beta},\dots,x_{k,\beta}}$  be the function obtained in Proposition 5.3 and set  $\alpha_{k,\beta} = \int_{\Omega} |Du_{k,\beta}^{-}|^2 dx$ . Then, for every positive integer  $\tilde{k}$ ,  $u_{k,\beta} \to -e_1$  in  $H^1_0(\Omega)$  as  $\beta \to +\infty$ ,

$$\lim_{\beta \to +\infty} \beta^{\frac{N-2}{2}} (\alpha_{k,\beta} - \lambda_1) = k \operatorname{cap}(\bar{r}_1) \left( \max_{\Omega} e_1 \right)^2,$$
(5.31)

$$\lim_{\beta \to +\infty} e_1(x_{i,\beta}) = \max_{\Omega} e_1 \qquad \forall i \in \{1, \dots, k\}$$
(5.32)

and

$$\lim_{\beta \to +\infty} \sqrt{\beta} |x_{i,\beta} - x_{j,\beta}| = \infty \qquad \text{for } i \neq j.$$
(5.33)

Moreover,  $u_{k,\beta}\left(\frac{x}{\sqrt{\beta}}+x_{i,\beta}\right) \rightarrow -\left[\lim_{|x|\to\infty}U(x)\right]^{-1}\max_{\Omega}e_1U(x) \quad \forall x \in \mathbb{R}^N, \quad \forall i \in \{1, \dots, k\} \text{ and the convergence is uniform on the compact subsets of } \mathbb{R}^N.$ 

For the proof, it suffices to argue as in the proof of Proposition 4.1, taking into account Lemmas 5.1 and 5.2.

As a direct consequence of Proposition 5.4 (see (5.31)), we can state the following corollary.

**Corollary 5.5** For all positive integer k and for  $\beta > \tilde{\beta}_k$ , let  $\alpha_{k,\beta}$  be as in Proposition 5.4. Then, there exists a sequence  $(b_k)_k$  such that

$$b_k \ge \hat{\beta}_k, \ b_k \le b_{k+1} \ and \ \alpha_{k,\beta} < \alpha_{k+1,\beta} \ \forall k \in \mathbb{N}, \ \forall \beta > b_{k+1}.$$
 (5.34)

**Proposition 5.6** Let  $b_k$  and  $\alpha_{k,\beta}$  be as in Corollary 5.5 for every positive integer k and for  $\beta > b_k$ . Then,  $\alpha_{k,\beta}$  depends continuously on  $\beta$  in  $]b_k, +\infty[ \forall k \in \mathbb{N}.$ 

**<u>Proof</u>** Taking into account Lemma 5.1, we have  $\alpha_{k,\beta} = \int_{\Omega} |Du_{k,\beta}^-|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\ldots,x_{k,\beta}) \quad \forall k \in \mathbb{N} \quad \forall \beta > b_k.$ 

Let us prove that  $\lim_{\beta \to \bar{\beta}} \alpha_{k,\beta} = \alpha_{k,\bar{\beta}} \,\forall \bar{\beta} \in ]b_k, +\infty[$ . First notice that, by lower semicontinuity arguments with respect to the weak  $H_0^1(\Omega)$  convergence, we have  $\liminf_{\beta \to \bar{\beta}} \alpha_{k,\beta}$  $\geq \alpha_{k,\bar{\beta}}$ . Then, arguing by contradiction, assume that there exists a sequence  $(\beta'_n)_n$  such that  $\lim_{n \to \infty} \beta'_n = \bar{\beta}$  and  $\lim_{n \to \infty} \alpha_{k,\beta'_n} > \alpha_{k,\bar{\beta}}$ , namely

$$\lim_{n \to \infty} \int_{\Omega} |Du_{k,\beta_n'}^-|^2 dx > \int_{\Omega} |Du_{k,\bar{\beta}}^-|^2 dx.$$
(5.35)

