UNIFORM UNIFORMISATION

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Introduction

Most misunderstandings in mathematics start in dimension 1. Probably the best example is class field theory, but another good one arose with R. Nevanlinna's use of the uniformisation theorem to prove a trivial yet hard isoperemtric inequality for the punctured plane. Unlike a classical inequality between area and length, Nevanlinna averaged the former, and so was able to replace the latter by its logarithm. In so doing, though, he used the full power of the complex structure, and rendered his study ill suited to soft generalisation. Nevertheless he did create value distribution theory, or better the study of the isoperemetric inequality in his sense, VI.2.2, on quasi-projective varieties. Depending on whether one studies discs or parabolic (in the sense of Ahlfor's) Riemann surfaces on one's variety this is more, respectively less, general than the isoperemetric inequality in Gromov's sense, albeit that under reasonable hypothesis, [Kl], even the less general version will imply the latter, and of course, all of these inequalities should be viewed as an integrated or large scale version of negative curvature.

At first glance this invocation of hard methods appears pointless. A brief reflection shows that holomorphic sectional curvature, even stricta dicta, does not admit an a priori interpretation in terms of algebraic geometry, except, of course, for curves where it coincides with Ricci curvature, whence the equidimensional Nevanlinna theory of Griffiths, Stoll et al. The said equidimensional study, however, confined itself, for largely historical reasons, to smooth varieties as opposed to what is now known to be the natural condition for negative Ricci curvature, i.e. Mori/Reid's canonical singularities. Indeed these can even be defined as precisely the singularities for which local versions of any of the above isoperemetric inequalities hold, and whence a conjectural equivalence between the isoperemtric inequality for all balls and canonical models of algebraic varieties, with the conjecture being the reverse implication by way of the minimal model programme.

Combined with examples this does seem to suggest that any attempt to pursue algebraic criteria for the isoperemtric inequality in a non-equidimensional setting is by definition hopeless. Indeed unlike the 'logarithmic surface area' of a ball which is naturally and uniquely computed by Nevanlinna's height applied to the canonical class there are countably many bundles that one might look at for discs, i.e. the tautological bundles on spaces of jets. At this point, however, Green & Griffiths, [GG], made the key observation that the law of large numbers may come to the rescue, implying, as it does, at least for algebraic surfaces when combined with the more subtle properties of the canonical class -essentially the existence of a Kähler-Einstein metric-, that for high enough jets the bundles in question are big when the Ricci curvature is negative. Developments in jet technology post op. cit. occasions a review of this in VI.1, but unlike the actual canonical class, however, it may not be possible to carry out surgery on either the surface or its jets to guarantee that these bundles are ample. Even worse there may be a base locus on which the restrictions of any of these bundles fails to be pseudo effective. On the positive side, though, the problem becomes equidimensional, albeit in the softest sense that current understanding of category theory permits, i.e. the linearisation of an nth order ODE corresponding to a component of the base is a foliation by curves, $\mathcal{F}$, on a n+1 dimensional variety X. Whence, up to some technical issues of definition occasioned by it's singularities, an identification of negative curvature with the relative positivity of the canonical class $K_{\mathcal{F}}$ of X over its classifying stack $[X/\mathcal{F}]$.

The issues posed by the foliation singularities are two fold. The one of no immediate relevance is for the definition of the classifying stack itself. This is, however, no worse than the problem of trying to view a punctured disc as a stack in the analytic topology with an infinite punctual stabiliser, and so is easily swept under the carpet. More critically, just as for Ricci curvature on varieties, one needs canonical foliation singularities before negative curvature, even locally, can be identified with a positivity property of $K_{\mathcal{F}}$. Indeed without this, one has, even for X smooth, a bundle which is only really a relative dualising sheaf rather than a genuine canonical class. Again, this can be taken as a definition of canonical singularities, which in turn is equivalent to the functorial extension of the Mori/Reid definition to this context, [M1]. Consequently the universality of our discussion depends on progress to a hypothesised resolution LCR, I.6.1, but this is rather promising, and while postponing our principle commentary on it till later, it is of immediate relevance to note that this cannot be achieved in the category of smooth spaces foliated by curves, but only in the 2-category of smooth algebraic (which will always mean Deligne-Mumford) stacks. Fortunately on replacing X by an algebraic stack $\mathcal{X}$ there are no more problems associated with the formation of $[\mathcal{X}/\mathcal{F}]$ than those we've already described, whence we will work with, and profit from, the soft flexibility that algebraic stacks afford, albeit that the presence of a taming form, i.e. the projectivity of the moduli, is essential.

Not all our initial preparation, however, is confined to blowing up, for we must also blow down, or more generally flip, the more obvious obstructions to negative curvature that we come across. In so doing we will be able to localise the $K_{\mathcal{F}}$ isoperemetric inequality of [M1] V.5.10, with the resulting model $(\mathcal{X}, \mathcal{F})$ enjoying $K_{\mathcal{F}}$ nef. Such a, so called minimal, model, which by op. cit. exists whenever the foliation isn't a pencil of conics, only eliminates any positively curved invariant subvarieties, and what is really desired is to carry out algebraic surgery until such times that we arrive to, a *canonical model*, i.e. not only $K_{\mathcal{F}}$ nef. but any solution curve $\mathcal{C}$ with $K_{\mathcal{F}}.\mathcal{C} = 0$ meets every effective Cartier divisor non-negatively, or equivalently: if in an act of wild optimism we suppose that the obstructions to negative curvature along our foliation will concentrate on something algebraic, then the algebraic object should move, giving our foliation a special form. Unlike minimality this is a substantially more difficult condition to guarantee. Indeed, every type of singularity that one can conceive of in actual Mori theory has a variant, in fact several, in the theory of canonical models of foliations, with plenty more beside, e.g. of arbitrary embedding dimension on 3 folds. The lack then of such a model, and with it the implied possibility of large algebraic sets of nil curvature will be a rather major difficulty in addressing our curvature problem to which we finally give a concrete form,

Question *Presented with a minimal algebraic stack $(\mathcal{X}, \mathcal{F})$ foliated by curves, can we find algebraic criteria to guarantee that $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} > 0$ for all transverse invariant measures $d\mu_{\mathcal{X}/\mathcal{F}}$?*

Of course the measures associated to bigger and bigger discs are not arbitrary, indeed they will, without loss of generality, also be nef., i.e. intersect every Cartier divisor non-negatively, so $K_{\mathcal{F}}$ big (general type) is certainly sufficient. Unlike the Ricci case, however, bigness is not a sine qua non for negative curvature, but rather for 'full' modular variation of hyperbolic leaves. A fact that is already evident in the theory of algebraic curves, but whose manifestations are not confined to such integrable examples. If, however, we could quantify the lack of bigness by appropriate smooth metricisation of $K_{\mathcal{F}}$ then the Bochner technique, [M2], would allow us to assert that a lack of modular variation causes the Kähler structure to split, and whence resolve our problem. Unfortunately this is very much a catch 22 scenario since the key to realising such a metricisation is the question itself. What we do learn, though, is that in each dimension n we should expect exactly one new case, i.e. not pulled back or fibred over lower dimension, which is hyperbolic yet not of general type, and it should satisfy $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} > 0$. We can thus refine our question by way of,

Antithesis *Can there exist a parabolic invariant measure, i.e. $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, together with some naturality conditions, V.I.2, on a minimal foliated stack with $K_{\mathcal{F}}^n = 0$, but $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} > 0$.*

To put the antithesis in perspective, one observes, quite generally, II.2.2, that a nef. bundle K on a n dimensional projective variety with $K^{n-1} \neq 0$ admits at most one class intersecting every effective Cartier divisor non-negatively, yet 0 with K, i.e. K^{n-1}. Needless to say such nefness is one of the naturality conditions on our measure $d\mu_{\mathcal{X}/\mathcal{F}}$, and so what our antithesis leads to is a special representative of $K_{\mathcal{F}}^{n-1}$. Plainly the particular thing about this representative is its invariance. It is important, however, to have some feeling for the nature of measures on proper stacks, which are a priori much better behaved than the simple patching of local data would seem to suggest. Indeed the algebraist who would be inclined to think of such as some sort of analytic 1-cycle will be much closer to the truth, than the analyst who carries a bag of local counterexamples to any reasonable statement that one might wish to make. In particular for every closed substack $\mathcal{Z}$ of $\mathcal{X}$ there are well defined segre classes $s_{\mathcal{Z},d\mu}$, or residual measures, on the projective normal cone, III.3, specialisation to the cone itself, III.5, and a well defined notion of *diffuseness*, i.e. lack of expressablity as a countable sum of direct images from lower dimensional stacks, V.3 & VI.3. The invariance, however, gives rise to a very different, a priori highly non-logarithmic, residue symbol, $Res(d\mu_{\mathcal{X}/\mathcal{F}})$, IV.1, expressing the lack of flatness of the bi-rational groupoid, IV.4, $\tilde{\mathfrak{f}} \rightrightarrows \mathcal{X}$ seen from $d\mu_{\mathcal{X}/\mathcal{F}}$.

The critical observation for relating these notions of residue comes from Connes, [Co], namely: $d\mu_{\mathcal{X}/\mathcal{F}}$ should be viewed as a realisation of motives over $[\mathcal{X}/\mathcal{F}]$ or if one prefers a Fredholm module over the ring of smooth relative correspondences. Either way the upshot is the same: relative analysis of $\mathcal{X}$ over $[\mathcal{X}/\mathcal{F}]$ is, abstractly, no more or less difficult than that on a Riemann surface, albeit that the presence of singularities requires a different approach via explicit formulae on the bi-rational groupoid to do the necessary harmonic theory, IV.5-7, than that of op. cit. Thus apart from casting significant functorial light on foliated residues, and indeed the precise relation between Conne's algebras and the classifying stack, IV.8, we deduce,

Lemma (IV.7.4) (Independently of LCR) *Suppose that the segre class of a transverse invariant measure around* $\mathrm{sing}(\mathcal{F})$ *is zero, then its residue class is zero, i.e. for all intents and purposes the infinitesimal groupoid* $\mathfrak{f} \rightrightarrows \mathcal{X}\setminus \mathrm{sing}(\mathcal{F})$ *seen from* $d\mu_{\mathcal{X}/\mathcal{F}}$ *has a flat completion across the singularities.*

Now if in the antithesis we further suppose the measure is diffuse, then a simple minded counting argument with global sections of bundles, V.1.4, V.2.2, implies,

corollary (VI.3.4/5) *For* $d\mu_{\mathcal{X}/\mathcal{F}}$ *diffuse the antithesis is rubbish. Indeed even* $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} \neq 0$ *will suffice.*

One's immediate reaction, therefore, is that our question has been answered modulo an induction the size of an olympiad problem book (compendium edition). Unfortunately, though, a new higher dimensional difficulty, V.4 & VI.4, emerges: obstructions to negative curvature can hide in $\mathrm{sing}(\mathcal{F})$. The difficulty here is not, for example, visible on the universal curve over the moduli stack of algebraic curves since it initially occurs when invariant discs on the smooth locus limit on $\mathrm{sing}(\mathcal{F})$, giving rise to singular solutions, I.4.10. As such it first occurs for leaves of the induced foliation in a weak branching formal stack (understood as the functorial extension, I.2 & I.6, of what one sees at 2-D saddles), followed by induction of the same.

This ultimate difficulty, which is precisely the obstruction to inducting the parabolicity of a measure through smaller substacks merits some comment. In the classical theory of saddles on surfaces the weak branch need not be convergent, but it's not far off, converging as it does after real blowing up. In dimension 3 this is false at what we've termed beasts, I.5.1, where, by definition, beasts develop as singularities degenerate. The initial saving grace is that a weak branch is a priori generically defined as an honest formal stack, whence some general nonsense combined with LCR (albeit this may be un-necessary) shows that it extends as a proper formal stack around singular components where it is generically defined, I.6.7. This does not, however, include singular components degenerating to such at beasts. The second thing is that a very general counting argument shows that any part of $d\mu_{\mathcal{X}/\mathcal{F}}$ which is relevant to a weak branch, VI.5.1/2, is in fact a diffuse measure on some honest substack in it's pseudo trace, i.e. the completion of the latter in appropriate components of $\mathrm{sing}(\mathcal{F})$ is non-empty and factors through the formal stack in question. The critical observation, V.4.1 & VI.5.3, is that the dynamics around the weak branch are so isolated from everything else that the totality of components held together by this infinitesimal glue inherits all the naturality properties, including nefness to the extent that it has sense, of a parabolic measure with the exception of the parabolicity condition itself. This latter requires further surgery, weak flops, specified by a final critical refinement, VI.6.5 & VI.7.3, of the cone theorem of $[\beta\mu]$. Such weak flops are rather natural since they terminate in a model on which every sequence of 'solutions' to the foliation, i.e. the co-normal bundle vanishes, converging to a disc with bubbles, converges to an honest disc. Flops, though, are to be understood as any non-contractile surgery from a minimal to a canonical model, as such, even weak ones are strictly more difficult than flips- klt foliated triples $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ are very rare. Thus to complete our machinery we could either make weak flops or extend the flip theorem to 'pseudo irreducible formal substacks'. The former is more conceptual, but leads to slightly worse ambient singularities, so, for expediency, we choose the latter, VI.6.10-12, to kill the last set of a priori obstructions to localising parabolicity.

Rather than undertaking a humongous induction to completely resolve our curvature query, let us take a couple of illustrative examples beginning with 3-folds, where the previous step in dimension 2 has been done, [M2], and we require to understand the hypothesised, necessarily non-diffuse, parabolic measure of the antithesis. The surface theory, combined with the above machinery, quickly allows us to deduce, V.2 & 4, that it must be a countable sum of measures of the following form: invariant measures on surfaces where the induced foliation is a conic pencil, rational curves which don't move, and divisors of elliptic fibre type inside a formal weak branching surface. The first and third possibilities are effectively finite, at least as far as the conic pencil itself is concerned. Plainly, we should really try and move the elliptic curve to show that it doesn't exist, but this is an aesthetic point, and what's essential is to exclude the possibility of infinitely many rigid rational curves invariant by the foliation and intersecting $K_{\mathcal{F}}$ in zero. A detailed local analysis, II.3-8, shows the curves in question can be flopped. If then it could be guaranteed that flopping preserved projectivity it would be easy to show that there were at most finitely many. The problem though is that it is rather difficult to relate this soft sense of flopping to the more generally accepted one involving a specific divisor. Indeed one could imagine being on an actual canonical model, but still finding infinitely many such contractible curves because the singularities became so bad that one simply ran out of Cartier divisors. The flopping operation, though, can only fail to be projective because there is no sense of positivity in Néron-Severi which if raised to the level of measures would be a contradiction since these must always have a sign irrespective of the existence of a taming form, and this is exactly what we do by way of some a priori soft, i.e. non-projective surgery, II.9, to ensure that the rigid curves are wholly isolated, and whence,
Finiteness in 3-D (V.2-4) *Let $d\mu_{\mathcal{X}/\mathcal{F}}$ be a transverse invariant measure on a three dimensional minimal foliated stack $(\mathcal{X}, \mathcal{F})$ not of general type with $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$ then there is a decomposition,*

$$d\mu_{\mathcal{X}/\mathcal{F}} = \sum_i d\mu_i + \sum_j L_j + \sum_k E_k$$

where, $d\mu_i$ are invariant measures on the finitely many invariant surfaces where the foliation is in conics, the L_j are a finite sum of invariant rational curves, and the E_k are the above, likely inexistent, elliptic curves.

In the case that $K_{\mathcal{X}}.K_{\mathcal{F}}^2 < 0$, $(\mathcal{X}, \mathcal{F})$ is actually fibred in rational curves over a foliated surface, so the elliptic bit doesn't exist, and the whole thing is an easily analysed Ricatti type object. In the antithesis case, the proof pretty much establishes that all the rational part can be killed by flopping, and contraction, so up to moving the elliptic object, and a slightly better understanding of the role $c_1^2 + c_2$ plays in Riemann-Roch, our curvature question is completely answered on 3-folds. To apply this to our initial problem on surfaces we need canonical resolution, albeit in local form, ILUT, I.1.3, together with [M4] which instantly gives,
corollary *Suppose ILUT, and let $(\mathcal{S}, \mathcal{D})$ be a 2-dimensional log-stack of general type with $c_1^2 > \frac{2}{3}c_2$ then the set of rational and elliptic curves (i.e. log-substacks étale covered by orbifolds of positive, respectively nil, topological Euler characteristic) form a proper substack $\mathcal{Z}$ outside of which the isoperemetric inequality holds in Nevanlinna's sense for arbitrary ramified covers of the line. In particular curves of genus g are bounded in moduli, and no holomorphic map from an affine algebraic curve can have Zariski dense image.*

More generally, our machinery is well adapted to investigating the isopermetric inequality on any surface. Indeed the semi-stability of the co-tangent bundle on a minimal surface with respect to it's canonical means that any ODE of order at least 1 once linearised and viewed as a foliation is either of general type, or satisfies the conditions of the antithesis. The extension to minimal models, [M1], of the fact that any rational map to a curve of positive genus is a morphism allows us to move easily between ODEs, and sub-ODEs, even if the latter isn't finite over it's natural image in the jet space, and to induct all the way down to order 0, i.e. curves on surfaces, VI.3. Arriving to hyperbolic curves is impossible, but for rational or elliptic we need [M4], which in turn needs the number of these to be finite. Whence, our second applications takes the form,
Theorem *Suppose LCR, then for $(\mathcal{S}, \mathcal{D})$ a 2-dimensional log-stack with canonical ($K_{\mathcal{S}} + \mathcal{D}$ sense) singularities and stable curve as boundary the following are equivalent,*

(a) $(\mathcal{S}, \mathcal{D})$ *is of general type without rational or elliptic curves.*
(b) $(\mathcal{S}, \mathcal{D})$ *satisfies the isoperemetric inequality in Nevanlinna's sense for ramified covers of the line.*
(c) $(\mathcal{S}, \mathcal{D})$ *satisfies the isoperemetric inequality in Gromov's sense for discs.*

More, or less, depending on your point of view, generally, the following are also equivalent under LCR,

(a) $(\mathcal{S}, \mathcal{D})$ *has general type, and the set of rational and elliptic curves form a proper closed substack* $\mathcal{Z}$.

(b) *The isoperemetric inequality for ramified covers of the line holds in Nevanlinna's sense on* $(\mathcal{S}, \mathcal{D})$ *outwith finitely many rational and elliptic curves*

(c) *The only ODEs on* $(\mathcal{S}, \mathcal{D})$ *admitting parabolic measures are rational and elliptic curves.*

(d) *The solution discs of any ODE on* $(\mathcal{S}, \mathcal{D})$, *other than a rational or elliptic curve, converge, in the compact open sense, modulo a proper algebraic set.*

To establish the a priori finiteness of elliptic curves on surfaces of general type, would appear to be a perfectly feasible refinement of the methodology. Despite the implied additional elegance, this isn't done because it won't change the basic fact that the methodology has a problem with the obstruction posed by parabolic algebraic curves. Indeed from the foliated point of view, a repeat of the 3-D finiteness theorem for rational curves looks quite hard, and plainly requires a fairly systematic attack in the direction of the construction of a canonical model, but with the same sort of caveats as for 3-folds that this isn't quite the same thing. In particular, it's not absolutely clear that the finiteness theorem holds for arbitrary foliated 4-folds, but at the same time it's unclear how to tailor things to the original surface.

Lack of LCR (log-canonical resolution) of foliation singularities should also be viewed as an obstruction rather than LCR as a hypothesis. Indeed if we have the convergence of invariant discs then a valuation for which LCR fails, which is necessarily approximable by discs outwith very special circumstances, will infact be a disc (strictly speaking a family of discs) and this gives the sort of contact structure which one needs. Plainly, making this completely rigorous is worth while, since we then arrive to a completely algebraic description of the validity or otherwise of the isoperemetric inequality on algebraic surfaces. My excuse for not having done this is that my recent trip to Valladolid convinces me that it is un-necessary, so I'll attempt a quick summary of the status quaestionis. In [C], F. Cano proved the local uniformisation theorem in dimension 3, and whence one would have imagined LCR in dimension 3. Unfortunately the corollary didn't follow since although the local global argument here goes back to Zariski, the local version in question was not sufficiently functorial with respect to the ideas, i.e. not all the centres respected the foliation, and what was worse there was even a case where the last centre was defined by a differential rather than an algebraic condition, I.1.2. This ultimate problem is intrinsic, and whence Cano proposed that the correct strategy was to take a root. Consequently the globalisation procedure is necessarily in the 2-category of algebraic stacks and/or $\mathbb{Q}$ gorenstien foliations if one wants to stick to spaces. The content of I.1 is the verification that Zariski's local global argument still works in these circumstances, which indeed it does, albeit in a slightly more delicate way, i.e. first use it to 'prepare' the divisor where one needs to take a root, then pass to stacks, then run it again. This discussion is not particularly specific to dimension 3, since combined with [BM] it will globalise any reasonable local theorem. The critical step then of local uniformisation looks likely to be accomplished by Cano and his collaborators by explicitly exploiting the one-dimensional features of the problem. Indeed just as the minimal model theorem can be viewed/is a large scale generalisation of semi-stable reduction of curves, they intend to view the problem as a large scale generalisation of plane curve desingularisation. The key point is a theorem of J. Cano, [Cj], see also [GS], that the Newton-Puiseux description around a valuation retains its validity for an arbitrary plane ODE. Thus although of almost immediate relevance to our situation on algebraic surfaces, it quite generally implies by a generic projection argument the existence of a contact structure, and the lack of any unpleasant surprises such as some Diophantine condition between coefficients of power series. Given that he's worked hard for it, and the solution is in sight, I would, therefore, anticipate that F. Cano's immediate goal is to complete the proof on 3-folds, so that my only real fear regarding the validity of LCR is that he stops short of arbitrary dimension, which in no essential way differs from 4.

I am, therefore, particularly indebted to Cano for explaining this to me, along with Bogomlov, Bonk, Gromov & Kontsevich for several other key contributions.

I. Singularities

I.1. Invariant Local Uniformisation

An example of F. Cano, c.f. [C], shows that canonical resolution of singularities cannot be achieved in dimension at least 3 by a sequence of blow ups in smooth centres. The difficulty, however, is resolved by working in the 2-category of stacks, and for convenience, we'll take this to mean with projective moduli. As such consider the following types of 'simple' morphisms between foliated logarithmic (i.e. with simple normal crossing boundary) stacks $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{D}, \mathcal{F})$, with, of course, $\tilde{\mathcal{D}} = \rho^{-1}\mathcal{D}_{red}$,

(P) *Blow up in a $\mathcal{F}_{\log}$ super-singular point, i.e. a negative discrepancy singular point* 0 *of $\mathcal{F}_{\log}$ such that the order of vanishing at* 0 *of a local generator is greater than that of any smooth invariant curve through it.*

(C) *Blow up in a smooth $\mathcal{F}_{\log}$-invariant curve, 'transverse' to $\mathcal{D}$, where, for want of a better word, 'transverse' means chosen so as so to preserve the simple normal crossing hypothesis, i.e. if the curve meets sing($\mathcal{D}$) then it's actually contained therein.*

where $\mathcal{F}_{\log}$ refers to the logarithmic vector field in $T_{\mathcal{X}}(-\log \mathcal{D})$ generating the foliation. Manifestly there is a smallest 2-category $\mathcal{C}$ generated by 'simple' morphisms given an initial object $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$. The key to deducing a global resolution statement from a local one involving sequences of 'simple' morphisms is well known to experts, i.e.

I.1.1 Claim *The above 2-category $\mathcal{C}$ has fibre products or, easier, the stack fibre product can be dominated by an element of $\mathcal{C}$.*

proof Plainly this reduces to looking at diagrams of the form,

$$\begin{tikzcd}
(\mathcal{X}_1, \mathcal{D}_1, \mathcal{F}_1) \arrow[d] & ? \arrow[l, "?"'] \arrow[d, "?"'] \\
(\mathcal{X}, \mathcal{D}, \mathcal{F}) & (\mathcal{X}_2, \mathcal{D}_2, \mathcal{F}_2) \arrow[l]
\end{tikzcd}$$

with 'simple' unquestioned arrows, and to show that we can take the questioned arrows in $\mathcal{C}$. Independently of the foliation this is absolutely trivial for unquestioned arrows of type (P) or (C). Indeed, everything reduces to the combination (P) & (C), so the questioned arrows are either blow up in the proper transform of the curve, or the fibre over the point. What is less trivial is to guarantee the invariance of the centres under $\mathcal{F}_{\log}$, or more correctly in the case of the fibres over the point. Nevertheless the definition of super-singular has been precisely constructed to guarantee this. □

As we've already remarked the operations (P) & (C) are insufficient to achieve log-canonical resolution in dimension 3, as the following example of F. Cano illustrates, viz:

I.1.2 Example

$$\partial = y\frac{\partial}{\partial z} + xz\frac{\partial}{\partial y} + x^{p+1}\frac{\partial}{\partial x},\ p \in \mathbb{N},\ +\ h.o.t.$$

with precision on the higher order terms to be found in [FF]. As such consider the following additional operation,

(R) *Extract a root of a smooth component of $\mathcal{D}$ if around the said component no 'improvement' is to be had by operations of type* (P) *or* (C).

Manifestly the word 'improvement' merits amplification. Plainly if a generator in $T_{\mathcal{X}}(-\log \mathcal{D})$ vanishes to order at least 2 at some point, then $\mathcal{F}_{\log}$ has a centre of positive discrepancy around it, and so, this can be improved. Consequently the only candidate for being non-improvable is something nilpotent in its linear part, and, more precisely, Cano's example. The said example can, however, be taken without loss of generality, to have $x = 0$ a component of $\mathcal{D}$, and is subsequently resolvable on extracting a root. As such consider,

I.1.3 ILUT (Invariant local uniformisation theorem) *Let v be a valuation of the function field of $\mathcal{X}_0$, then there is a sequence,*

$$(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n) \underset{\rho_n}{\longrightarrow} (\mathcal{X}_{n-1}, \mathcal{D}_{n-1}, \mathcal{F}_{n-1}) \longrightarrow \ldots\ldots \longrightarrow (\mathcal{X}_1, \mathcal{D}_1, \mathcal{F}_1) \underset{\rho_1}{\longrightarrow} (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$$

with ρ_i of type (P), (C), *or* (R) *around the centre of v on $\mathcal{X}_{i-1}$ such that the centre of v on $\mathcal{X}_n$ has log-canonical foliation singularities.*

The consequence of ILUT as opposed to LUT is, or course,

I.1.4 Fact *Suppose* ILUT, *then for any $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$ there is a sequence,*

$$(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n) \underset{\rho_n}{\longrightarrow} (\mathcal{X}_{n-1}, \mathcal{D}_{n-1}, \mathcal{F}_{n-1}) \longrightarrow \ldots\ldots \longrightarrow (\mathcal{X}_1, \mathcal{D}_1, \mathcal{F}_1) \underset{\rho_1}{\longrightarrow} (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$$

with ρ_i globally of type (P), (C), *or* (R) *such that $(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n)$ has log-canonical foliation singularities.*

proof Suppose otherwise and augment $\mathcal{C}$ to the smallest 2-category, $\tilde{\mathcal{C}}$ generated by $\mathcal{C}$ and global (R). Now form,

$$(\hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\mathcal{F}}) := \varprojlim_{\lambda \in \tilde{\mathcal{C}}} (\mathcal{X}_\lambda, \mathcal{D}_\lambda, \mathcal{F}_\lambda)$$

Of itself $(\hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\mathcal{F}})$ is not dominated by the Zariski-Riemann surface of the function field of $\mathcal{X}_0$, but, its points are. Better still under morphisms in $\tilde{\mathcal{C}}$, non log-canonical points map to non-log-canonical points, so there must be a valuation v of $\mathbb{C}(\mathcal{X}_0)$ whose centre is never log-canonical on any $(\mathcal{X}_\lambda, \mathcal{D}_\lambda, \mathcal{F}_\lambda)$. This doesn't quite contradict ILUT since its statement permits (R) locally, and, we need global. ILUT does, however, tell us that there is a $(\mathcal{X}_{\lambda_1}, \mathcal{D}_{\lambda_1}, \mathcal{F}_{\lambda_1})$ such that the centre of $v_1(= v)$ on $\mathcal{X}_{\lambda_1}$ is the origin in a Cano example, and, without loss of generality, the $x = 0$ divisor, defined by I.1.2, is a component $\mathcal{B}$ of $\mathcal{D}_{\lambda_1}$. As such, we have to get ourselves into a position where we can take global (R). To this end consider,

I.1.5 Sub-claim *Suppose in the notation of* I.1.4 *we replace $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$ by its germ around $\mathcal{B}$, and each $(\mathcal{X}_i, \mathcal{D}_i, \mathcal{F}_i)$ with its germ around the proper transform $\mathcal{B}_i$ of $\mathcal{B}$ then we may suppose that $(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n)$ is log-canonical.*

Sub-proof The basic problem that we face is the strong restriction imposed by global (R), whereas, we, manifestly, want to augment $\mathcal{C}$ by,

Arbitrary (R)*: Extract a square root of a component of $\mathcal{D}$*

In a global situation this operation is inadmissible, since, if we augment $\mathcal{C}$ to $\mathcal{C}^*$ by way of adding in Arbitrary(R), we may no longer be able to form fibre products à la I.1.1. The problem, in the notation of op. cit., is that if $\rho_1 : \mathcal{X}_1 \to \mathcal{X}$ is the extraction of a root and, $\rho_2 : \mathcal{X}_2 \to \mathcal{X}$ a blow up in Z then after blowing up in $\rho_1^{-1}(Z)_{red}$ we may obtain additional non-scheme like structure not supported on the root, and, this locus, in which we must blow up to dominate $\mathcal{X}_2$, may not be invariant. Locally around $\mathcal{B}$, however, the only global root that we have to worry about is $\mathcal{B}$ itself, and, as we've said, there is no problem around the proper transform of $\mathcal{B}$. By way of detail let $\tilde{\mathcal{X}_1}$ be the blow up of $\mathcal{X}_1 = \mathcal{X}(\sqrt{\mathcal{B}})$ in $\rho_1^{-1}(Z)_{red}$, with $\mathcal{E}_1$ the exceptional divisor, then around the proper transform $\tilde{\mathcal{B}}$ of $\mathcal{B}$, $\tilde{\mathcal{X}_1}$ already maps to $\mathcal{X}_2$. To see this map is in $\mathcal{C}^*$, one notes that it's the same as blowing up in the intersection of $\mathcal{B}$ with the exceptional divisor on $\mathcal{X}_2$, then extracting a root of the proper transform of $\mathcal{B}$. Consequently, in this semi-local setting, $\mathcal{C}^*$ has fibre products, and, whence, the sub-claim by LUT, and the compactness of the Zariski-Riemann surface. $\Box$

Now let's use the sub-claim to clean up $\mathcal{B}$, by supposing that for all $(\mathcal{X}_\nu, \mathcal{D}_\nu, \mathcal{F}_\nu) \in \tilde{\mathcal{C}}$ over $(\mathcal{X}_{\lambda_1}, \mathcal{D}_{\lambda_1}, \mathcal{F}_{\lambda_1})$ there are non-canonical points other than in the component of the singular locus through v_1. Again appealing to ILUT, we find, $(\mathcal{X}_{\lambda_2}, \mathcal{D}_{\lambda_2}, \mathcal{F}_{\lambda_2})$, and a valuation v_2 with centre the origin of another Cano example on the proper transform $\mathcal{B}_2$ of $\mathcal{B}$. This cannot, however, by the sub-claim, continue ad infinitum, so we may eventually suppose that everything on $\mathcal{B}$ is log-canonical apart from finitely many Cano examples. At which point we have many choices, but a convenient one is to separate the v_i for $i > 1$ from $\mathcal{B}$ by blowing up in points, so that everything is canonical apart from the component of the singular locus through v_1, which, itself, is necessarily isolated since the log-canonical points are open in sing($\mathcal{F}$), and indeed, without loss of generality, both smooth with the field enjoying everywhere non-zero linear part by a minor variant of the above. Whence, we've eventually arrived to the situation where we may apply global (R), and so contradict ILUT for v_1. $\square$.

Having profited from the greater flexibility afforded by the 2-category of algebraic stacks to obtain a resolution even of an initially wholly scheme like object $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ there are many issues remaining, viz: is this really the kind of resolution we want ? how far away is it from being projective/a scheme etc. Fortunately, these have all been addressed in [M1] I.3, I.6 & I.7, with a convenient conclusion being,

I.1.6 Corollary *Let (X, D, F) be a foliated 3-dimensional log-triple, then there is a projective bi-rational modification $\rho : (\tilde{X}, \tilde{D}, \tilde{F}) \to (X, D, F)$ such that the latter has log-canonical foliation singularities, and $\tilde{X}$ has at worst $\mathbb{Z}/2$ quotient singularities. In particular if X is projective (respectively a scheme), $\tilde{X}$ is projective (resp. a scheme.)*

proof The various projective/scheme conclusions arise from the representability of a stack with projective moduli as a groupoid with finite étale source and sink, [M1] I.3.2, a condition which is plainly not altered by any number of applications of global (R). Everything else is in op. cit. I.6 & I.7 albeit that its worth re-visiting I.1.2 by way of another manifestation of the Cano example whereby we see that $\mathbb{Z}/2$ quotient singularities is best possible, i.e.

$$\partial = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z^{2p}\frac{\partial}{\partial z},\ x \leftrightarrow y,\ z \mapsto -z,\ p \in \mathbb{N} \qquad \square$$

I.2. Weak 3-D Branching

Even in the presence of canonical/log-canonical singularities sing($\mathcal{F}$) can still be complicated. The basic reason for this is that log-canonical singularities only assure the non-nilpotence of the 1st order linearisation of a local generator around a point, so for example, there may well be less than the co-dimension of sing($\mathcal{F}$) eigenvalues. In dimenson 3 we have two possibilities, i.e. 1 or 2 -always to be understood with multiplicity- eigenvalues around an isolated singularity, or 1 eigenvalue in the non-isolated case. To understand the former requires only Jordan decomposition, while in the latter we require to perform Jordan decomposition uniformly around sing($\mathcal{F}$). Plainly this is a local and purely scheme like question, so let Z be an irreducible component of sing($\mathcal{F}$), and $\hat{X}$ =Spf$\mathcal{O}$ a formal scheme complete in the I_Z adic topology, with ∂ a generator of the foliation on $\hat{X}$. Our objective is,

I.2.1 Claim *Suppose the formal germ $(\mathcal{O}, \partial)$ is log-canonical with 1 eigenvalue at the generic point of Z, then there is an invariant formal subscheme W of $\hat{X}$ which at every point $z \in Z(\mathbb{C})$ has for completion around z the formal subscheme defined by the equation $x = 0$, where, at z the semi-simple part of ∂ is $x\frac{\partial}{\partial x}$.*

Notice that at z, although the function x is non-unique, the sub-scheme $x = 0$ is unique, and dependent only on $\mathcal{F}$. To prove the claim observe,

I.2.2 Reduction *We may, without loss of generality, suppose that Z is smooth.*

proof Certainly we can find a resolution of singularities $\rho : \tilde{X} \to \hat{X}$, or better a formal blow up, c.f. [M1] IV.2, such that ∂ lifts to $\tilde{X}$, and $\tilde{X}$ is complete around the proper transform $\tilde{Z}$ of Z. To deduce the claim from here one appeals to the co-herence of push-forward of co-herent sheaves under proper maps of formal schemes, or, slightly more correctly, the proof of the same, cf. [EGA],III, III.3.4.2. $\square$

Of course we've profited here, and previously, from the openness of the 2-eigenvalue condition, and all that remains is to make a few observations beginning with,

I.2.3 Fact *Let D be an endomorphism of a rank 2 vector bundle E over a curve Z which is rank 1 at every point $z \in Z$, then every point of Z has an open neighbourhood U over which we may write D as,*

$$\begin{bmatrix} \lambda(z) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\lambda(z)$ is the eigenvector at z.

proof Both Ker$D$ and Ker$D - \lambda(z)$ are rank 1 sub-bundles of E which are everywhere disjoint. □

Everything being local, we may therefore shrink Z so as to suppose mod I_Z^2, ∂ has the form,

$$\partial = x\frac{\partial}{\partial x} + f(z)y\frac{\partial}{\partial z}$$

with $x = y = 0$ the defining equations of Z. A simple induction mod I_Z^n therefore produces an element ξ of $\mathcal{O}$ vanishing on Z such that $\partial\xi = \xi(1+u)$, for $u \in I_Z$, and since the local weak branches of completions at points depend only on the foliation, we may, without loss of generality suppose that $\partial x = x$ in $\mathcal{O}$. This already proves the claim, but we may also note in passing,

I.2.4 Fact *If Z is smooth then every point $z \in Z(\mathbb{C})$ has a I_Z complete neighbourhood U over which we can find coordinates $x, y, z, \in \mathcal{O}$ such that the foliation is given by a field of the form,*

$$\partial = x\frac{\partial}{\partial x} + a(y,z)\frac{\partial}{\partial y} + b(y,z)\frac{\partial}{\partial z}$$

proof Proceed by induction modulo I_Z^n. □

Notice, however,

I.2.5 Z Warning *Both I.2.1 & 1.2.4 are false for completions in* sing($\mathcal{F}$), *i.e.* if sing($\mathcal{F}$) *is not wholly contained in the weak scheme $x = 0$ of the Jordan decomposition of a point $z \in$* sing($\mathcal{F}$), *then we can neither find a uniform Jordan decomposition in the completion, nor even, the weak sub-scheme $x = 0$. The basic example of this is when the local Jordan decomposition at a point has the form $x\frac{\partial}{\partial x} + zy\frac{\partial}{\partial y}$.*

There are, of course, no such problems at isolated points. Furthermore, an isolated point with weak branching becomes a singular curve with weak branching after blowing up except,

I.2.6 Fact *Let* 0 *be an isolated singularity with weak branching, then, if after arbitrary blowing up in* sing($\mathcal{F}$) *and the proper transform thereof this phenomenon persists, we may, after blowing up, suppose for $\lambda \in \mathbb{C}^\times$ that the semi-simple part ∂_S of the Jordan decomposition of a local generator has the form,*

$$\partial_S = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$$

I.3 Log-Flatness

Let $(\mathcal{X}, \mathcal{F})$ be a foliated smooth stack with canonical singularities, then we may apply the algorithmic resolution procedure of [BM] to obtain a modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ by a sequence of blow ups in invariant centres to obtain an invariant simple normal crossing divisor $\mathcal{E}$ which contains every point of the induced singular locus. As such we have an ideal $\mathcal{I}_{nlf}$ of non-log flat points according to the surjection,

$$\Omega_{\tilde{\mathcal{X}}}(\log \mathcal{E}) \longrightarrow \rho^* K_{\mathcal{F}} \mathcal{I}_{nlf} \longrightarrow 0$$

For many problems, such further modification is neither here nor there, so we will adopt,

I.3.1 Convention *We will often assume with little or no warning, that such a divisor $\mathcal{E}$ is present*

In particular, and in so much as as we do permit such modification, we can bring the singularities of $\mathcal{F}$, especially in dimension 3, into better shape. The discussion is local, so without loss of generality scheme like, and we begin with,

I.3.2 Fact *If at $z \in \operatorname{sing}(\mathcal{F})$ the semi-simple part of the Jordan decomposition has 2 non-zero eigenvalues and $\operatorname{sing}(\mathcal{F})$ has dimension 1 then $\mathcal{F}$ is log-flat at Z and, indeed, $\operatorname{sing}(\mathcal{F})$ is smooth at z. Better still there is at most one other formal invariant hypersurface through z containing the generic point of $\operatorname{sing}(\mathcal{F})$.*

proof Notice that any generic point of $\operatorname{sing}(\mathcal{F})$ is contained in a component of $\mathcal{E}$, so there is at least 1 smooth invariant invariant hypersurface with local equation y, say, at z containing the said generic point.