Let us set  $\bar{u}_n = -u_{\bar{k},\beta'_n}^- + \sum_{i=1}^k (u_{\bar{k},\beta'_n}^+)_i \| (u_{\bar{k},\beta'_n}^+)_i \|_{L^2(\Omega)}^{-1}$ . Since  $\int_{\Omega} |D(\bar{u}_n^+)_i|^2 dx = \beta'_n \, \forall n \in \mathbb{N}$ ,  $\bar{u}_n$  converges to a function  $\bar{u} \in E_{x_{1,\bar{\beta}},\dots,x_{k,\bar{\beta}}}^{\bar{\beta}}$  in  $L^2(\Omega)$ , weakly in  $H_0^1(\Omega)$  and a.e. in  $\Omega$ . It follows that  $\int_{\Omega} |D\bar{u}_i^+|^2 dx \leq \bar{\beta}$  and  $\|\bar{u}_i^+\|_{L^2(\Omega)} = 1 \, \forall i \in \{1,\dots,k\}$ . Therefore, if (5.35) holds, one can find a function  $\tilde{u} \in E_{x_{1,\bar{\beta}},\dots,x_{k,\bar{\beta}}}^{\bar{\beta}}$  such that  $\int_{\Omega} |D\tilde{u}_i^+|^2 dx = \bar{\beta}, \|\tilde{u}_i^+\|_{L^2(\Omega)} = 1$  $\forall i \in \{1,\dots,k\}$  and

$$\lim_{n \to \infty} \int_{\Omega} |Du_{k,\beta_n'}|^2 dx > \int_{\Omega} |D\tilde{u}^-|^2 dx.$$
(5.36)

Now, let us consider the function  $\tilde{u}_n \in E_{x_{1,\beta'_n},\dots,x_{k,\beta'_n}}^{\beta'_n}$  such that  $(\tilde{u}_n^+)_i(x) = \tilde{u}_i^+(\sqrt{\beta'_n\bar{\beta}^{-1}})$  $(x - x_{i,\beta'_n}) + x_{i,\bar{\beta}} \quad \forall x \in \Omega, \forall i \in \{1,\dots,k\}, \forall n \in \mathbb{N} \text{ and } \tilde{u}_n^- \text{ is the nonnegative function}$ in  $H_0^1(\Omega)$  such that  $\tilde{u}_n^-(x) = 0 \quad \forall x \in \bigcup_{i=1}^k \operatorname{supp}(\tilde{u}_n^+)_i, \|\tilde{u}_n^-\|_{L^2(\Omega)} = 1$  and

$$\int_{\Omega} |D\tilde{u}_n^-|^2 dx = \min\left\{\int_{\Omega} |Du|^2 dx : u \in H_0^1(\Omega), \ u \ge 0 \text{ in } \Omega, \\ u(x) = 0 \ \forall x \in \bigcup_{i=1}^k \operatorname{supp}(\tilde{u}_n^+)_i, \ \|u\|_{L^2(\Omega)} = 1\right\} \ \forall n \in \mathbb{N}.$$
(5.37)

Notice that  $\lim_{n\to\infty} \int_{\Omega} |D\tilde{u}_n^-|^2 dx = \int_{\Omega} |D\tilde{u}^-|^2 dx$ ; moreover, since  $\|(\tilde{u}_n^+)_i\|_{L^2(\Omega)}^{-2} \int_{\Omega} |D(\tilde{u}_n^+)_i|^2 dx = \beta'_n$ , we have

$$\alpha_{k,\beta'_n} = \int_{\Omega} |Du_{k,\beta'_n}|^2 dx = \varphi_{k,\beta'_n}(x_{1,\beta'_n},\dots,x_{k,\beta'_n}) \le \int_{\Omega} |D\tilde{u}_n|^2 dx \quad \forall n \in \mathbb{N}$$
(5.38)

and, as  $n \to \infty$ ,

$$\lim_{n \to \infty} \int_{\Omega} |Du_{k,\beta_n'}|^2 dx \le \lim_{n \to \infty} \int_{\Omega} |D\tilde{u}_n|^2 dx = \int_{\Omega} |D\tilde{u}^-|^2 dx,$$
(5.39)

in contradiction with (5.36).