Now let ξ, η, ζ be Jordan coordinates at z, with ∂ a local generator affording semi-simple part $\partial_S = \xi\frac{\partial}{\partial\xi} + \lambda\eta\frac{\partial}{\partial\eta}$, $\lambda \neq 0$. Plainly $\partial\xi$, $\partial\eta$ are in $I_{\operatorname{sing}(\mathcal{F})}$, while the former generate the ideal (ξ, η), so by our hypothesis on the dimension $\partial\zeta \in (\xi, \eta)$, and $\operatorname{sing}(\mathcal{F})$ is smooth at z.

Suppose, therefore, that $f : V \to \hat{X}$ is the normalisation of an invariant formal hypersurface containing $\operatorname{sing}(\mathcal{F})$, with $\hat{X}$ the completion at z, so that in particular $f^{-1}\operatorname{sing}(\mathcal{F})$ is generically a bunch of Cartier divisors around which V is smooth, and we pick one of these, W, say. A priori neither V nor $\hat{X}$ are schemes, but everything is local around z, and,

$$\operatorname{Hom}(\mathcal{O}_{\hat{X}}, \mathcal{O}_V) \supset \operatorname{Hom}(V, \hat{X})$$

so we still have a map on replacing $\operatorname{Spf}\mathcal{O}_{\hat{X}}$ by $\operatorname{Spec}\mathcal{O}_{\hat{X}}$ etc., and, better still, $\xi = 0$ or $\eta = 0$ are invariant equations of schemes if they are so in $\hat{X}$. Profiting from this let $w \in \operatorname{sing}(\mathcal{F})$ be a nearby point where V, $f^{-1}\operatorname{sing}(\mathcal{F})$ and the induced foliation are smooth around the pre-image of w, and let ${\partial_S}^w + {\partial_N}^w$ be a Jordan decomposition of ∂ in the completion at w. In particular if ${\partial_S}^w = \xi^w\frac{\partial}{\partial\xi^w} + \lambda(w)\eta^w\frac{\partial}{\partial\eta^w}$ then we observe,

I.3.3 Possibilities *In the completion at w either,*

(a) $\lambda(w) \notin \mathbb{N} \cup \mathbb{N}^{-1}$ *and V completed at w is one of the two necessarily invariant formal schemes $\xi^w = 0$ or $\eta^w = 0$*

(b) $\lambda(w) \in \mathbb{N} \cup \mathbb{N}^{-1}$*, so, say $\lambda(w) = n$, and $\xi^w = 0$ invariant with either*

(i) *There is no nilpotent part, and $\eta^w = 0$ is invariant.*

(ii) *A component of $\partial_N\eta^w = 0$ defines the only other invariant hypersurface through w containing* $\operatorname{sing}(\mathcal{F})$*, so something of the form, $t^p(\xi^w)^n + \eta^w$, or $(\xi^w)^n + t^p\eta^w$, for an appropriate local function $t = 0$.*

(iii) *There are no other invariant hypersurfaces, i.e. $\partial_N = \xi^w\frac{\partial}{\partial\eta^w}$.*

Similarly one has either,

(a)' $\lambda \notin \mathbb{N} \cup \mathbb{N}^{-1}$ *and both $\xi = 0$, $\eta = 0$ are invariant.*

(b)' $\lambda \in \mathbb{N} \cup \mathbb{N}^{-1}$ *and we have (i), (ii), or (iii) as above.*

As such if we have (a)' we cannot have (b)(iii), and, indeed, for very general w, without loss of generality $\xi = 0$, $\xi^w = 0$ (respectively $\eta = 0$, $\eta^w = 0$) define the same hypersurface, so V is either $\xi =)$ or $\eta = 0$. The same is true if we have (b)'(i), while if we have (b)'(ii) we know the form of an invariant hypersurface through z, and, likewise for (b)'(iii).

Consequently we're pretty much done, beyond noting that our initial equation $y = 0$ is smooth, so that it cannot be the singular hypersurface that may occur in (b)'(ii), as such $\frac{\partial y}{y}(z) \neq 0$, and even here things are log-flat. It's also worth noting down,

I.3.4 Sub-fact *In the notations of I.3.2 and its proof, any invariant hypersurface in the completion of X in* $\mathrm{sing}(\mathcal{F})$ *through z containing* $\mathrm{sing}(\mathcal{F})$*, is, on completion at a very general point w of the unique component of* $\mathrm{sing}(\mathcal{F})$ *through z, of one of the hypersurfaces $\xi^w = 0$, or $\eta^w = 0$ defined by ∂_S^w.*

Manifestly the situation presented by I.3.2 is not only reasonable, but, best possible. As such before proceeding let us consider to what extent we can improve the other singularities. To this end, we have,

I.3.5 Lemma *Let $x = 0$ be the weak branch W through a not necessarily isolated singularity 0 with $\partial = \partial_S + \partial_N$ the Jordan decomposition of a local generator, then necessarily $\partial_S = x\frac{\partial}{\partial x}$ and ∂_N is a non-log-canonical field in y and z. Suppose further that ∂_N is at least quadratic in y and z then for $\rho : \tilde{X} \to X$ the blow up in 0, a singularity p on the induced foliation satisfies either,*

(a) $\rho^*\partial$ has an isolated singularity with semi-simple part of full rank, and p is not in the proper transform of W.

(b) p is non-isolated and the induced weak branch is the proper transform $\tilde{W}$ of W through p.

proof Outside of the exceptional divisor the lemma is clear, and one just calculates the exceptional divisor by hand. $\square$

The usefulness of the lemma derives from,

I.3.6 Further Lemma *Again let $x = 0$ be the weak branch W, but through a non-isolated singularity 0 such that the induced foliation together with* $\mathrm{sing}(\mathcal{F}) \cap \mathrm{W}$ *does not have log-canonical singularities then the nilpotent part of a local generator is indeed at least quadratic.*

proof We can write a generator as $\partial = x\frac{\partial}{\partial x} + f(y,z)\delta$ for δ a saturated plane field in y and z with $f(0) = 0$. As such either,

(i) δ is singular, and we're done.

(ii) δ is smooth, but f is not even simple normal crossing, whence at least quadratic. $\square$

The only possibility not covered by this discussion is therefore,

I.3.7 Exceptional Lemma *As ever $x = 0$ the weak branch W, but here through an isolated singularity with ∂_N a saturated nilpotent plane field in y and z, then for $\rho : \tilde{X} \to X$ the blow up in the origin of the germ around 0, a singularity of the induced foliation is either,*

(a) Isolated with semi-simple part of full rank, and p not in the proper transform $\tilde{W}$ of W.

(b) p is in $\tilde{W}$ which is still the weak branch.

proof Again proceed by direct calculation on the exceptional divisor. $\square$

To describe the results of these lemmas, let us introduce,

I.3.8 Definition *The weak branching locus of $(\mathcal{X}, \mathcal{F})$ is the set of points in* $\mathrm{sing}(\mathcal{F})$ *where the rank of the semi-simple part of a local generator is less than the co-dimension of* $\mathrm{sing}(\mathcal{F})$*. We take only reduced scheme structure over the said locus and denote it $WB(\mathcal{F})$.*

Applying the lemmas we obtain,

I.3.9 Fact *Let $(\mathcal{X}, \mathcal{F})$ have canonical singularities and dimension 3, then there is a sequence of modifications in singular points of the foliation $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ such that,*

(a) Any isolated points in $WB(\tilde{\mathcal{F}})$ have semi-simple part of rank 2.

(b) Not just $WB(\tilde{\mathcal{F}})$ but the non-isolated points of sing$\tilde{\mathcal{F}}$ form in their reduced structure a curve with singularities at worst nodes. Better still the induced foliation in the weak branch of $\tilde{\mathcal{F}}$ around the non-isolated points, together with the divisor sing$\tilde{\mathcal{F}} \cap W$ has log-canonical singularities.

proof At an isolated singularity of rank 2 or 3 we do nothing. Likewise at a non-isolated singularity having rank 2. If, however, the singularity is isolated of rank 1, we blow up in points of W and its proper transform until we have a log-canonical situation on the same. By I.3.6 & I.3.7 we never introduce any other singularities, except isolated ones of full rank. Similarly if we begin at a non-isolated singularity by I.3.5 &

I.3.7 the weak branch on a punctual blow up in the same is its own proper transform, so we proceed to a log-canonical situation, with any other singularity being isolated of full rank. This proves everything when combined with I.3.5-7 except possibly at points where the singular locus goes out of W, but there, W is smooth and the Jordan form looks like, $\partial = x\frac{\partial}{\partial x} + y^p z^q \frac{\partial}{\partial z}$, $p, q \in \mathbb{N}$, so sing$(\mathcal{F})$ is at worst nodal. $\Box$

To describe the isolated points we have,

I.3.10 Further Fact *On blowing up a foliation $(\mathcal{X}, \mathcal{F})$ with canonical singularities, $\mathcal{X}$ smooth of dimension 3, in points we may suppose any isolated singularity has either,*

(a) Semi-simple part of full rank, and is log-flat.

(b) Semi-simple part $\partial = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$, $\lambda \in \mathbb{C}^\times$, with $y = 0$ in $\mathcal{E}$ if it is log-flat, or $z = 0$ the local equation for $\mathcal{E}$ otherwise.

proof The co-tangent bundle of any component of $\mathcal{E}$ is an invariant subspace of the 1st order linearisation, so (a) is automatic, as is (b). $\Box$

The non-isolated points of full rank have already been described in 1.3.2-4, and so we have,

I.3.11 Final Fact *Let $(\mathcal{X}, \mathcal{F})$ be a smooth foliated stack with canonical singularities of dimension 3, then after a sequence of blow ups in points we obtain a modification $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ such that any singularity not described by I.3.9 or I.3.10 is a point of a non-isolated component of $WB(\tilde{\mathcal{F}})$. Furthermore any connected component of $WB(\tilde{\mathcal{F}})$ is either everywhere log-flat or not at all. In either case if $\mathcal{E}'$ is the part of $\mathcal{E}$ other than components which are weak branches then as reduced stacks, $\mathcal{E}' \cap W(\tilde{\mathcal{F}}) = WB(\tilde{\mathcal{F}}) = sing\tilde{\mathcal{F}} \cap W(\tilde{\mathcal{F}})$ and is a simple normal crossing divisor in $\tilde{W}$. Finally for $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{0\}$ the Jordan form at such points is one of,*

(a) $\partial = x\frac{\partial}{\partial x} + \nu(y, z) y^p z^q \frac{\partial}{\partial y}$, $\nu(0) \neq 0$.

(b) $\partial = x\frac{\partial}{\partial x} + \nu(y, z) y^p z^q \frac{\partial}{\partial z}$, $\nu(0) \neq 0$.

(c) *y, z are Jordan coordinates for a canonical plane field δ with*

$$\partial = x\frac{\partial}{\partial x} + \nu(y, z) y^p z^q \delta \, , \, \nu(0) \neq 0.$$

In all cases $y = 0$ is a component of $\mathcal{E}'$, and $z = 0$ too when $q > 0$, except for (a) when $p = 1$ and the singularity is log-flat.

To finish this section let us make,

I.3.12 Remark/Definition *Notice that in respect of the various descriptions 1.3.9-11 we do not claim, nor care, whether the said forms enjoy any sort of stability under further blowing up. The descriptions are, however, convenient, and so we introduce: Any of the singularities described in I.3.9-11 will be labeled convenient. Furthermore we will extend the definition to further suppose that a log-flat singularity of type I.3.10(b) also has the plane $z = 0$ in $\mathcal{E}$. This, as ever, can be achieved by blowing up in points, albeit that this may create singularities of type I.3.2 if $\lambda \in \mathbb{N}$.*

I.4. Invariant Curves in 3-D

We wish to tabulate the possibilities for invariant curves through the singularities of a foliated 3-fold $(\mathcal{X}, \mathcal{F})$ with convenient singularities. The discussion is local and even formal in the completion of points of sing$(\mathcal{F})(\mathbb{C})$, so throughout let X be a complete scheme like étale neighbourhood of 0, with ∂ a local generator of the foliation affording a Jordan decomposition $\partial_S + \partial_N$. A more general discussion of the possibilities for a germ C of an invariant curve is already in [M1] V.4, so we content ourselves to a quick resumé with a little extra detail, taking only an interest in invariant curves not factoring through $\mathcal{E}$. To begin with,

I.4.1 Case 1 *0 is isolated of full rank.*

Here things are log-flat, so take $y = 0$ in $\mathcal{E}$, and normalise by way of $\partial_S(y) = y = \partial(y) = y$ with λ, μ the eigenvalues of Jordan coordinates x, z. Notice,

I.4.2 Fact *If λ, respectively μ, is not in $\mathbb{Q}_+$ then x, respectively z, vanishes on C. In particular if $\lambda, \mu \notin \mathbb{Q}$ or $\lambda, \mu \in \mathbb{Q}_{<0}$ then C is the necessarily smooth curve $x = z = 0$.*

Plainly this leads to,

I.4.3 Sub-case 1(a) *$\lambda \in \mathbb{Q}_+$, $\mu \in \mathbb{Q}_-$. Renormalise by way of $\partial_S(y) = \partial(y) = py$, $\partial_S(x) = qx$, $\partial_S(z) = -rz$; $(p,q,r) = 1$, $p,q,r \in \mathbb{N}$. As such the plane z is invariant, and the induced foliation is either semi-simple or it is not. If not, whence $p = 1$ there are no curves, otherwise they're of the form,*

$$z = 0,\ y^{\tilde{q}} = cx^{\tilde{p}},\ c \in \mathbb{P}^1(\mathbb{C})\backslash\{0\},\ \tilde{p} = \frac{p}{(p,q)},\ \tilde{q} = \frac{q}{(p.q)}$$

Similarly we have,

I.4.4 Sub-case 1(b) *$\lambda, \mu \in \mathbb{Q}_+$, so re-normalise as above, but with $\partial_S(z) = rz$. The only canonical singularities admitting a curve not in $\mathcal{E}$ are,*

(I) *$q = r$, and $\partial_N = x\frac{\partial}{\partial z}$.*

(II) *$r = ip + jq$, $i,j \in \mathbb{N}$, $(p,q) = 1$, i minimal amongst such representations of r, and, $\partial_N = f(x,y)\frac{\partial}{\partial z}$, where,*

$$f(x,y) = \sum_{0 \leq d \leq [j/p]} a_d y^{i+dq} x^{j-dp} \quad \text{not a power of } y$$

In both cases C is contained in $\partial_N(z) = 0$, so 1(b)(I) is like 1(a) modulo interchanging x & z. On the other hand 1(b)(II) is a bit fastidious, so let's punctually blow up to improve it. Under such a modification, for $p \neq q$, there are 3 isolated singularities, two of type 1(a) and one of type 1(b). When eventually $p = q = 1$, we get one isolated singularity, and a non-isolated one without weak branching, so we can subordinate this to other cases to be discussed presently. Before this observe,

I.4.5 Case 2 *∂ is isolated with ∂_S of rank 2. If it is log-flat, there are no curves outwith $\mathcal{E}$, otherwise the only such curve is the weak branch itself.*

As to the next case, which contains 1(b)(II) after blowing up,

I.4.6 Case 3 *∂ is non-isolated of rank 2.*

Again we normalise in this necessarily log-flat situation by $\partial_S(y) = \partial(y) = y$, $y = 0$ in $\mathcal{E}$, take x to be the non-zero eigenfunction of ∂_S with eigenvalue λ, and z the nil-function. Consequently z is zero on every curve, and,

I.4.7 Fact *If $\lambda \notin \mathbb{Q}_+$, then C is the necessarily smooth and invariant curve $x = z = 0$. For $\lambda \in \mathbb{Q}_+$, the plane $z = 0$ is invariant, and for the normalisation $\partial(y) = \partial_S(y) = py$, $\partial_S(x) = qx$; $p,q \in \mathbb{N}$, $(p,q,) = 1$. The description is as per I.4.3.*

This leaves us with weak branching to consider around curves, i.e.

I.4.8 Case 4 *∂ has a non-isolated singularity in the non-log-flat weak branching locus.*

Consequently the curves lie in the weak branch W, and are described by,

I.4.9 Fact *Things as above, with C not contained in* $\mathrm{sing}(\mathcal{F})$*, then* $\mathrm{sing}(\mathcal{F})$ *is smooth and everywhere tangent to the induced foliation in W, i.e. the Jordan form is $x\frac{\partial}{\partial x}+\nu(y,z)y^{p+1}\frac{\partial}{\partial y}$, $p\in\mathbb{N}$, $\nu(0)\neq 0$, and C is the curve $x=z=0$. Plainly there is a unique such curve for every such point of* $\mathrm{sing}(\mathcal{F})$.

One should not, however, be fooled by this/develop a false sense of security, i.e.