Thus, we can conclude that  $\alpha_{k,\beta}$  depends continuously on  $\beta$  in  $]b_k, +\infty[$ .

q.e.d.

<u>Proof of Theorem 2.1</u> For every positive integer k, for  $\beta > 0$  large enough so that  $\Omega_{k,\beta} \neq \emptyset$  and for  $\varepsilon > 0$ , let us consider a point  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon}) \in \Omega_{k,\beta}$  and a function  $u_{k,\beta,\varepsilon} \in M_{x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}}^{\beta,\varepsilon}$  such that  $f_{\beta,\varepsilon}(u_{k,\beta,\varepsilon}) = \varphi_{k,\beta,\varepsilon}(x_{1,\beta,\varepsilon},\ldots,x_{k,\beta,\varepsilon}) = \max_{\Omega_{k,\beta}} \varphi_{k,\beta,\varepsilon}$  (here we apply Propositions 3.1 and 3.2).

As  $\varepsilon \to 0$  (up to a subsequece)  $(x_{1,\beta,\varepsilon}, \ldots, x_{k,\beta,\varepsilon})$  tends to a point  $(x_{1,\beta}, \ldots, x_{k,\beta}) \in \Omega_{k,\beta}$ and  $u_{k,\beta,\varepsilon}$  converges in  $H_0^1(\Omega)$  to a function  $u_{k,\beta} \in E_{x_{1,\beta},\ldots,x_{k,\beta}}^{\beta}$  which, for  $\beta > 0$  large enough, satisfies the equation  $\Delta u - \alpha_{k,\beta}u^- + \beta u^+ = 0$  in  $\Omega$  with  $\alpha_{k,\beta} = f_{\beta}(u_{k,\beta}) = \int_{\Omega} |Du_{k,\beta}^-|^2 dx = \varphi_{k,\beta}(x_{1,\beta},\ldots,x_{k,\beta}) = \max_{\Omega_{k,\beta}}\varphi_{k,\beta} > \lambda_1$  (here we apply Lemmas 5.1 and 5.2 and Proposition 5.3).

Thus  $(\alpha_{k,\beta},\beta)$  belongs to the Fučík spectrum  $\Sigma$  for  $\beta > 0$  large enough. Moreover, from Proposition 5.4 we infer that, for every positive integer  $k, \alpha_{k,\beta} \to \lambda_1$  as  $\beta \to +\infty$ while  $u_{k,\beta} \to -e_1$  in  $H_0^1(\Omega)$ . Corollary 5.5 guarantees the existence of a nondecreasing sequence  $(b_k)_k$  of positive numbers such that  $\alpha_{k,\beta} < \alpha_{k+1,\beta} \forall \beta > b_{k+1}$ . Proposition 5.6 shows that  $\alpha_{k,\beta}$  depends continuously on  $\beta$  in  $]b_k, +\infty[$ .

All the other assertions in Theorem 2.1 follow directly from Proposition 5.4 as one can easily verify.

q.e.d.

**Remark 5.7** Assume that the domain  $\Omega$  satisfies in addition the following condition: there exists an open subset A of  $\Omega$  such that  $\sup_{\partial A} e_1 < \sup_A e_1$ . Then, the method used to prove Theorem 2.1 may be easily adapted in order to construct eigenfunctions  $u_{k,\beta}$  as in Theorem 2.1, with k bumps localized near k concentration points  $x_{1,\beta}, \ldots, x_{k,\beta}$ , with rescaled bumps having the same asymptotic profile (still described by the radial solution U of (2.1)), but with the concentration points that, as  $\beta \to +\infty$ , approach maximum points of  $e_1$  in A (i.e.  $x_{i,\beta} \to x_i$  as  $\beta \to +\infty$ , with  $x_i \in A$  and  $e_1(x_i) = \max_A e_1$  for  $i = 1, \ldots, k$ ). **Remark 5.8** Notice that (as we show in a paper in preparation) one can also obtain infinitely many curves of the Fučík spectrum  $\Sigma$ , asymptotic to the lines  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$  and corresponding to eigenfunctions of different type, with bumps localized near points of the boundary of  $\Omega$  (while the eigenfunctions  $u_{k,\beta}$  given by Theorem 2.1 present k bumps localized near the maximum points of  $e_1$ ).