I.4.10 Remark/Definition/Warning *The curves in the weak branching locus which are invariant by the foliation induced on W behave in many respects like the closure of invariant curves in the smooth part. Unlike other curves in* $\mathrm{sing}(\mathcal{F})$*, the map from the co-normal bundle of $\mathcal{F}$ to the curve vanishes, and they merit a name such as* **singular solutions** *or* **singular leaves***. At a generic point of such the Jordan decomposition is $x\frac{\partial}{\partial x}+\nu(y,z)y^{p}\frac{\partial}{\partial z}$, $\nu(0)\neq 0$.*

I.4.11 Case 5 *∂ has a non-isolated singularity at the weak branching locus, which is supposed log-flat.*

As ever we take $y=0$ a component of $\mathcal{E}$, and normalise by $\partial(y)=\partial_S(y)=y$, with y, x, z Jordan coordinates. Consequently the curve is given by $x=z=0$. A useful description of this curve may be given in terms of invariant hypersurfaces. Indeed suppose $f_n=0$ is an irreducible local equation of such with the property that,

$$f_n=g_n(x,z)+y^n h_n,\ n\in\mathbb{N}$$

then by virtue of the invariance under ∂_S, we obtain,

$$y^n\{nh_n+y\frac{\partial h_n}{\partial y}\}\in(f_n)$$

Things are factorial, so $h_n=c_nf_n-y\frac{\partial h_n}{\partial y}$ for some function c_n, and whence,

$$f_{n+1}:=f_n(1-c_ny^n)=g_n+y^{n+1}h_{n+1}(=-\frac{\partial h_n}{\partial y})$$

the infinite product $\prod(1-c_ny^n)$ comfortably converges at the formal level, so if our hypersurface isn't $y=0$ it can be described by an equation $g(x,z)=0$. This must be invariant under ∂_N, so we conclude,

I.4.12 Fact *Let the singularity be as per I.4.11 (or even I.3.11) then there is at least 1, and at most 2, invariant hypersurfaces through the singularity other than the weak branch itself. Together these hypersurfaces form a simple normal crossing divisor, so that, in particular, an invariant hypersurface is determined uniquely by the image of its co-normal sheaf in the residual co-tangent bundle. When there are two such hypersurfaces, their intersection is the curve not in $\mathcal{E}$.*

I.5. Convergence & the 3-D Beast

By definition a beast occurs when the foliation passes from weak branching to a semi-simple part of full rank, and in doing so fails to be log-flat. Consequently,

I.5.1 Definition *In the presence of convenient singularities the Jordan form of a 3-D beast in appropriate coordinates is $\partial=x\frac{\partial}{\partial x}+z^py\frac{\partial}{\partial y}$, where $y=0$ is a local equation for $\mathcal{E}$.*

At first glance this looks pretty trivial, but in reality it's anything but, since the Jordan form may fail to exist in anything other than the completion of a point. To see this, let us discuss in somewhat greater generality, the behaviour of the Jordan form around non-isolated, non-weak branching singularities. As ever the discussion is local, whence scheme like, and log-flat, so there is a coordinate y with $y=0$ describing a component of $\mathcal{E}$, and x another coordinate such that $x=y=0$ defines $\mathrm{sing}(\mathcal{F})$, or more accurately the non-weak branching component Z, with ∂ a local generator of $\mathcal{F}$. The preferable way to normalise is to take ∂ to be the identity on the co-normal bundle of Z in $y=0$, so $\partial x=x+a(z)y$, $\partial y=\lambda(z)y$, supposing of course, $\lambda\neq\infty$. Consequently we can diagonalise the linearisation of ∂ in $\mathrm{End}(N_{Z/X})$ iff $1-\lambda|a$. In neighbourhoods of the beast this is not a problem since λ is close to zero, otherwise, and quite generally if

v_p is the valuation at a point $p \in Z$, we can take a such that $v_p(a) < v_p(1-\lambda)$. Similarly we want a good coordinate z on Z, so consider ∂ as an element of $\mathrm{End}(\Omega_X \otimes \mathcal{O}_Z)$, then a priori, $\partial z = b(z)x + c(z)y$, $\mathrm{mod} I_Z^2$. To find z with $\partial z = 0$ mod I_Z^2 is only obstructed if $\lambda = 0$, but since we're supposing that the weak branch is in convenient form this obstruction doesn't present itself, so that apart from around $\lambda = 1$, all points admit an étale neighbourhood such that ∂ is semi-simple in $\mathrm{End}(\Omega_X \otimes \mathcal{O}_Z)$, while at points p with $\lambda(p) = 1$ we have the situation,

$$\begin{bmatrix} 1 & 0 & 0 \\ a & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with a locally polynomial of degree $< v_p(1-\lambda)$.

Now let's consider the situation under completion in Z, with $\hat{X}$ the corresponding étale formal neighbourhood, and $\hat{X}_{an}$ the same in the analytic topology. On the latter we can consider a neighbourhood defined by $\lambda \notin \mathbb{Q}$, with for the moment λ non-constant, and we have,

I.5.2 Fact *On $\hat{X}_{an} \backslash \lambda^{-1}(\mathbb{Q})$ we may find coordinates x, y, z such that,*

$$\partial x = x,\ \partial z = 0,\ \textit{and}\ \partial y = \lambda(z) y$$

proof We've already done this mod I_Z^2, and one proceeds by induction mod I_Z^n. $\square$

Apart from this, the situation is rather more complicated. There are of course, subcases determined by $\mathbb{Q}_-$ and $\mathbb{Q}_+$, so let V_- and V_+ be open sectors around either of these, then as per I.5.2 we obtain,

I.5.3 Fact *Over V_- there is a coordinate X defining a hypersurface containing* $\mathrm{sing}(\mathcal{F})$*, everywhere transverse to $y = 0$ such that $\frac{\partial x}{x} = 1$ (mod I_Z).*

Any improvement beyond this is highly local, i.e.

I.5.4 Fact *Let $p \in \hat{X}_{an}$ with $\lambda(p) = -m/n$, $m, n \in \mathbb{N}$, $(m.n) = 1$ then for $v = v_p(\lambda + m/n)$ we have Jordan coordinates x, y, z such that,*

$$\partial x = nx\{1 + N(z, x^m y^n)\},\ \partial y = -my\{1 + M(z, x^m y^n)\},\ \partial z = L(z, x^m y^n)$$

where L, M, N have degree $< v$ in z, and $x^m y^n$ divides L.

The situation around V_+ is both better and worse, i.e.

I.5.5 Fact *Around V_+ we can find coordinates x, y, z such that $\partial z = 0$, $\partial y = \lambda(z) y$, and,*

$$\partial x = x + \sum_{n=1}^{\infty} a_n(z) y^n$$

Furthermore if $p \in \mathrm{sing}(\mathcal{F})$, then we can take $\partial x = x$ in a neighbourhood of p unless $\lambda \in \mathbb{N}^{-1}$. In the latter case with $\lambda(p) = 1/n$, this further simplification is possible iff $v_p(a_n) \geq v_p(\lambda - 1/n)$.

Before interpreting these results let's observe that λ is defined globally on the component Z. Indeed $\frac{\partial y}{y}$ defines a global section of $H^0(Z, K_{\mathcal{F}})$ as does the trace, which gives a map $\lambda : Z \to \mathbb{P}^1$, and in neighbourhods of infinity we have,

I.5.6 Fact *Renormalise at infinity, so $\delta(y) = y$ for a possibly different field δ, then there is a smooth invariant hypersurface in the completion of $\mathcal{X}$ in $\mathcal{Z}$ given locally by $x = 0$, with $x = y = 0$ defining $\mathcal{Z}$.*

As such the existence of $x = 0$ in the completion $\hat{\mathcal{X}}$ of $\mathcal{X}$ around $\mathcal{Z}$ is obstructed only at 0 and $\mathbb{N}^{-1}$. Returning therefore to our local set up we observe that one of two things happens, i.e.

I.5.7 Fact *Suppose indeed $\lambda(p) = 1/n$, $n \in \mathbb{N}$, and in the notation of I.5.5, $v_p(a_n) < v_p(\lambda - 1/n)$ with m the difference, then on a neighbourhood $\Delta \ni p$ of $\hat{X}$ there is an invariant hypersurface through p of the form $y^n + t^m x$ for a suitable parameter t at p.*

proof In the situation as described we can certainly find Δ so that $\partial x = x + a_n y^n$ for a possibly different x to that in I.5.5, with $v_p(a_n) < v_p(\lambda - 1/n)$. The desired surface is then the component of,

$$\frac{x + a_n y^n}{\lambda(z)} - nx$$

which contains $\mathrm{sing}(\mathcal{F})$.

Consequently we arrive to,

I.5.8 Fact/Definition *Let $\hat{\mathcal{X}}^*_{an}$ be an appropriate analytic neighbourhood of $\hat{\mathcal{X}}$ minus its beasts, then there is a formal substack $\mathcal{W}$ of $\hat{\mathcal{X}}^*_{an}$ defined uniquely by the condition that its completion at some point of a real blow up in some beast is the completion of the weak branch of the beast in the same. Furthermore at a given beast, taken in the notations of I.5.5, to be $x = y = z = 0$, the following are equivalent,*

(a) *$\mathcal{W}$ extends to a neighbourhood of 0 in $\hat{\mathcal{X}}_{an}$, and we say that the beast is* **tame**.

(b) *$a_n(p) = 0$, for almost all p, with $\lambda(p) = 1/n$.*

(c) *For almost all p with $\lambda(p) = 1/n$, there are two invariant smooth hypersurfaces through p containing $\mathrm{sing}(\mathcal{F})$.*

(d) *For almost all p with $\lambda(p) = 1/n$, there is an invariant curve through p which does not lie in $\mathcal{E}$.*

proof That $\mathcal{W}$ exists in $\hat{\mathcal{X}}^*_{an}$ is a consequence of the uniqueness discussion in I.3.3. Condition (b) implies (a) by direct calculation. Conversely if we have (a) then I.3.2 gives (b), which is certainly equivalent to (c) by the same. As for (d), for almost all p with $\lambda(p) = 1/m$, $v_p(\lambda - 1/n) = 1$ so we have (c) iff (d) by I.3.3. $\square$

Regrettably, therefore, most beasts will not be tame and we deduce,

I.5.9 Final Fact *If the beast is not tame, it is not possible to express it's Jordan form convergently. Not only does this apply (unsurprisingly) to convergence in the strict sense, it also applies after real blowing up. Indeed there are not even finitely many sectors determined by $\arg(y)$, $\arg(z)$ (which modulo further blowing up, can always be supposed to define any asymptotic expansion about the origin) on which the weak branch $x = 0$ can be supposed convergent.*

I.6. Higher Convenience

We will proceed as per the convention of I.3.1, so that $(\mathcal{X}, \mathcal{F})$ is a foliated stack with log-canonical singularities, and $\mathcal{E} \supset \mathrm{sing}(\mathcal{F})$ an invariant divisor that we will augment at will by appropriate modification. The dimension is now arbitrary, and to understand this situation we will need a higher dimensional resolution theorem/hypothesis, i.e.

I.6.1 Hypothesis *In the 2-category of formal stacks over $\mathbb{C}$ consider the following simple modifications $\rho : (\mathcal{X}, \mathcal{E}, \mathcal{F}) \to (\mathcal{Y}, \mathcal{D}, \mathcal{G})$ between foliated log-stacks,*
(a) ρ is a blow up in an invariant centre.
(b) ρ is the extraction of a root of a component of $\mathcal{D}$.
Then for any formal stack $(\mathcal{Y}, \mathcal{D}, \mathcal{G})$ there is a sequence of simple modifications,

$$(\mathcal{X}, \mathcal{E}, \mathcal{F}) = (\mathcal{Y}_n, \mathcal{D}_n, \mathcal{G}_n) \underset{\rho_n}{\longrightarrow} (\mathcal{Y}_{n-1}, \mathcal{D}_{n-1}, \mathcal{G}_{n-1}) \longrightarrow \ldots\ldots \longrightarrow (\mathcal{Y}_1, \mathcal{D}_1, \mathcal{G}_1) \underset{\rho_1}{\longrightarrow} (\mathcal{Y}_0, \mathcal{D}_0, \mathcal{G}_0) = (\mathcal{Y}, \mathcal{D}, \mathcal{G})$$

such that $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ has log-canonical singularities with $\mathcal{X}$ smooth and $\mathcal{E}$ simple normal crossing.

Notice that pseudo-reflecting monodromy around invariant divisors is irrelevant to any of the enunciated properties of $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ so infact,

I.6.2 Fact *Given $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ as above we may kill any extra pseudo-reflecting monodromy that we may have introduced to get a map $\pi : (\mathcal{X}, \mathcal{E}, \mathcal{F}) \to (\mathcal{X}_0, \mathcal{E}_0, \mathcal{F}_0)$ through which the composition of the ρ's factors, $(\mathcal{X}_0, \mathcal{E}_0, \mathcal{F}_0)$ has all the desired properties, and in addition is an honest birational modification of $(\mathcal{Y}, \mathcal{D}, \mathcal{G})$.*

Just as per dimension 3, the critical object is the weak branching locus. From the higher dimensional standpoint, we only really care about points which are non-log flat so introduce,

I.6.3 Definition *A point ξ of $\mathcal{X}$ is said to be in the (very) weak branching locus, $(V)WB(\mathcal{F})$ if it is not log-flat, and the linearisation of a generator ∂ of the foliation at ξ has semi-simple part of rank strictly less than the co-dimension of* $\mathrm{sing}(\mathcal{F})$.

Notice that the use of the word very is simply intended to avoid confusion with the possibility that log-flat points may admit weak branching. In practice we're unconcerned about this so, as the parenthesis suggests, the word very may well be omitted. In any case, observe,

I.6.4 Fact *After further modification by blowing up in invariant centres the non-log flat locus and very weak branching locus are synonymous.*

proof Indeed suppose otherwise, then at a point ξ we have as many eigenvalues as the co-dimension, so the eigenfunctions actually define the singular locus, which must, therefore, be smooth and after blowing up in this centre every point over ξ becomes log-flat. The global proposition reduces to the above local discussion by the compactness of the Zariski-Riemann surface and the existence of algorithmic resolution. $\square$

The redeeming feature of the weak branching locus is,

I.6.5 Fact (cf. [M1] V.4) *Let $\mathfrak{X}$ be an affine formal scheme complete in the I_Z-adic topology of the singular locus Z of a foliation by curves such that,*

(a) $\mathfrak{X}$ and Z are smooth.

(b) there is a local generator of the foliation whose linearisation mod I_Z^2 admits a non-nilpotent Jordan decomposition.

then there is a unique formal invariant subscheme $\mathfrak{W}$, the weak branch, whose co-normal bundle is given by the vanishing of the non-zero eigenvectors mod I_Z^2. In particular for $\hat{\mathcal{X}}$ the completion of $\mathcal{X}$ in a component Z of, say, $VWB(\mathcal{F})$ there is an open formal algebraic stack $\mathfrak{U}$ (containing at least the points of Z where the number of eigenvalues is that of the generic point) and a closed irreducible formal substack $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ of $\mathfrak{U}$, the weak branching stack, which has exactly this property at the generic point.

Observe, in particular, by the definition of formal localisation that $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ may very well be defined apriori by functions with essential singularities, and its Zariski closure could easily be not just all of $\mathcal{X}$ but even $\hat{\mathcal{X}}$ when closure is understood formally. Given LCR, however, this doesn't happen and we'll establish,

I.6.6 Claim *Suppose* LCR *then every weak branching stack has a formal Zariski closure of the same dimension, and by a sequence of simple maps we can achieve.*

I.6.7 Emb(beded)LCR *Not only may we suppose after a sequence of simple modifications that $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ is smooth, $\mathcal{E}$ has simple normal crossings and $\mathcal{F}$ has log-canonical singularities, but we may suppose that every (very) weak branching stack considered as a formal substack of $\mathcal{X}$ completed in the appropriate component of* $\mathrm{sing}(\mathcal{F})$ *has the same properties.*

proof We proceed by a double induction in the 2-category of formal stacks. Firstly we induct on the dimension n say, of $\mathcal{X}$, so suppose the whole statement for anything of dimension at most n, and then induct by the co-dimension of the weak branching stack. In particular if $\mathcal{Z}$ is the substack of $WB(\mathcal{F})$ where there is exactly one eigenvalue then as per I.2.1, it's immediate that we can find a smooth invariant formal sub-stack $\mathcal{W}_1$ of $\mathcal{X}$ completed in $\mathcal{Z}$. Now apply EmbLCR to $\mathcal{W}_1$ and we're off and running, i.e. suppose we have not only proved that every weak branch of co-dimension p has a well defined equidimensional formal Zariski closure, but that we've also got the embedded resolution property for each of these.

Whence let $\mathcal{Z}$ be a component of $WB(\mathcal{F})$ where the weak branch has co-dimension $p+1$, and as per I.2.2, we can suppose that $\mathcal{Z}$ is smooth if our initial interest is to prove that it admits an equidimensional Zariski closure in the completion of $\mathcal{X}$ in $\mathcal{Z}$, which we may aswell suppose is synonymous with $\mathcal{X}$. The existence of the Zariski closure, however, is local at a point $\zeta \in \mathcal{Z}$. Furthermore, where the point is generic, in fact when the number of eigenvalues is exactly $p+1$, there's nothing to prove, so we may in fact suppose that the number of eigenvalues is $q \leq p$, so after, possibly additional, completion in $\mathcal{X}$ to say $\hat{\mathcal{X}}$ we have a well defined $\mathcal{W}_q \ni \zeta$. On the other hand $\mathcal{Z}$ is smooth, and the weak branch is already generically defined, so the existence of an equidimensional Zariski closure at ζ is really a question of the surjectivity of some maps between reflexive $\mathcal{O}_{\mathcal{Z}}$-modules, and since completion is faithfully flat we may suppose that $\mathcal{X}$ not only is $\hat{\mathcal{X}}$ but further complete as we please. In particular by way of EmbLCR for $\mathcal{W}_q$ we find a smooth formal substack of co-dimension $p+1-q$ which is precisely the Zariski closure of $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ at ζ. Plainly $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ has dimension smaller than $\mathcal{X}$, and we may apply EmbLCR to it to conclude. $\Box$

II. Rational Curves in 3-D

II.1. Cone Theorems

We proceed from the previous local discussion I.4 to the global study of curves in dimension 3, or more precisely the extremal sub-cone $\mathrm{NE}_{K_{\mathcal{F}}=0} := \{\alpha \,|\, K_{\mathcal{F}}.\alpha = 0\}$. Necessarily for $(\mathcal{X}, \mathcal{F})$ a gorenstein foliated 3-fold with projective moduli and $K_{\mathcal{F}}$ nef, any extremal ray in $\mathrm{NE}_{K_{\mathcal{F}}=0}$ is an extremal ray in $\mathrm{NE}_1(\mathcal{X})_{\mathbb{R}}$, e.g. [K] II.4.10.3. We begin by filtering $\mathrm{NE}_{K_{\mathcal{F}}=0}$ by co-dimension, i.e. define cones by,

II.1.1 Definition *Let things be as above, and put,*

(1) $P_+ = \{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \,|\, D.\alpha \geq 0, \forall D \in \mathrm{NE}^1(\mathcal{X})\}$

(2) $P_1 =$ linear span of $\{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \,|\, \exists$ an invariant divisor D with $D.\alpha < 0\}$

(3) $P_1 =$ linear span of $\{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \,|\, \exists$ a non-invariant divisor D with $D.\alpha < 0\}$

Plainly $P_+ + P_1 + P_2$ is $\mathrm{NE}_{K_{\mathcal{F}}=0}$. Furthermore a class in P_2, or more correctly an extremal ray R not in $P_+ + P_1$ is necessarily represented by the class of a map $f : \mathcal{L} \to \mathcal{X}$ with invariant image, albeit that we should apriori include the possibility that the image is completely singular. Indeed since the D of P_2 is non-invariant, the tangency locus $\mathcal{O}_D(K_{\mathcal{F}} + D)$ is a Cartier divisor on D which must contain the ray, then $K_{\mathcal{F}} + 2D$ is the tangency with the tangency, etc., so R must be an invariant curve with $K_{\mathcal{F}}.R = 0$. Plainly the number of such classes is countable, and we have to do similarly for P_1, i.e.

II.1.2 Claim *There are countably many invariant divisors D_i and invariant curves R_j such that,*

$$\mathrm{NE}_{K_{\mathcal{F}}=0} = P_+ + \sum_i (i_{D_i})_* \{\mathrm{NE}(D_i)_{K_{\mathcal{F}}=0}\} + \sum_j \mathbb{R}_+ R_j$$

proof Since the chow scheme of the projective moduli X of $\mathcal{X}$ exists, and has countably many components, with invariance being a closed condition, the only non-trivial case occurs when $(\mathcal{X}, \mathcal{F})$ has a meromorphic first integral. After a sequence of blow ups $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ we can assume that the integral is resolved by a map, so a generic member of the family of divisors in question has trivial normal bundle, whence on $\tilde{\mathcal{X}}$ we only need countable. However, $P_+(\tilde{\mathcal{X}})$ pushes forward to $P_+(\mathcal{X})$, so on $\mathcal{X}$ we only need countable too. □

Notice that we can further refine the situation within a given invariant divisor D_k or better invariant substack $\mathcal{D}$. To begin with, notice that,

$$\mathrm{NE}_{K_{\mathcal{F}}=0} = P_+ + \sum_i (i_{D_i})_* \{\mathrm{NE}(D_i)_{K_{\mathcal{F}}=0}\} + \sum_j \mathbb{R}_+ R_j$$

is again a closed cone, essentially because the intersection of divisors is a curve and the moduli has an ample divisor on it. Better still the cone $\mathrm{NE}(\mathcal{D})_{kf=0})$ is itself polyhedral, except in some degenerate cases that we proceed to investigate. A priori $\mathcal{D}$ isn't even normal, so let $\tilde{\mathcal{D}}$ be it's normalisation with $\mathcal{G}$, $\tilde{\mathcal{G}}$ the induced foliations. For $\tilde{\mathcal{D}}$ it makes sense to talk about a relatively (foliated) minimal resolution $\rho : \mathcal{Y} \to \tilde{\mathcal{D}}$, so for $\mathcal{H}$ the foliation on $\mathcal{Y}$,

$$K_{\mathcal{H}} = \rho^* K_{\tilde{\mathcal{G}}} - \sum_i a_i E_i$$

with $\bigcup E_i$ are the curves contracted by ρ_i and $a_i \in \mathbb{Q}_+$. In particular we may thus go from $(\mathcal{Y}, \mathcal{H})$ to a canonical model $(\mathcal{Y}_0, \mathcal{H}_0)$, cf. [M2], by ρ_0 with,

$$K_{\mathcal{H}} = \rho_0^* K_{\mathcal{H}_0} + \sum_j b_j C_j$$

where $\bigcup C_j$ is contracted by ρ_0, and $b_j \in \mathbb{Q}_{\geq 0}$. Finally $K_{\tilde{\mathcal{G}}}$ is related to $K_{\mathcal{F}}$ by,

$$K_{\tilde{\mathcal{G}}} = K_{\mathcal{F}} - Z$$

for a possibly empty curve supported on $\mathrm{sing}(\mathcal{F})$, where here, and throughout, we'll be notationally a bit loose about the difference between a divisor on $\mathcal{D}$ and it's pull-back to $\tilde{\mathcal{D}}$.

Plainly we can immediately exclude the possibility that $\mathcal{H}$ is of general type, since although $\mathrm{NE}_{K_{\mathcal{H}}=0}$ may be non-trivial, it's generated by finitely many invariant curves. The other cases are more fastidious, so let's suppose as much as we can, namely: $\mathcal{D}$ is non-redundant, so that there is an extremal ray R supported in $\mathcal{D}$, and a supporting function H_R vanishing on it, but strictly positive on the closed sub-cone obtained by summing over the invariant divisors other than $\mathcal{D}$.

Now, since everything is gorenstein, any curves around which $\mathcal{D}$ isn't generically smooth are invariant, and since we can plainly suppose that R has no support on such curves, there is an unambiguous sense to R as a class in $\mathrm{NS}_1(\tilde{\mathcal{D}})$ understood in Mumford's sense, and we divide by cases, i.e.

II.1.3 case (I) $R^2 < 0$.

This is rather easy. Indeed as per [K] 4.12.3, we may suppose that R is an honest curve C with non-invariant proper transform $\tilde{C}$. Consequently the adjunction formula of [M2] gives $(K_{\mathcal{H}} + \tilde{C}).\tilde{C} \geq 0$, so that for a possibly empty divisor E contracted by ρ,

$$(K_{\tilde{\mathcal{G}}} + C).C = K_{\tilde{\mathcal{G}}}.(\tilde{C} + E) + C^2 \geq K_{\tilde{\mathcal{G}}}.E + (C^2 - \tilde{C}^2) \geq 0$$

and we conclude to the absurdity, $K_{\mathcal{F}}.R \geq -R^2 > 0$.

II.1.4 Case (II) $R^2 > 0$

This can only happen if every curve in $\mathcal{D}$ becomes equivalent when pushed forward to $\mathcal{X}$, and of course $\mathcal{H}$ must have Kodaira dimension 0. Better still there can be no invariant curves on $(\mathcal{Y}_0, \mathcal{H}_0)$, $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ has dimension at most 0, and $(\mathcal{Y}, \mathcal{H})$ coincides with $(\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$. In particular $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$ actions are excluded, or even extensions of elliptic curves by $\mathbb{G}_a$ or $\mathbb{G}_m$. Notice in addition that if $(\mathcal{Y}, \mathcal{H}) = (\mathcal{Y}_0, \mathcal{H}_0)$ is a smooth foliation (e.g. the remaining case of an abelian surface) then $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D} = \emptyset$. Indeed otherwise by blowing up in the singular point, and normalising around the exceptional divisor, we would construct a prime divisor of $\mathbb{C}(Y)$ of zero discrepancy contrary to the hypothesised terminality of $(\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$. Furthermore singularities of $\mathcal{X}$ or $\mathcal{Y}$ (actually in this situation they coincide) where $\mathcal{F}$ is smooth must give rise to invariant curves, so everything may be supposed smooth. Observe, however, that either $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ or $\mathcal{O}_{\mathcal{D}}(-\mathcal{D})$ is effective, or both (i.e. $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ is numerically trivial, which gives us what we want anyway). To elaborate, there is at most one real class in co-homology tangent to the foliation, and since there is an effective such coming from invariant measures, we find that either $\mathcal{D}^2$ or $-\mathcal{D}^2$ is in P_+, which is nonsense.

II.1.5 Case (III) $R^2 = 0$.

This requires sub-division to discuss the behaviour,

sub-case (a) $(\mathcal{Y}_0, \mathcal{H}_0)$ *has numerical Kodaira dimension 1.*

The index theorem gives $K_{\mathcal{H}_0}, K_{\mathcal{F}}$ & R all parallel. Consequently if the foliation is parallel to the Kodaira fibration, R is elliptic and we're done. Furthermore R is extremal, so we conclude that $\mathcal{Y} = \tilde{\mathcal{D}}$ with $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ of dimension at most 0. In addition if $\mathcal{Y} \neq \mathcal{Y}_0$, and the Kodaira fibration exists, then the fibration considered on $\mathcal{Y}$ has an invariant fibre, which is nonsense, so we reduce to the possibility that $(\mathcal{Y}, \mathcal{H}) = (\mathcal{Y}_0, \mathcal{H}_0)$ is perfectly smooth, and as above so are $\mathcal{D}$ and $\mathcal{X}$, since otherwise we would get invariant curves parallel to R. More than this we cannot really say except to note that $R = K_{\mathcal{F}}.\mathcal{D}$, and that H_R is a supporting function.

The sub-sub-case of Hilbert-Modular surfaces is less satisfactory, since $(\mathcal{Y}, \mathcal{H})$ may not coincide with $(\mathcal{Y}_0, \mathcal{H}_0)$, and $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ or $\mathrm{sing}(\mathcal{Y})$ or $\mathrm{sing}(\mathcal{X})$ could have support on the curves contracted by ρ_0. Nevertheless R is still $K_{\mathcal{F}}.\mathcal{D}$, and again H_R is a supporting function.

sub-case (b) $(\mathcal{Y}_0, \mathcal{H}_0)$ *has Kodaira dimension 0.*

Arguing as above, and noting that the cones for $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$ actions are generated by invariant curves, we reduce to $(\mathcal{Y}_0, \mathcal{H}_0) = (\mathcal{Y}, \mathcal{H}) = (\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$, with $\mathcal{F}$ smooth around $\mathcal{D}$, and, if they exist, $\mathrm{sing}(\mathcal{X})$ and $\mathrm{sing}(\mathcal{Y})$ coinciding in an invariant curve. As such there are the following sub-sub-cases up to étale covering,

(i) *Things arise from the extension of an elliptic curve by $\mathbb{G}_a$ or $\mathbb{G}_m$, so R is a non-invariant rational curve.*

(ii) *The foliation is a product $E \times B$, with E elliptic, and R is a non-invariant section $e \times B$.*

(iii) *The foliation is of Kronecker type on an abelian surface. As such there is a risk that the cone may cease to be polyhedral if the surface isn't irreducible, or worse the product of a complex multiplication curve with itself.*

Consequently H_R is a supporting function except possibly in (iii) if the cone is round. In all these cases, however, we can play the $\mathcal{D}^2$ or $-\mathcal{D}^2$ is effective trick to find a special class in P_+, and this is what forces H_R to be a supporting function when the abelian surface of (iii) has a 2 dimensional cone.

sub-case (c) $(\mathcal{Y}_0, \mathcal{H}_0)$ *is a conic bundle.*

The statement itself is something of an over simplification, since what one should really do is divide Z_{red} into its parts $Z' + Z''$ which are invariant and non-invariant for $\tilde{\mathcal{G}}$, then work at the level of the log-canonical bundle $K_{\tilde{\mathcal{G}}} + Z''$. When the log-Kodaira dimension is non-negative, the essential deduction $(\tilde{\mathcal{D}}, \tilde{\mathcal{G}}, Z'') = (\mathcal{Y}, \mathcal{H}, Z'') = (\mathcal{Y}_0, \mathcal{H}_0, Z'')$ continues to work, and we deduce that we have an isotrivial family of quasi projective curves after an étale covering. Consequently if R isn't invariant its a section, and coincides with Z'' which is invariant in $\mathcal{X}$. In the remaining case, we observe that Z'' cannot be empty since $K_{\mathcal{F}}$ is nef, and by I.4.11 necessarily has just 1 component with invariant curves factoring through the weak branching scheme. As such, by op. cit., the degenerations of an irreducible fibre (which itself can only intersect $\mathrm{sing}(\mathcal{F})$ in Z'') only intersect $\mathrm{sing}(\mathcal{F})$ in Z''. In particular the quadratic form of the degeneration of the generic rational curve has no non-zero entries, so a minor variant of the standard lemma shows that a positive sum of Z'' and the components of degenerations is equal to R, since, incidentally, the extremality of R implies $K_{\mathcal{H}}.R \geq 0$.

To summarise, therefore, we have,

II.1.6 Fact *Let $\tilde{P}_+$ be the necessarily closed cone defined by,*

$$P_+ + \sum_k NE_1(\mathcal{D}_k)$$

where $\mathcal{D}_k$ is an invariant Weil divisor whose normalisation (equal to itself if $\mathcal{X}$ is smooth around $\mathcal{D}_k$)is an étale quotient of an abelian surface, with $\mathcal{D}_k \cap \mathrm{sing}(\mathcal{F}) = \emptyset$, and the induced foliation of Kronecker type, then,

$$NE_{K_{\mathcal{F}}=0} = \tilde{\mathcal{P}}_+ + \sum_i \mathbb{R}_+ R_i$$

where the R_i are extremal rays having one of the following forms,

(a) *R_i is a $\mathcal{F}$ invariant, including possibly completely singular, curve.*

(b) *$R_i = K_{\mathcal{F}}.\mathcal{D}_i$ for $\mathcal{D}_i$ a Weil divisor whose normalisation $\tilde{\mathcal{D}}_i$ is the minimal model of a foliated surface of numerical Kodaira dimension 1, and which intersects $\mathrm{sing}(\mathcal{F})$ in at most points. Indeed if $\tilde{\mathcal{D}}_i$ has actual Kodaira dimension then all of $\mathcal{X}$, $\mathcal{F}$, $(\mathcal{D}_i, \mathcal{F})$ are completely smooth.*

(c) R_i *is either a rational curve not invariant by* $\mathcal{F}$*, or transverse to the foliation on a divisor whose normalisation is a quotient of a trivial elliptic fibration co-inciding with the induced foliation.*

In addition the rays $R_i \notin \tilde{P}_+$ *are locally discrete, i.e. the cone is locally polyhedral.*

proof/elaboration Notice that all the closure/discreteness statements are based on the same tick that if $R \notin P_+$ then there is a divisor with $D.R < 0$. □

II.2. An Index Theorem

A good clarification of the nature of the cone P_+ arises from wholly general considerations about numerical Kodaira dimension. Indeed let X be a normal projective variety of dimension n, say over a field, but of arbitrary characteristic, H an ample bundle, and K a nef. divisor with $K^n = 0$, but $K^{n-1} \neq 0$. Necessarily K^{n-1} is a class in the cone $A_1(X)$ of nef. curves, and indeed in the extremal face $A_1(X)_{K=0}$. Now suppose that it's not everything in this face, then there is a not necessarily effective divisor L positive on K^{n-1}, vanishing on some ray R of $A_1(X)_{K=0}$. We'd like to believe that a small perturbation of K by L has lots of sections, and rather surprisingly this turns out to be easy, i.e.

II.2.1 Fact (cf. [B&] 4.4) *Suppose* $\epsilon > 0$ *is sufficiently small and rational, then for* $d \in \mathbb{N}$ *sufficiently large and divisible,*

$$h^0(X,\, K^d L^{\epsilon d}) \gtrsim \frac{d^n \epsilon}{(n-1)!} K^{n-1}.L$$

proof Choose H a priori so that $L_+ := H + L$ is ample, then iterated application of exact sequences, $p < \epsilon d$,

$$0 \to H^0(X, \mathcal{O}_X(dK + \epsilon d L_+ - (p+1)H)) \to H^0(X, \mathcal{O}_X(dK + \epsilon d L_+ - pH)) \to H^0(\mathcal{O}_H(dK + \epsilon d L_+ - pH))$$

gives the estimate,

$$\begin{aligned} h^0(X,\, K^d L^{\epsilon d}) &\geq h^0(X,\, K^d L_+^{\epsilon d}) - \sum_{p=0}^{\epsilon d - 1} h^0(H,\, K^d L_+^{\epsilon d}) \\ &\gtrsim \frac{d^n}{n!}(K + \epsilon L_+)^n - \frac{\epsilon d^n}{(n-1)!}(K + \epsilon L_+)^{n-1}.H \\ &= \frac{d^n \epsilon}{(n-1)!}(K^{n-1}.L + O(\epsilon)) \end{aligned}$$
□

Certainly therefore there is a $\delta > 0$ such that $dK + \epsilon L \geq \delta H$ as effective $\mathbb{Q}$ divisors whence the absurdity that $R = 0$, which we'll note explicitly by,

II.2.2 Fact If $\alpha \in A_1(X)$, and $K.\alpha = 0$, then $\alpha \in \mathbb{R}_{\geq 0} K^{n-1}$.

Notice also that an even more trivial variation of the same argument shows,

II.2.3 Claim *If infact* K *is nef, and* $K^n > 0$*, then* $A_1(X) \cap (K = 0) = 0$.

Presumably the general situation of numerical Kodaira dimension $\nu \in \{0, \ldots, n\}$, is described by $\alpha \in A_1(X) \cap (K = 0)$ iff $\alpha = K^\nu.\beta$, where β is a class of 'Demailly-Peternell' type, i.e. the intersection of $n - 1 - \nu$ amples on a modification $\tilde{X} \to X$. The other extremal case of $\nu = 0$ is of course [B&]. Regardless the discussion shows that in our main case of interest for foliations the cone is polyhedral, i.e.

II.2.4 Claim *Suppose that* $(\mathcal{X}, \mathcal{F})$ *is a foliated gorenstein 3-fold with* $K_{\mathcal{F}}$ *nef, and* $\mathbb{Q}$*-gorenstein in the usual sense, i.e.* $K_{\mathcal{X}}$ *is a* $\mathbb{Q}$*-bundle, and that the cone generated by* $K_{\mathcal{F}}$ *and* $K_{\mathcal{X}}$ *contains a big, then,*

$$\mathrm{NE}_{K_{\mathcal{F}}=0} = P_+ + \sum_i \mathbb{R}_+ K_{\mathcal{F}}.\mathcal{D}_i + \sum_j \mathbb{R}_+ R_j$$

where the R_j are invariant curves with $K_{\mathcal{F}}.R_j = 0$; the invariant divisors $\mathcal{D}_i$ are as described in II.1.6(b), and for $K_{\mathcal{F}}^2 \neq 0$, P_+ is nil if $K_{\mathcal{F}}^3 \neq 0$, and $\mathbb{R}_+ K_{\mathcal{F}}^2$ otherwise.

proof Notice that the condition on $K_{\mathcal{F}}$ & $K_{\mathcal{X}}$ implies that things like II.1.6(c) don't happen, i.e. in the notation of op. cit. one has $\mathcal{D}^2$ or $-\mathcal{D}^2 \in P_+$, but these classes intersect $K_{\mathcal{X}}$ in zero, which is nonsense. □

It's probably also worth observing that there's a nicer looking cone defined, for example, inside positive measures or co-homology, or even projected into Néron-Severi, but generated by transverse invariant measures. Let's give some notation to this such as $\mathrm{NE}(X/\mathcal{F})_{K_{\mathcal{F}}=0}$, then inside this cone the only terms that one finds in general are of the form P_+, $\mathbb{R}_+ R_j$ for R_j an invariant curve, V.2, proving this relies on some even more delicate results in dimension 2, but unfortunately it's a priori difficult to relate extremality here to extremality in $\mathrm{NE}_{K_{\mathcal{F}}=0}$, and the latter is what underpins flopping.

II.3. Tabulation

Plainly as one can see from II.1.6 or II.2.4 we aren't actually interested in all the invariant rational curves, but only those which intersect $K_{\mathcal{F}}$ in zero, and are not contained in the singular locus. Consequently let $(\mathcal{X}, \mathcal{F})$ be a foliated gorenstein stack with $K_{\mathcal{F}}$ nef, and $f : \mathcal{L} \to \mathcal{X}$ the normalisation of an invariant 1-dimensional substack intersecting $K_{\mathcal{F}}$ in zero. Furthermore let G be the generic stabiliser of $\mathcal{L}$, then there is a 1-dimensional orbifold $\tilde{\mathcal{L}}$ and a group stack $\mathbb{G}$ over $\mathcal{L}$ generically equal to G such that $\mathcal{L} = [\tilde{\mathcal{L}}/\mathbb{G}]$. On $\tilde{\mathcal{L}}$ we have two types of distinguished points, viz:

(a) $l_i \in \tilde{\mathcal{L}}(\mathbb{C})$ maps to $\mathrm{sing}(\mathcal{F})$, with m_i the order of it's stabiliser.

(b) $l_j \in \tilde{\mathcal{L}}(\mathbb{C})$ does not map to $\mathrm{sing}(\mathcal{F})$, but has non-trivial stabiliser of order n_j.