In fact, under the same assumptions as in Theorem 2.1, there exists a nodecreasing sequence  $(\bar{b}_k)_k$  of positive numbers, having the following properties. For all  $\beta > \bar{b}_k$ there exists  $\bar{\alpha}_{k,\beta} > \lambda_1$  and  $v_{k,\beta} \in H^1_0(\Omega)$ ,  $v^+_{k,\beta} \neq 0$  and  $v^-_{k,\beta} \neq 0$ , such that (1.1), with  $\alpha = \bar{\alpha}_{k,\beta}$  and  $u = v_{k,\beta}$ , is satisfied for all  $\beta > \bar{b}_k$ . Moreover, for every  $k \in \mathbb{N}$ ,  $\bar{\alpha}_{k,\beta}$ depends continuously on  $\beta$ ,  $\bar{\alpha}_{k,\beta} < \bar{\alpha}_{k+1,\beta} \ \forall \beta > \bar{b}_{k+1}$  and  $\bar{\alpha}_{k,\beta} \to \lambda_1$ , as  $\beta \to +\infty$ , while  $v_{k,\beta} \to -e_1$  in  $H_0^1(\Omega)$ . Furthermore,  $v_{k,\beta}$  present k bumps that, as  $\beta \to +\infty$ , concentrate near k points approaching the boundary of  $\Omega$ ; the concentration rate is greater than the approaching rate between two distinct concentration points or between the concentration points and the boundary (so that the k bumps remain quite distinct). The eigenfunctions  $v_{k,\beta}$  have lower energy and they have a different variational nature compared to the eigenfunctions  $u_{k,\beta}$ . In fact, their bumps present a different asymptotic profile which is not described by the function U, as it happens for the eigenfunctions  $u_{k,\beta}$  (see Theorem 2.1). Notice that, since  $v_{k,\beta}$  has lower energy than  $u_{k,\beta}$ , we can also say that, even in the case k = 1, Theorem 2.1 does not give the first curve of the Fučík spectrum (see for istance [15]) because, for all  $\beta > b_1$ , the pair  $(\alpha_{1,\beta},\beta)$  does not belong to the first curve; the eigenfunctions corresponding to pairs  $(\alpha, \beta)$  of the first curve have lower energy and only one bump which, for  $\alpha$  or  $\beta$  large enough, is localized near the boundary of  $\Omega$  (see [31] and [32]).

**Remark 5.9** It is interesting to know from where the curves of the Fučík spectrum we obtain come from. They might come from bifurcations of the first curve of the Fučík spectrum, which emanates from the pair  $(\lambda_2, \lambda_2)$ , or they might come from pairs  $(\lambda_i, \lambda_i)$  of higher eigenvalues, or might be they do not meet the line  $\{(\alpha, \beta) \in \mathbb{R}^2\}$ :  $\alpha = \beta$ , etc. .... The fact that the corresponding eigenfunctions present several nodal regions (as the Fučík eigenfunctions related to pairs  $(\alpha, \beta)$  close to pairs  $(\lambda_i, \lambda_i)$ of higher eigenvalues) seems to suggest that they might be curves emanating from the pairs  $(\lambda_i, \lambda_i)$ . However notice that, for the Fučík eigenfunctions we obtained in this paper, only the positive part presents several nodal regions while the negative part has only one nodal region (on the contrary, it is natural to expect that for the Fučík eigenfunctions corresponding to pairs  $(\alpha, \beta)$  close to pairs  $(\lambda_i, \lambda_i)$ , both positive and negative parts present several nodal regions); on the other hand, also in the case N > 1, one can find simple examples of curve in the Fučík spectrum that pass through pairs  $(\lambda_i, \lambda_i)$  of higher eigenvalues and are asymptotic to lines  $\{\lambda\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda\}$ whith  $\lambda > \lambda_1$ . Thus, the problem is widely open and might give rise to interesting results. Most probably, if  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  with N > 1, for each pair  $(\lambda_i, \lambda_i)$  of eigenvalues, the smallest curve of the Fučík spectrum emanating from  $(\lambda_i, \lambda_i)$ ,