Consequently if we let g denote the genus of the moduli of $\mathcal{L}$, with I, J index sets for points of type (a) or (b), then the adjunction formula of [M1] I.8 reads,

$$0 = \#G\, K_{\mathcal{F}}.{}_f\mathcal{L} = (2g-2) + \#I + \sum_{i\in I} \frac{s_i - R_i - 1}{m_i} + \sum_{j \in J} 1 - \frac{1}{n_j}$$

where s_i, R_i are the segre class of $\mathrm{sing}(\mathcal{F})$, and the ramification calculated on an étale scheme like neighbourhood of l_i, cf. op. cit. I.8.6. Necessarily, $s_i - R_i - 1 \geq 0$, so we deduce,

II.3.1 Fact *If the moduli isn't rational, $\tilde{\mathcal{L}}$ is an elliptic curve, with $\mathcal{L}$ wholly contained in the smooth locus.*

Thus, supposing rational moduli, we profit from the inequality $\#I + \#J \leq 2$. In particular,

II.3.2 Similar Fact *If $\#I = 0$, then $\mathcal{L}$ is wholly contained in the smooth locus, and admits an étale cover by an elliptic curve.*

The most important case though is,

II.3.3 Fact/Definition *If $\#I = 2$ (so $\#J = 0$), then $s_i = R_i + 1$, for both points l_i, $i \in I$, which will be refered to as the ends.*

As a quasi sub-case this contains,

II.3.4 Fact *If $\#J = 2$, and $\#I \neq 0$, whence $= 1$, $s_i = R_i + 1$ at the unique point l_i mapping to $\mathrm{sing}(\mathcal{F})$. Consequently the universal cover of $\mathcal{L}$ is $\mathbb{P}^1$, and we get a map $f : \mathbb{P}^1 \to V$, for $V \to \mathcal{X}$ étale such that f is as per II.3.3*

To take care of the remaining cases with $\#I = 1$ observe that a local generator ∂ of $\mathcal{F}$ around the singular point $f(l)$ lifts to a vector field on scheme like étale neighbourhoods of l on $\mathcal{L}$ and the order of vanishing v of ∂ applied to a parameter at l on the same is well defined and satisfies,

$$s_i - R_i - 1 \geq v - 1 \geq 0$$

In particular we cannot have $v = 1$, and [M1] IV.10 gives further restrictions, i.e.

II.3.5 Fact *If $\#I = \#J = 1$, then $n_j | m_i$, and $v = m_i/n_j + 1$. As such, if the singular point is scheme like in $\mathcal{X}$, $\mathcal{L}$ is everywhere scheme like and $v = 2$.*

In dimension 3 we can of course make,

II.3.6 Assumption *$(\mathcal{X}, \mathcal{F})$ is a smooth foliated stack with convenient singularities with the only non-scheme like ones being of Cano type, i.e. with Jordan form*

$$x\{1 + a(xy, z)\}\frac{\partial}{\partial x} - y\{1 + b(xy, z)\}\frac{\partial}{\partial y} + z^{2p}\{1 + c(xy, z)\}\frac{\partial}{\partial z}$$

and $\mathbb{Z}/2$ acting via $x \leftrightarrow z$; $z \mapsto -z$, $p \in \mathbb{N}$. We will further assume that convention I.3.1 is in force.

Consequently we deduce,

II.3.7 Fact *Things as per II.3.6, then at least after further blowing up in points, and up to finitely many exceptions we may assume that any invariant curve, rational or otherwise, not factoring through $\mathcal{E}$ meets* $\mathrm{sing}(\mathcal{F})$ *in a non-isolated convenient singularity which is scheme like.*
proof The only isolated possibilities which may not be finite in number are I.4.3 & 4, and these occur in a unique formal hypersurface through the singularity in which the induced foliation is log-canonical rather than canonical. We can resolve this singularity to canonical, at which point the proper transform of the surface contains a non-isolated component of $\mathrm{sing}(\mathcal{F})$ through which all but finitely many of the original curves must pass. With a little more blowing up, we may suppose that this new non-isolated singularity is everywhere convenient. This reduces us to non-isolated singularities, where it is possible for non-scheme like singularities to exist, but these are easily removed. Specifically,
(a) The gorenstein cover of the moduli may have the property that non-isolated points of $\mathrm{sing}(\mathcal{F})$ meet the non-scheme like points in a point. This doesn't happen after blowing up in the appropriate component of $\mathrm{sing}(\mathcal{F})$, which is necessarily of the form II.3.6.
(b) $\mathcal{X}$ is ramified in invariant divisors over its gorenstein cover, in which case there is a $\mathcal{X}_0$ between $\mathcal{X}$ and the cover, cf. [M1] I.7.4, with the same $K_{\mathcal{F}}$, so we'll work on $\mathcal{X}_0$ instead. $\square$

Putting this together with II.3.3-5 we obtain,

II.3.8 Fact *With finitely exceptions and $(\mathcal{X}, \mathcal{F})$ of dimension 3, any curve $f : \mathcal{L} \to \mathcal{X}$ invariant by $\mathcal{F}$ not factoring through $\mathcal{E}$ meeting* $\mathrm{sing}(\mathcal{F})$ *with $K_{\mathcal{F}}.{}_f\mathcal{L} = 0$ has a scheme like analytic neighbourhood (formal would be wholly sufficient) $X \to \mathcal{X}$ such that $\mathbb{P}^1$ is the normalisation of the pre-image of $\mathcal{L}$ and indeed we have a fibre square,*

$$\begin{tikzcd}
(X, \mathcal{F}) \arrow[d] & \mathbb{P}^1 \arrow[l, "f"'] \arrow[d] \\
(\mathcal{X}, \mathcal{F}) & \mathcal{L} \arrow[l, "f"']
\end{tikzcd}$$

Furthermore $f(\mathbb{P}^1)$ meets $\mathrm{sing}(\mathcal{F})$ *in either one end 0, or two ends 0 and ∞ both of which lie in non-isolated convenient components, and the the possibilities are described as follows,*

(I) *There is only one end $f(0)$, and f factors at $f(0)$ through a non-log flat weak branch W. As such f has smooth image and the Jordan form at $f(0)$ looks like, $x\frac{\partial}{\partial x} + \nu(y, z)y^2\frac{\partial}{\partial y}$, $\nu(0) \neq 0$.*

(II) *There are two ends albeit that $f(0)$ may equal $f(\infty)$. In the latter case the branches over $f(0)$ & $f(\infty)$ are no worse than plane cusps, and both these branches lie in the same plane. Equally both ends may be disjoint cusps, or disjoint smooth, or a combination. Regardless the singularities at $f(0)$ & $f(\infty)$ are log-flat, and the parameter v prior to* II.3.5 *is always 1.*

proof Applying II.3.4 we may assume that the only non-scheme like points on $f(\mathcal{L})$ have stabiliser $\mathbb{Z}/2$, and there are 2 of these at points other than the ends. Consequently the fundamental group of $f(\mathcal{L})$ is either $\mathbb{Z}/2$ or $< a, b : b^2 = 1 >$, and in either case the étale covering of $\mathcal{L}$ by $\mathbb{P}^1$ descends to $f(\mathcal{L})$. Furthermore (II) is valid locally in the case where $f(\mathcal{L})$ has singular image by I.4.11, i.e. $f(\mathcal{L})$ is regularly embedded in $\mathcal{X}$, whence homotopic to a tubular neighbourhood in the same. Everything else is just a corollary of the previous tabulations combined with II.3.6-7. $\square$

II.4. One end

We get the case of one end out of the way first, and in the process establish some notation with,

$$N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n), \ m \geq n \in \mathbb{Z}$$

the co-normal bundle of our invariant curve L in our formal scheme X. On $\mathbb{P}^1$ we take homogeneous coordinates $[S, T]$, $s = S/T$ the standard coordinate at $o = [0, 1]$ etc., and over the cone $\mathbb{V}(N^{\vee}_{L/X})$, take x, y local generators of $\mathcal{O}(m)$, $\mathcal{O}(n)$ respectively at 0 with ξ, η the same at infinity, so that $x = t^m\xi$, $y = t^n\eta$. The condition of one end means that on the cone we may suppose that the foliation is given by $s^2\frac{\partial}{\partial s}$. As such, to start we do some linear algebra, i.e. write,

$$\partial \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and see how the matrix transforms, from which we deduce that we can write,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - ms & 0 \\ c(s) & d - ns \end{bmatrix}, \, ad = 0 \, , \deg c \leq m - n$$

Equally, however, not both a and d can be zero. Now what's important is to deduce that the $x = 0$ plane is the weak branch, so suppose this is false, and form the blow up $\rho : \tilde{X} \to X$ in the reduced structure of $\mathrm{sing}(\mathcal{F})$ which to first order would be $y = s = 0$, with $\tilde{L}$ the proper transform of L. Consequently $N^{\vee}_{\tilde{L}/\tilde{X}}$ contains as a sub-bundle $\mathcal{O}(m)$, and the H-N filtration/Grothendieck decomposition is,

$$N^{\vee}_{\tilde{L}/\tilde{X}} = \mathcal{O}(m) \oplus \mathcal{O}(n+1)$$

even if $m = n$ since $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. The condition $\partial x = -msx \bmod I^2_L$ is completely stable under blowing up & proper transform, but after $(m - n) + 1$ blow ups, our matrix becomes,

$$\begin{bmatrix} \lambda - (m+1)s & O \\ 0 & -ms \end{bmatrix}, \ \lambda \neq 0$$

and to 1st order, inside the weak branch, $\partial = -ms\frac{\partial}{\partial x} + s^2\frac{\partial}{\partial s}$, so $m = 0$, which, incidentally, even proves $n = 0$ if we were in the desired case, from which we deduce,

II.4.1 Fact *Either* mod I^2_L *the germ of the weak branch coincides with $x = 0$, and $n = 0$, or $N_{L/X} = \mathcal{O} \oplus \mathcal{O}(b)$, $b \in \mathbb{N} \cup \{0\}$ and L moves in a covering family.*

The latter case is invariably to be considered good and will be passed over without comment. As such we're reduced to the situation $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}$, $m \in \mathbb{N}$ with the $\mathcal{O}(m)$ function $x = 0$ coinciding with the weak branch. In this situation, and wholly generally, an easy induction gives,

II.4.2 Fact *There is a unique bundle $\mathcal{O}_X(1)$ on X lifting $\mathcal{O}_{\mathbb{P}^1}(1)$ and generating* Pic *of the same. Furthermore, for any thickening $X_p = \mathrm{Spec}\,\mathcal{O}_X/I_L^p$, $p \in \mathbb{N}$, sections of $\mathcal{O}_{X_p}(q)$, $q \geq -1$ can be lifted to X.*

What in general, however, is rather false is that sections of $\mathcal{O}_{X_p}(q)$, $q \leq -2$ can be lifted. Indeed even from X_2 to X_3. This can be interpreted at the level of blowing up. Specifically, the canonical map $\mathcal{O}(m) \to N^\vee_{L/X}$ determines a unique curve $\tilde{L}$ in the exceptional divisor E of the blow up $\tilde{X}$ in L, and we have an exact sequence,

$$0 \longrightarrow \mathcal{O}_{\tilde{L}}(-E) = \mathcal{O}_{\tilde{L}} \longrightarrow N^\vee_{\tilde{L}/\tilde{X}} \longrightarrow N^\vee_{\tilde{L}/E} = \mathcal{O}_{\tilde{L}}(-m) \longrightarrow 0$$

Of course $\tilde{L} \xrightarrow{\sim} L \xrightarrow{\sim} \mathbb{P}^1$ is the unique negative section of a Hizerbruch surface, and being able to lift to 2nd order, amounts to asking that the H-N filtration of $N^\vee_{L/X}$ is $\mathcal{O}(m) \oplus \mathcal{O}$. Should this happen, it makes sense to ask the same question on $\tilde{X}$ blown up in $\tilde{L}$, so in the first instance we might say that the H-N filtration is *stable after 1 blow up*, in the second *after 2*, and so forth. Whence we formulate,

II.4.3 Claim *Suppose quite generally (i.e. nothing to do with $\mathcal{F}$), $N^\vee_{L/X} = \mathcal{O} \oplus \mathcal{O}(m)$, $m \in \mathbb{N}$ is stable after p blow ups, then the unique map $\mathcal{O}(m) \to N^\vee_{L/X}$ can be lifted to X_{p+2}.*

proof For $\pi : \tilde{X} \to X$ as above, $N^\vee_{\tilde{L}/\tilde{X}} = \mathcal{O} \oplus \mathcal{O}(m)$ is stable after $p-1$ blow ups, so $\mathcal{O}(m) \to N^\vee_{\tilde{L}/\tilde{X}}$ lifts to $\tilde{X}_{p+1}$. We denote the nilpotent sub-scheme that it cuts out by $\tilde{V}_{p+1}$, with E_k the kth-thickening of E by itself, and $\tilde{V}_k$ the kth neighbourhood of $\tilde{L}$ in $\tilde{V}_{p+1}$ for $k \leq p+1$. Now consider the following diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{\tilde{L},E}(-(p+1)E) & \longrightarrow & \mathcal{O}_E(-(p+1)E) & \longrightarrow & \mathcal{O}_{\tilde{L}}(-(p+1)E) = \mathcal{O}_{\tilde{L}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{\tilde{V}_{p+1},E_{p+1}}(-E) & \longrightarrow & \mathcal{O}_{E_{p+1}}(-E) & \longrightarrow & \mathcal{O}_{\tilde{V}_{p+1}}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{\tilde{V}_p,E_p}(-E) & \longrightarrow & \mathcal{O}_{E_p}(-E) & \longrightarrow & \mathcal{O}_{\tilde{V}_p}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Notice that $\pi_*\mathcal{O}_{E_{p+1}}(-E) = I_{L,X}/I^{p+2}_{L,X}$, and we assert that $\pi_*\mathcal{I}_{\tilde{V}_{p+1},E_{p+1}}(-E)$ defines the required ideal. To see this one checks by induction on $0 \leq k \leq p+1$, that $R^i\pi_*\mathcal{I}_{\tilde{L},E}(-kE) = 0$, $i > 0$, so that the whole diagram is still exact after π_*. Consequently the said ideal cuts out a unique regularly embedded subscheme, $\pi_* V_{p+1}$, say, of X_{p+2} in which L has trivial normal bundle. $\square$.

Needless to say the claim is well adapted to our needs, Indeed back at our $\mathcal{F}$ invariant curve L in X with one end, on blowing up in it the curve $\tilde{L}$ is necessarily invariant. Better still on $\tilde{X}$ the singularities consist of the proper transform of sing$\tilde{\mathcal{F}}$ and an isolated point. A local calculation shows that $\tilde{L}$ passes through the former and not the latter. In addition the function s is still non-zero on $\tilde{L}$, so by I.4.11, $\tilde{L}$ factors through the induced weak branch, which is in fact the proper transform $\tilde{W}$ of W. Now our linear algebra prior to II.4.1 applies to deduce that the H-N filtration of $N^\vee_{L/X}$ is stable after 1 blow up, and by induction arbitrarily so. Consequently,

II.4.4 Fact *There is a smooth formal subscheme W_L of X whose completion at the end coincides with the weak branch, and $N_{L/W_L} \xrightarrow{\sim} \mathcal{O}_L$. In particular when X arises from the global context II.3.8, W_L is algebraic, with the induced foliation being in conics.*

This passes over some subtleties in the deformation theory of algebraic stacks, but an adequate solution to this is explained in [M1] III.2.

II.5. Two movable ends

We continue to address the case where the image L of $\mathbb{P}^1$ in II.3.8 is smooth. Our next goal concerns not only two ends, but a not wholly negative normal bundle, i.e. $N^\vee_{L/X}$ has H-N filtration $\mathcal{O}(m)\oplus\mathcal{O}(n),\ m\in\mathbb{N},\ n\leq 0$. Now, independently of the extra hypothesis we retake the notation of II.4, only now a generator of the foliation on the cone is $\partial = s\frac{\partial}{\partial s}$, and for $m\geq n$ the foliation is described by a matrix,

$$\partial\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\ a,d\in\mathbb{C},\ \deg c\leq m-n$$

It may happen that c cannot be taken to be zero. This occurs if $d-a\in\{0,\dots,m-n\}$, in which case it's a constant times s^{d-a}. In any of our two end situations $ad=0$, and there is at most one non-zero eigenvalue λ (which for convenience is allowed to be zero), with a well defined semi-simple matrix,

$$A_S(0) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

at zero, transforming as,

$$A_S(\infty) = \begin{bmatrix} -m-\lambda & 0 \\ 0 & -n \end{bmatrix} \text{ or } \begin{bmatrix} -m & 0 \\ 0 & -n-\lambda \end{bmatrix}$$

for the standard orientation of loops at zero and infinity. Equally, however, the matrix at infinity must be singular, and always taking $m\in\mathbb{N}$ we have the following,

II.5.1 Possibilities *Either, without loss of generality* (I) $n=0$ *or indeed* (II) $n\neq 0$ *albeit* $\lambda=-m$ *and in any case,*

$$A_S(0) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

Notice, however, that in the second hypothesis the matrix at ∞ has the eigenvalue in the bottom right corner. In the particular case of $n<0$ we eliminate this by blowing up. Proceeding as II.4.3, with $\rho:\tilde{X}\to X$ the blow up in the component of $\mathrm{sing}(\mathcal{F})$ through ∞ with $\tilde{L}$ the proper transform, we have an exact sequence,

$$0\longrightarrow\mathcal{O}(m)\longrightarrow N^\vee_{\tilde{L}/\tilde{X}}\longrightarrow\mathcal{O}(n+1)\longrightarrow 0$$

which necessarily splits, so the H-N filtration is stable under this operation - which, incidentally, it may not be if we try the same stunt at 0 or punctually. Plainly we need to make n blow ups to get what we want, and with the final one the singularity changes from that of I.4.7 to log-flat weak branching. Regardless the situation is still convenient but now, $N^\vee_{L/X}=\mathcal{O}(m)\oplus\mathcal{O},\ m>0$, and whether here, or in the previous, we can think about employing II.4.3 with $\pi:\tilde{X}\to X$ the blow up in L, and $\tilde{L}$ the canonical section of the resulting Hizerbruch surface. Under this operation $\tilde{L}$ meets the induced singular locus in the proper transform of $\mathrm{sing}(\mathcal{F})$ at an end of type I.4.6, and in another log-flat end, with weak branching if that's what we had to start with. As such we get two ends of the same type with $A_S(0)\mapsto\tilde{A}_S(0)$, say, $A_S(\infty)\mapsto\tilde{A}_S(\infty)$, without change. The H-N filtration may not, however, be stable. Should this occur then $\lambda=-a$, $a\in\{1,\dots,m-1\}$, as such,

II.5.2 Possibilities *The applicability of II.4.3, or otherwise, is described by,*

(I) $-\lambda\notin\{1,\dots,m-1\}$, *and the situation is stable after infinite blowing up, so by op. cit.,* L *moves in the* $x=0$ *direction.*

(II) $-\lambda \in \{1, \ldots, m-1\}$, *but it still succeeds in being stable under blowing up, so, again L moves in the $x = 0$ direction.*

(III) $-\lambda \in \{1, \ldots, m-1\}$, *and the situation is only stable under $p \in \mathbb{N} \cup \{0\}$ blow ups.*

Notice that the smooth formal divisor D in which L moves, has $D.L = -m < 0$, so it must be invariant, since otherwise the tangency locus between $\mathcal{F}$ and D gives a section of $H^0(D, \mathcal{O}_D(K_{\mathcal{F}} + D))$. Using I.3.3 & I.4.11 we can describe its form. The situation is as follows,

II.5.3 Fact *Let D be the invariant divisor constructed above in cases (I) or (II), or, better, $\tilde{D}$ if we started with $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n)$, $n < 0$, and effected $-n$ blow ups in* $\mathrm{sing}(\mathcal{F})$ *through the end at ∞ to work on $\rho : \tilde{X} \to X$ with D its image in X. Furthermore let $\mathcal{D}, \tilde{\mathcal{D}}, \tilde{\mathcal{X}}$ etc. be the necessarily global objects occuring when things arise from the a priori global situation* II.3.8. *Then in all cases the induced foliation on D is a pencil of conics, and either,*

(a) $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}$, *and the completion of D in the end at zero is the unique plane not in $\mathcal{E}$ containing the given component of* $\mathrm{sing}(\mathcal{F})$, *and idem at infinity if $\lambda \neq -m$.*

(b) $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n)$, $n < 0$, *and, again, the completion of D in the end at zero is the unique plane not in $\mathcal{E}$ containing the given component of* $\mathrm{sing}(\mathcal{F})$, *while at ∞ we have a singularity of type* I.4.7, *but the completion of D at ∞ is locally the divisor $z = 0$ of* I.4.7 *with the equations of the conics those of op. cit.*

proof At zero, let $X = 0$ be the local equation of a Jordan coordinate distinct from s, and $x = 0$ the local equation for D. By construction x is invariant with non-zero eigenvalue equal to that of X. Furthermore in the $\xi = 0$ plane the semi-simple foliation is smooth, given by a fibration Y =constant, which we may also take to be Jordan. Whence we can write, $x = x_0(z) + Xh$, and apply ∂_S to conclude,

$$(\lambda h + \partial_S h)X = x\nu, \ \nu(0) \neq O$$

so $X|x$, and deduce that they're the same. A quick check of possible Jordan forms reveals, however, that the nilpotent part at zero lies in the ideal of D, so the end at zero has a removable singularity for the induced foliation in D. Whence D, or slightly more precisely, $\mathcal{D}$, is a pencil of conics. This completely proves (a), and much of (b). As to what remains: on $\tilde{X}$ the induced foliation has a log-flat singularity with weak branching. Consequently if the singularity at ∞ is removable in D then either we have an invariant rational curve for every point of a component of $\mathrm{sing}(\mathcal{F})$ that contracts to the end $f(\infty)$, and this is what is asserted, or we don't. If, though, we don't the induced foliation in D would have a saddle node, which is nonsense for a conic pencil, or the curves stop intersecting the weak branching locus at a tame beast on $\tilde{X}$. In the latter situation, the fact that L is smooth implies, I.3.7 & I.4.7, that D is smooth on X, which would contradict (a) for nearby curves. Furthermore the induced singularity in $\tilde{D}$ at infinity must indeed be removable, since otherwise, we'd again get a conic pencil with a saddle. $\square$

II.6. Rigid Ends

From our previous discussion we see that to study rational curves L which do not move we may assume that we have $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n)$, $m, n \in \mathbb{N}$ with two ends at which the eigenvalues are $-m$ and $-n$ respectively. One should, however, not be lulled into a false sense of security since,

II.6.1 Remark/Warning *We are only studying curves not in $\mathcal{E}$ up to finitely many exceptions, and some of these exceptions may be rigid yet have both positive and negative terms in $N^{\vee}_{L/X}$. Furthermore we'll eventually have to be more precise about* II.5.2(III), *since this only reduces to the current scenario after an operation depending on L.*

Being that as it may, in our current situation we can by [A] effect a formal contraction $\rho : X \to X_0$, with $(X_0, \mathcal{F})$ Gorenstein. By the theorem of formal functions $\Gamma(\mathcal{O}_{X_0})$ is complete in the filtration $F^p\Gamma(\mathcal{O}_{X_0})$, where,

$$\frac{F^p\Gamma(\mathcal{O}_{X_0})}{F^{p+1}\Gamma(\mathcal{O}_{X_0})} = H^0(L, S^p N^\vee_{L/X})$$

By definition $(X_0, \mathcal{F})$ still has canonical singularities, and for ∂ a generator of the foliation on X_0, equivalently all of X, we can make a Jordan decomposition $\partial = \partial_S + \partial_N$ of the action on $\Gamma(\mathcal{O}_{X_0})$, or, if one prefers, globally on X. In particular, with the notations of II.4, there are functions F, G on X with the following properties,

(a) $\partial_S F = -mF$, F lifts $x = 0$ over a standard affine patch around 0, and mod I_L^2 is $t^m\xi$ at infinity.

(b) $\partial_S G = nG$, G lifts $\eta = 0$ over a standard affine neighbourhood of infinity, and mod I_L^2 is $s^n y$ at zero.

with this in mind, the assertion is,

II.6.2 Claim *Assuming, as we may, the normalisation $\partial s = s$, $t = 1/s$, then $t^m|F$ at infinity, while $s^n|G$ at zero.*

proof If in a minor abuse of notation t, ξ, η are Jordan coordinates at ∞, the eigenfunctions of $-m$ are $t^{m+k\eta}\eta^k$, and similarly around zero. □

Consequently the (formal) Zariski closure of $F = 0$ on a neighbourhood of zero, respectively of $G = 0$ on a neighbourhood of ∞, are smooth invariant divisors D_0, D_∞ defined by global sections of $\mathcal{O}_X(-m)$, and $\mathcal{O}_X(-n)$ respectively. Furthermore, by the previous argument, I.3.2, around 0, D_0 is necessarily the unique invariant plane at 0 not in $\mathcal{E}$ containing $\mathrm{sing}(\mathcal{F})$, and similarly at infinity. In particular we can perform a formal flop.

The operation of formal flopping is described as follows. Firstly, let's aim for the standard notation, so let X_- be our original X, X what was the contraction X_0, and ρ_- the map. Now form the formal stack, $X_-(\sqrt[m]{D_0}, \sqrt[n]{D_\infty})$, and blow up in the pull-back of L. The moduli of this $\tilde{X}$ is a formal scheme, in fact a weighted blow up of X, but the given stack $\tilde{\mathcal{X}}$ is rather nicer, since we can blow down in the other direction to a smooth formal stack, which we simplify by putting a minimal smooth structure to obtain a formal stack $\hat{\mathcal{X}}_+$, étale over its moduli X_+ in co-dimension 1. As such if $D_{0,-}$, $D_{\infty,-}$ (i.e. our old D_0, D_∞) go to $D_{0,+}$, $D_{\infty,+}$ they no longer meet the flopped curve L_+, or at the stack level $\mathcal{L}_+$, which itself is an invariant generalised projective stack $\mathbb{P}^1(m,n)$, so potentially not generically scheme like if $(m,n) \neq 1$. Notice also that $X_-, X_+ \to X$ are at least formally relatively projective, i.e. they're Prof's, cf. [M1] IV.2.3, and whence we summarise the situation as follows,

II.6.3 Fact *Suppose the image of $f : \mathcal{L} \to \mathcal{X}$ is smooth, with $N^\vee_{\mathcal{L}/\mathcal{X}} > 0$, and on the scheme like formal neighbourhood $X_- \to \mathcal{X}$, $\lambda \neq -1$, then in fact $\mathcal{L} = L$ is everywhere scheme like, and there is a formal flop,*

$$\begin{tikzcd}[column sep=small] & (\tilde{X}, \tilde{\mathcal{F}}) \arrow[dl] \arrow[dr] & \\ (X_-, \mathcal{F}_-) \arrow[dr, "\rho_-"', "-D_{0,-};\, -D_{\infty,-} \text{ relatively ample}"] & & (X_+, \mathcal{F}_+) \arrow[dl, "\rho_+", "D_{0,+};\, D_{\infty,+} \text{ relatively ample}"'] \\ & (X, \mathcal{F}) & \end{tikzcd}$$

Up to some obvious technical lacunae, i.e. relative projectivity of the ρ's, and algebraicity of the D's; this equally describes the global flop. It is furthermore an anti-flop for $\mathcal{E}_-$, with $\mathcal{E}_+$ a weighted blow up of $\mathcal{E}_-$ containing the flopped curve as exceptional divisor. Nevertheless $\mathcal{E}_+$ has simple normal crossings, and even the convenience of the singularities is preserved.

proof/elaboration Since we're working under rather restrictive hypothesis, the only possibility for f to have non-scheme like image is the generalised weighted projective stack $\mathcal{P}^1(2,2)$, or, what amounts to the same thing, $[\mathbb{P}^1/\mathbb{Z}/2]$ for the standard $x \leftrightarrow -x$ action. In this case our scheme like formal neighbourhood $X \to \mathcal{X}$ comes equipped with a $\mathbb{Z}/2$ action, and we can arrange for our global generator ∂ of $\mathcal{F}$ over X to satisfy $\partial^\sigma = -\partial$ for σ the generator of $\mathbb{Z}/2$. In particular, $\partial = -\partial_S^\sigma - \partial_N^\sigma$ is still a Jordan decomposition, so $\partial_S^\sigma = -\partial_S$, and eigenfunctions go to eigenfunctions. Whence this can only happen if $\lambda = -1$. This establishes all that's been said, but also more. Indeed, we must by uniqueness, I.3.2, have $D_{0,-}^\sigma = D_{\infty,-}$, and idem for the germs $\mathcal{E}_{0,-}$; $\mathcal{E}_{\infty,-}$ of $\mathcal{E}$ through 0 and ∞. Also the fixed points of σ for our conventional choice of coordinates are $1, -1$, which are still distinct from the singularities of the flipped curve. Plainly at the level of the completion $\hat{\mathcal{X}}$ of $\mathcal{X}$ in $f(\mathcal{L})$ there is only one divisor $\mathcal{D}_-$, say, and the flopped stack $\mathcal{L}_+$ is again a $\mathcal{P}^1(2,2)$ with a single point in $\mathrm{sing}(\mathcal{F})$, which is scheme like. Better still the intermediary flap $\tilde{\mathcal{X}}$ is just the blow up in $\mathcal{L}_-$, and we summarise by the obvious diagram,

II.6.4 Fact *In the missing case where $f(\mathcal{L})$ is non-scheme like, we have a flop in formal stacks described by,*

$$\begin{tikzcd}[column sep=small]
 & (\tilde{\mathcal{X}},\tilde{\mathcal{F}}) \arrow[dl] \arrow[dr] & \\
(\hat{\mathcal{X}}_-,\mathcal{F}_-) \arrow[dr, "-\mathcal{D}_- \text{ relatively ample } \rho_-"'] & & (\hat{\mathcal{X}}_+,\mathcal{F}_+) \arrow[dl, "\rho_+ \ \mathcal{D}_+ \text{ relatively ample}"] \\
 & (\hat{\mathcal{X}},\mathcal{F}) &
\end{tikzcd}$$

Again it is a $\mathcal{E}_-$ anti-flop, the singularities are still convenient, and everything globalises, modulo the aforesaid technical lacunae understood at the level of the moduli.

II.7. Singular curves

Our considerations so far have ignored the possibility that the map $f : \mathcal{L} \to \mathcal{X}$ is ramified, or has intersecting branches in its image. As such, suppose to begin with, that the image of L in the scheme like neighbourhood X has a node, with by definition, both branches smooth. The only way this can occur, as ever up to finitely many exceptions with no factoring through $\mathcal{E}$, is at singularities of type I.4.7, with the branches described by I.4.3. In particular the eigenvalue is in $\mathbb{N}$, and by blowing up we reduce to the case that it's equal to 1, then by a further blow up contradict the simple connectedness of $\mathbb{P}^1$.

Consequently let's turn to cusps but under the initial hypothesis that the two ends 0 and ∞ are disjoint. The unique possibility here that we need to deal with occurs when at least one of the ends has Jordan form,

$$\partial = ps\frac{\partial}{\partial s} + qx(1+a(y))\frac{\partial}{\partial x} \ , \ a(0) \neq 0$$

and $p, q \in \mathbb{N} \setminus \{1\}$. If we resolve the singularity around our cusp which we necessarily suppose of the form, $(s^p, cs^q, 0)$, $c \in \mathbb{C}^\times$, then the induced end at 0 has a nil matrix for it's semi-simple part in the normal direction. Whence the cusp must move, so that in the limit $c \to 0$, we find an invariant rational curve given around $f(0)$ by $x = y = 0$. Now the ends are disjoint, so we may aswell suppose that at ∞, both our original curve, and its smooth deformation limit are smooth. However, by our previous considerations the deformation limit must move in the divisor $x = 0$, so in fact the curve even moves in the whole space. Notice though,

II.7.1 Remark *Families of cusps with $K_\mathcal{F}.\mathcal{L} = 0$ in the $y = 0$ plane can occur, but only when there are non-scheme like points in* $\mathrm{sing}(\mathcal{F})$.

This leaves the possibility of $f(0) = f(\infty)$, with at least one branched cusp. So, say, one branch smooth, one branch cusped. On resolving the cusp we either get a node, which we know doesn't happen, or we find that we again have a nil semi-simple matrix in the normal direction at the resolved cusp, so we obtain an inexistent node as a deformation limit. Both branches cusped reduces to this case by way of a deformation limit, and so,

II.7.2 Fact *Apart from finitely many exceptions (all in the weak branching locus by the way) if there is an invariant singular 1-dimensional substack* $\mathcal{L}_0$ *not wholly in* $\mathrm{sing}(\mathcal{F})$ *with* $K_{\mathcal{F}}.\mathcal{L}_0 = 0$*, then* $\mathcal{L}_0$ *(or some multiple of it) moves in a family covering* $\mathcal{X}$*.*

II.8. Constant eigenvalue

So far we've managed to conclude that an invariant rational type stack will, up to finitely many exceptions, move unless the formal scheme like neighbourhood X of L ($\xrightarrow{\sim} \mathbb{P}^1$) has eigenvalues $-a$ and $-b$ at the ends, with $N^\vee_{L/X}$ either $\mathcal{O}(a) \oplus \mathcal{O}(b)$ or $\mathcal{O}(a+b) \oplus \mathcal{O}$. In the latter case we have a further integer of interest p, which is the number of times we need to blow up in L, and the extremal horizontal curve in the exceptional divisor, before equating this situation with the former. Slightly more conveniently this latter situation can be described in terms of a weighted blow up $X(L) \to X$, which is equivalently the blow down of the first $p-1$ exceptional divisors in the direction from whence they came. If we put a minimal smooth stack structure on $X(L)$, say, $X(L)^p$, then around the extremal curve L_p in the exceptional divisor P (albeit that it may not be globally extremal) everything is scheme like, with the monodromy concentrating on another horizontal curve, denoted L_p^∞, around which it is identically $\mathbb{Z}/p$. Manifestly the construction of $X(L)^p$ globalises with no loss of projectivity of the moduli. This construction allows us to focus on the $\mathcal{O}(a) \oplus \mathcal{O}(b)$ case, which we'll call *rigid*, while the latter case we'll denote *frigid*.

Regardless, as we've said, in either case we have a Jordan decomposition, $\partial_S^L + \partial_N^L$ of a local generator ∂ of the foliation on X. In the rigid case this is particularly easy to describe since we have embedding coordinate $x_0, \dots, x_a; y_0 \dots y_b$ on the contraction of L, equivalently generators of $\Gamma = \Gamma(\mathcal{O}_X)$, and,

II.8.1 Formula $\partial_S^L = (i-a)x_i \frac{\partial}{\partial x_i} + (b-j)y_j \frac{\partial}{\partial y_j}$, $\partial_N^L = y_b x_i \alpha(x_a, y_b) \frac{\partial}{\partial x_i} + x_a y_j \beta(x_a, y_b) \frac{\partial}{\partial y_j}$

for a priori arbitrary functions of two variables α and β. Equally the Jordan decomposition around L must restrict to a Jordan decomposition at the ends. Furthermore when one of the ends, say ∞, has constant eigenvalue we equally have a Jordan decomposition of the foliation around the singular component $Z_\infty \ni \infty$, or, more correctly, a neighbourhood of ∞ after completion in Z_∞ such that around both L and Z_∞ we can simultaenously find a generator of the foliation. Indeed the issue is as follows: Jordan decomposition is a notion of fields not of foliations, so while it's true that the semi-simple direction is largely independent of the choice of generator, the nilpotent direction may not be. As such fix an affine neighbouhood U_∞ of Z_∞ with trivial Picard group, and consider the formal scheme Ξ, obtained by completion, whose trace is $U_\infty \cap Z_\infty$, and finitely many rigid or frigid curves L_i with an end in $U_\infty \cap Z_\infty$. Now contract all the L_i, so that we get a formal scheme $\pi : \Xi \to \Xi^\#$. Plainly, if we can trivialise $K_{\mathcal{F}}$ around $\Xi^\#$ then it trivialises over Ξ. By construction, however, the trace of $\Xi^\#$ is affine, so the question is of first order, and since the singularity of the trace is a plane node, we have an isomorphism,

$$\mathcal{O}^\times_{\Xi_0} \xrightarrow{\sim} \mathcal{O}^\times_{U_\infty \cap Z_\infty} \coprod_i \mathcal{O}^\times_{L_i}$$

Furthermore, to give a line bundle on the trace of $\Xi^\#$ is to give a line bundle on Ξ_0 which is trivial on each L_i, so, indeed, we can find a generator ∂ of the foliation, which is equally a generator around both $U_\infty \cap Z_\infty$ and the L_i on some formal neighbourhood, so that we get a uniform Jordan decomposition $\partial_S + \partial_N$ on the said neighbourhood, which in turn gives rise to,

II.8.2 Case (a) $\partial_N \neq 0$

In this case one observes by the explicit formula II.8.1 that if the curve L is rigid then it is defined as the intersection of a pair of divisors - necessarily the D_0, D_∞ post II.6.2 if the eigenvalue at 0 is unramified- around which ∂_N is non-saturated. Since there are only finitely many such divisors, the number of possibilities for a rigid L is finite. Similarly, if it's frigid the same statement is true on $X(L)$, and the only extra possibility for a divisor along which ∂_N is non-saturated is the exceptional divisor P. On the other hand, this is transverse to D_0 at 0, and transverse to D_∞ at ∞, so for unramified eigenvalue this is irrelevant, and the number of frigid curves is constant too.