corresponding to lower energy eigenfunctions, is asymptotic to  $\{\lambda_1\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda_1\}$ while the other curves passing through  $(\lambda_i, \lambda_i)$  are asymptotic to lines  $\{\lambda\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\lambda\}$  whith  $\lambda > \lambda_1$ .

**Remark 5.10** The difference between the case of dimension N = 1 and the case N > 1 becomes even more evident if in (1.1) we replace the Dirichlet boundary condition by the Neumann condition  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial \Omega$ .

In fact, if we denote by  $\tilde{\lambda}_1 < \tilde{\lambda}_2 \leq \tilde{\lambda}_3 \leq \ldots$  and by  $\tilde{\Sigma}$ , respectively, the eigenvalues of  $-\Delta$  and the Fučík spectrum with Neumann boundary conditions, we have  $\tilde{\lambda}_1 = 0$ and, if N = 1, no curve of  $\tilde{\Sigma}$  is asymptotic to the lines  $\{0\} \times \mathbb{R}$  and  $\mathbb{R} \times \{0\}$ . Indeed, if N = 1, a direct computation shows that the Fučík spectrum consists of the lines  $\{0\} \times \mathbb{R}$  and  $\mathbb{R} \times \{0\}$  and of infinitely many curves  $C_2, C_3, \ldots$  having the following properties: for every  $i \geq 2$ ,  $C_i$  is a smooth, unbounded, decreasing curve, emanating from  $(\tilde{\lambda}_i, \tilde{\lambda}_i)$  and asymptotic to the lines  $\{\frac{\tilde{\lambda}_i}{4}\} \times \mathbb{R}$  and  $\mathbb{R} \times \{\frac{\tilde{\lambda}_i}{4}\}$  (notice that  $\frac{\tilde{\lambda}_i}{4}$  is an eigenvalue of  $-\Delta$  in  $H^1(\Omega)$  if and only if i is an odd positive integer and, in this case,  $\frac{\tilde{\lambda}_i}{4} = \lambda_{(i+1)/2}$ ). Therefore, if N = 1, no curve of  $\tilde{\Sigma}$  is asymptotic to the lines  $\{0\} \times \mathbb{R}$ and  $\mathbb{R} \times \{0\}$  and every nontrivial pair  $(\alpha, \beta)$  of  $\tilde{\Sigma}$  satisfies  $\alpha > \frac{\tilde{\lambda}_2}{4}$  and  $\beta > \frac{\tilde{\lambda}_2}{4}$  (with  $\tilde{\lambda}_2 > \tilde{\lambda}_1 = 0$ ).

On the contrary, the situation is quite different in the case N > 1. In fact (as we show in a paper in preparation) in this case there exist infinitely many curves contained in  $\tilde{\Sigma}$  and asymptotic to the lines  $\{0\} \times \mathbb{R}$  and  $\mathbb{R} \times \{0\}$ ; the corresponding eigenfunctions have an arbitrarily large number of bumps which may be localized in the interior of  $\Omega$ or near prescribed connected components of  $\partial\Omega$ ; both, interior and boundary bumps, present the same asymptotic profile (still described by the function U).

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