II.8.3 Case (b) $\partial_N = 0$

In this case the situation is uniformly semi-simple at both ends, so if we have infinitely many curves, we may sub-divide so that they're all either frigid or rigid, and we obtain a formal substack $\mathcal{Y}$ of the completion of $\mathcal{X}$ in the component Z_0 of $\mathrm{sing}(\mathcal{F})$ through the end at zero, and as large a collection of curves $\mathcal{L}_i$ as we may choose such the the slopes of the H-N filtration of $N_{\mathcal{L}_i/\mathcal{X}}$ are constant in i. Now we simply approximate $\mathcal{Y}$ around Z_0 by way of global sections of some ample bundle H, supposing, as we may, that $\mathcal{Y}$ is non-algebraic. Consequently for $n \in \mathbb{N}$ we can find divisors $D \in |nH|$ whose restriction to $\mathcal{Y}$ vanishes on Z_0 to order n^2, so if $D|_{\mathcal{L}_i} \neq 0$ then, $\deg_H(\mathcal{L}_i)$ is at least of order n. In particular D vanishes on all of the $\mathcal{L}_i$ which enjoy $\deg_H(\mathcal{L}_i) \leq O(n)$. Whence on $\mathcal{Y}$ we get a Cartier divisor $D|_{\mathcal{Y}}$ with cycle at least,

$$n^2 Z_0 \ + \sum_{\deg_H(\mathcal{L}_i) \leq O(n)} m_i \mathcal{L}_i$$

On the other hand the slopes of $N_{\mathcal{L}_i/\mathcal{X}}$ are constant, so we must have,

$$m_i \geq n^2 (1 - \frac{\deg_H \mathcal{L}_i}{n}), \quad \deg_H \mathcal{L}_i \leq O(n)$$

and whence by Bézout's theorem, or better, its various refinements, [F] 12.3, an inequality of $\mathbb{Q}$ cycles,

$$H^2 >> \sum_{\deg_H(\mathcal{L}_i) \leq O(n)} (1 - \frac{\deg_H \mathcal{L}_i}{n}) \mathcal{L}_i$$

so the degree of the $\mathcal{L}_i$'s is uniformly bounded, whence the absurdity that $\mathcal{Y}$ is algebraic. Thus, to summarise, we have,

II.8.4 Fact *Suppose the eigenvalue of a rigid or frigid curve is constant at one of its ends, i.e. if, say, the end is zero with Z_0 the component of* $\mathrm{sing}(\mathcal{F})$ *through the same, the map $Z_0 \to |K_{\mathcal{F}}^{\otimes 2}|$ given by symmetric functions is constant, then there are only finitely many rigid or frigid curves with an end in Z_0.*

II.9 The Primitive Beast

The situation that remains to be understood occurs when we are presented with two (possibly identical) components Z_0 and Z_∞ of $\mathrm{sing}(\mathcal{F})$, together with an infinite (infact Zariski dense) set of frigid or rigid curves $\mathcal{L}_i$ with ends in the same. Furthermore, we may suppose that in terms of the two unique formal divisors S_0, containing Z_0 or T_∞, D_∞ containing Z_∞ the $\mathcal{L}_i$ factor through D_0 and D_∞ at the ends. The primitive situation that we wish to achieve is that every invariant curve meeting all but finitely many of our $\mathcal{L}_i$'s is itself rigid or frigid. As we've currently set up the problem this is impossible since the divisors S_0, S_∞ coincide with components of our background divisor $\mathcal{E}$ through $\mathrm{sing}(\mathcal{F})$. Manifestly components of $\mathcal{E}$, and other things, will have to be eliminated. As a consequence we'll have to allow a more general set up, which to begin with we describe over the Gorenstein covering stack $(\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$ of the moduli. In particular it may occur that around Z_0 and/or Z_∞ the situation is non-gorenstein (actually the monodromy here will be $\mathbb{Z}/2$, and will only occur at one of the ends) then over and above this we will have some cyclic monodromies $\mathbb{Z}/n_0$, $\mathbb{Z}/n_\infty$ corresponding to the extra monodromy at the generic points of Z_0 and Z_∞ respectively. The non-gorenstein case is a bit more fastidious, so let's ignore it for the moment. As a result if d is the gcd of n_0 and n_∞ a rigid, or frigid, curve in this scenario will (up to finitely many exceptions) admit an étale covering by the generalised weighted projective stack $\mathcal{P}^1(\frac{n_0}{d}, \frac{n_\infty}{d})$. Furthermore, since we'll allow no more extra non-scheme structure than this, any smooth invariant curve not factoring through $\mathrm{sing}(\mathcal{F})$ is either étale covered by $\mathbb{P}^1$, a weighted projective stack of the form $\mathcal{P}^1(n)$, or, again, a $\mathcal{P}^1(\frac{n_0}{d}, \frac{n_\infty}{d})$. In any case, since all these objects are simply connected with co-homology and Picard groups as expected, cf. [M1] I.9, the whole of the previous discussion goes through up to an appropriate change in the definition of $\mathcal{O}(1)$. As such the only really important thing to notice is the change in the behaviour of the eigenvalue which we normalise by a choice of standard loops at the ends, i.e. $\frac{\partial_0 s}{s} = 1$. and $\frac{\partial_\infty t}{t} = 1$ for s, t local equations for S_0, S_∞ respectively with $p = n_0/d$, $q = n_\infty/d$ a rigid or frigid curve has eigenvalues $-a/q$, $-b/p$; $a, b \in \mathbb{N}$ at 0 and ∞, while its normal bundle is either $\mathcal{O}(\frac{a}{pq}) \oplus \mathcal{O}(\frac{b}{pq})$ or $\mathcal{O}(\frac{a+b}{pq}) \oplus \mathcal{O}$ for $\mathcal{O}(\frac{1}{pq})$ the generator of $\mathrm{Pic}\mathbb{P}^1(p,q)$, and, of course, this is to be understood on an étale neighbourhood if $d \neq 1$.

Now let's consider how this more general set up relates to our initial situation, and convenient singularities. In particular if we're not a priori supposing I.3.1, then we ask whether the invariant curves intersecting $K_{\mathcal{F}}$ in zero, and factoring through S_0 are finite or infinite in number. In the former case, we don't care, and do nothing, in the latter case, either they're all frigid or rigid, and, again, we do nothing, or they move, in which case the Zariski closure, say $\bar{S}_0$, is a divisor, and the induced foliation is a pencil of conics. Actually, rather better, the induced log-object $(\bar{S}_0, Z_0 \cup W_0, \mathcal{F})$, where Z_0, W_0 are the components of $\mathrm{sing}(\mathcal{F})$ not invariant by the induced foliation is generically an isotrivial family of $\mathbb{G}_m$'s over Z_0 and/or W_0. Now since the eigenvalue is non-constant along Z_0 it's equally so along W_0, and around W_0 we have the canonical formal invariant planes, i.e. S_0 and an other S_1, containing it. Consequently we may ask the same question for S_1 as we did for S_0, and continue if we find lots of invariant curves leading to a well defined $\bar{S}_1$. Notice, in particular, on continuing by induction, that inside any $\bar{S}_i$ we never find curves of which a neighbourhood is non-gorenstein. Indeed an open subset of any $\bar{S}_i$ containing a dense set of its invariant curves can be contracted to an open neighbourhood U_i of some component W_i of $\mathrm{sing}(\mathcal{F})$. On this component we can put a smooth stack structure such that the foliation has canonical singularities, but is non-gorenstein over its moduli, i.e. there is a non-trivial character χ of the monodromy group G of the generic point of W_i such that $\partial^\sigma = \chi(\sigma)\partial$ for an appropriate local generator ∂ of the foliation. The co-dimension of the singular locus is, however, two so $\chi = 1$ or -1, and the eigenvalue is identically -1. As such associated to S_1 there is a genuinely different component W_1 of $\mathrm{sing}(\mathcal{F})$, etc., and in any case, the creation of the W_i's plainly terminates. if the process terminates in Z_∞ then there is nothing more to do. Otherwise we ask the same questions creating a bunch of invariant divisors $T_{\infty - i}$ and components V_i of $\mathrm{sing}(\mathcal{F})$ with non-constant eigenvalue. Plainly there is an open set containing the frigid and/or rigid initial curves, and all but finitely many of the new ones, which can be contracted to Z_0 and/or Z_∞, and lead to our goal of a primitive neighbourhood of the beast, i.e. one where any $K_{\mathcal{F}}$ nil invariant curves meeting the rigid or frigid ones have the same property. Nevertheless we want to do this in a reasonably global way that avoids causing too much trouble. Furthermore, we're happy to do it even at the cost of losing projectivity of the moduli, not to mention a lot of arbitrariness away from the frigid or rigid curves. Consequently, we may even start with some blowing up so that the induced log-object $(\bar{S}_0, Z_0 \cup W_0, \mathcal{F})$ is smooth with canonical singularities, so, in particular, there is a projection σ_0 to Z_0. As a result, we can write,

$$K_{\mathcal{F}}|_{\bar{S}_0} = Z'' + C$$

where Z'' is the part of $\mathrm{sing}(\mathcal{F}) \cap \bar{S}_0$ invariant by the induced foliation, and C is contracted to a not necessarily unique canonical model. Indeed, the non-uniqueness has its advantages since if Γ is any irreducible invariant curve not factoring through Z'' we can insist that Γ does not intersect the support of C, so it cannot intersect Z'' either, i.e. Z'' is a bunch of fibres of σ_0. Now we want to flop curves in the support of C, but the problem arises that for the curve C_i which should be contracted in $\bar{S}_0$, to flop it, we'd prefer $\bar{S}_0.C_i < 0$. On the other hand let's just guarantee this by blowing up enough an appropriate curve in the same fibre of σ_0. Now we can flop: C_i is contractible, whence there is a well defined semi-simple foliation on a formal neighbourhood of it in $\mathcal{X}$, and since $\bar{S}_0.C_i < 0$ the divisor associated to it is not nil for the semi-simple foliation so we can find a formal divisor F such that C_i has a split neighbourhood (cf. [M1] IV.4) defined by $\bar{S}_0$ and F. Once we flop C_i we preserve the smoothness of our new $\bar{S}_0$ by explicit computation of what this kind of flop looks like. Around Z'' the situation is a bit different. To see why a singular curve flops, observe that by I.4.11/12, S_0 must infact be a weak branch about the curve. It will suffice to work on an étale neighbourhood, so say the curve is a generalised weighted projective line $\mathbb{P}^1(a, b)$ with a, b relatively prime, and $\mathcal{O}(\frac{1}{ab})$ the generator of its Picard group. There is a distinguished case (which is actually easier) of log-type, but the principle is the same, so we'll ignore it and concentrate on the 'general' case where the curve meets exactly one other curve in the fibre of S_0. As such we can arrange that around 0, i.e. $\mathbb{Z}/a$ monodromy, we have coordinates s, y_0 in S_0, and at infinity coordinates t, y_∞ such that the generator ∂ (necessarily existent by II.4.2) has the form,

$$\partial|_{S_0} = y_0^u s^v (s\frac{\partial}{\partial s} - \frac{n}{b} y_0 \frac{\partial}{\partial y_0})$$

Now we require to compute mod (x_0, y_0^{u+1}), where x_0 is the local equation for the weak branch, so we can suppose $x_\infty = z^m x_0,\ y_\infty = z^n x_0$ are equations at infinity mod $(x_\infty, y_\infty^{u+1})$ with z the transition function for $\mathcal{O}(\frac{1}{ab})$, and $m, n \in \mathbb{N}$. Consequently for ∂ to exist we must have,

$$\frac{\partial x_0}{x_0} = 1 + \sum_{i=1}^{u-1} f_i(s) y^i + f(s) y^u$$

where apart from the irrelevant $\deg f_i \le n_i/b$, and by the way $vb = nu + a$, the leading term in f which is precisely of degree v is $-\frac{m}{b} s^v$, and infact to this precision the Jordan form at 0 can be taken to be,

$$x_0(1 - \frac{m}{b} s^v y_0^u)\frac{\partial}{\partial x_0} + y_0^u s^v (s\frac{\partial}{\partial s} - \frac{n}{b} y_0 \frac{\partial}{\partial y_0})$$

From which it's easy to see that the flop exists. Indeed profiting from the possibility of taking roots, we can suppose that $n = 1$, blow up in the curve, repeat the same analysis, and conclude by induction on m, which has now gone down to $m - 1$, in the spirit of II.4.3. Whence, not only is there a flop, but it's a flap, arising from extracting a mth root of x_0 and a nth root of y_0. We therefore eventually reduce to the situation where σ_0 expresses $\bar{S}_0$ as a $\mathbb{P}^1$-bundle over Z_0, which we then contract. The only effect of this is to make the new Z_0 non-scheme like, but otherwise wholly smooth, and we just continue, with the only change being that we contract at the next stage a bundle of weighted projective stacks.

We have thus modified our original situation as follows: there is a stack $\rho : \tilde{X} \to \mathcal{X}$ obtained by a sequence of blow ups in invariant centres, whence necessarily with projective moduli, together with a proper map $\pi : \tilde{X} \to \mathcal{X}_{pr}$, where the latter may not have projective moduli. All these maps are isomorphisms in a neighbourhood containing the generic point of all but finitely many of our original rigid and/or frigid curves determined by the ends Z_0, Z_∞, with $\mathcal{X}_{pr}$ having the wholly desirable property that any curves in the planes S_0^{pr}, T_∞^{pr} intersecting $K_{\mathcal{F}}$ in zero, must themselves be frigid or rigid, and of course, these are the only possibilities for curves meeting our original curves. Notice, of course, that when the ends Z_0 and Z_∞ are the same component in $\mathcal{X}_{pr}$ then apart from the non-gorenstein case with $\mathbb{Z}/2$ monodromy, the curves now have nodes.

Next, let's just write down the Jordan decomposition of a generator ∂ at 0 in the rigid case, where we take standard coordinates on our weighted projective stack analogous to II.4, with $x = 0$, $y = 0$ for the H-N directions, then,

$$\begin{aligned}
\partial_S^0 &= \frac{\partial}{\partial s} - \frac{a}{q}\frac{\partial}{\partial x} \\
\partial_N^0 &= xy^{\tilde{p}}\alpha(s^{\tilde{a}}x^{\tilde{q}}, y^{\tilde{p}})\frac{\partial}{\partial x} + s^{\tilde{a}}x^{\tilde{q}}y\beta(s^{\tilde{a}}x^{\tilde{q}}, y^{\tilde{p}}\frac{\partial}{\partial y}
\end{aligned}$$

where $\tilde{a} = a/(a,q)$, $\tilde{q} = q/(a,q)$ etc., and the local monodromy is $\mathbb{Z}/p$ acting by way of $(s, x, y) \mapsto (\theta s, \theta x, y)$, $\theta^p = 1$. In particular one sees that either the eigenvalue at zero or infinity ramifies at its end unless, $\tilde{p} = \tilde{q} = 1$, and since this can only happen finitely many times, we may suppose that it doesn't occur, so that,

$$\frac{\partial_N}{xy} = \alpha(s^{\tilde{a}}x, y)\frac{\partial}{\partial x} + s^{\tilde{a}}\beta(s^{\tilde{a}}x, y)\frac{\partial}{\partial y}$$

Consequently if the eigenvalue doesn't ramify in zero, ∂_N/xy is a smooth foliation with tangent direction transverse to our curve, but tangent to any other invariant curve through this end, and similarly at infinity. The frigid situation is even more desirable. Indeed if $d \in \mathbb{N}$ is the number of times that we have to blow up to make it rigid, with $\tilde{a}$ as above, then modulo $(d+2)$th powers of the ideal of the curve, the eigenvalue looks like,

$$-\tilde{a} + (s^{\tilde{a}}x - y^{d+1})f$$

for f a function and s, x, y appropriate coordinates at 0. Whence, this ramifies, and there can only be finitely many such curves.

Apart, therefore, from a smooth rigid curve in $\mathcal{X}_{pr}$ with no other curves at its ends, we must eliminate,

II.9.1 Outstanding Cases *That may involve non-ramified eigenvalues*

(i) *The ends at* 0 *and* ∞ *meet, and the curve forms a node in* $\mathcal{X}_{pr}$.

(ii) *There is another smooth rigid curve joining the distinct ends at* 0 *and* ∞.

(iii) *There is only one end, i.e. we're in the* $\mathbb{Z}/2$ *monodromy situation, and there is another curve through the end with the same property.*

In all these cases, a neighbourhood, whether of the node, or the two curves simultaenously, is elliptic, so we cannot a priori guarantee the existence of a single generator of the foliation on the whole neighbourhood, and whence of a unique well defined nilpotent direction. Nevertheless, the situation is not arbitrary, and in the respective cases the following occur,

(i) Organise the reduction to $\mathcal{X}_{pr}$ so that at the penultimate stage there is no extra monodromy at zero, and only $\mathbb{Z}/q$ at infinity. Then there are $a, b \in \mathbb{N}$ equal to the eigenvalues at the end, and another positive integer d, satisfying, $q^2 = a(d - 1/b)$

(ii) Both the eigenvalue and its inverse at either end are in $\mathbb{Z}$.

(iii) Exactly as per (ii).

Since we're supposing non-constant eigenvalue at either end, we therefore conclude,

II.9.2 *Let* $(\mathcal{X}, \mathcal{F})$ *be a given foliated gorenstein normal 3-fold with* $K_{\mathcal{F}}$ *nef., then any map* $f : \mathcal{L} \to \mathcal{X}$ *from a smooth 1 dimensional stack bi-rational at its generic point (more generally a bundle of classifying stacks over the same) and invariant, with* $0 = K_{\mathcal{F}}\cdot_f\mathcal{L} > -\chi(\mathcal{L})$ *either moves in a family covering* $\mathcal{X}$ *or has bounded degree or on* $\mathcal{X}_{pr}$ *defines a smooth rigid curve which meets no other invariant curve whose intersection with* $K_{\mathcal{F}}$ *is zero.*

proof/elaboration We've discussed everything, except maybe when $\mathcal{L}$ moves in a family which covers $\mathcal{X}$. Depending on how big the normal bundle of the generic member of the family is, i.e. either ample and the foliation is given by a vector field, ample$\oplus$trivial and we have a family of vector fields, or trivial, so we get some sort of Ricatti thing, we conclude. $\square$

III. Residual Measures

III.1 Stoke's Theorem

We'll require Stoke's theorem in a little more generality than is standard, albeit that the following remarks undoubtedly constitute the sort of thing that is well known to experts. A more than sufficient degree of generality amounts to currents with measure coefficients on a smooth proper analytic stack $\mathcal{X}/\mathbb{C}$. Evidently a minor variant of [KM] implies the existence of a moduli map $\pi : \mathcal{X} \to X$, with $X/\mathbb{C}$ a proper complex analytic space. Equally we have well defined sheaves of smooth forms $\mathcal{A}^{*,*}$ and normal currents $\mathcal{N}_{*,*}$, i.e. $L_1^{*,*} + d(L_1^{*,*})$, where the lower numeration indicates indexing by dimension, albeit that at times a co-dimension indexing $\mathcal{N}^{*,*}$ may at times be convenient/appear. In any case the local nature of the Poincaré lemma and standard considerations of hypercohomology immediately imply the usual sort of smoothing in co-homology, i.e.

$$H^q(\mathcal{X}_{an}, \mathbb{C}) = \frac{ZA^q(\mathcal{X})}{dA^{q-1}(\mathcal{X})} = \frac{ZN^q(\mathcal{X})}{dN^{q-1}(\mathcal{X})}$$

where A^*, N^* are global forms and normal currents respectively, while the Z at the front indicates those of the same which are closed. Consequently given a globally closed normal dimension p current $d\mu$ we can find a smooth $2\dim_{\mathbb{C}}\mathcal{X} - p$ form τ and a L_1 function γ such that,

$$d\mu = \tau + d\gamma$$

As a result if $d\mu$ is a positive (p,p) current, then in fact $d\gamma$ has measure coefficients, i.e.

$$|d\gamma(f)| << \|f\|_\infty$$

where f is a smooth test form, and, as ever $<<$ is less than up to an inessential constant. Now observe that for U_i, $i \in I$ a properly nested sequence of open sets indexed by some Lebesgue measurable I with the property that the union of the boundaries has positive Lebesgue measure, the measure of the boundary of U_i with respect to $d\mu \wedge \omega^p$, ω smooth positive, is zero with probability one. Consequently we may without difficulty,

(a) 'Cover' $\mathcal{X}$ by disjoint analytic neighbourhoods Δ_α such that $\mathcal{X} \setminus \bigcup \Delta_\alpha$ has not only zero Lebesgue measure, but zero $d\mu \wedge \omega^p$ measure.

(b) Apply the said partition in order to write,

$$d\mu = \sum_\alpha \mathbb{I}_{\Delta_\alpha} d\mu = \tau + \sum_\alpha d(\mathbb{I}_{\Delta_\alpha}\gamma)$$

(c) Take the Δ_α to be polydiscs, regularises the $\mathbb{I}_{\Delta_\alpha}\gamma$ in the usual way, so as to obtain a sequence of regularisation $\gamma_n \to \gamma$ such that for U_i, $i \in I$ as before,

$$d\mu(\mathbb{I}_{U_i} f) = \lim_n \int_{U_i} f d\mu_n$$

with probability one independently of the test function f, where needless to say, $d\mu_n = \tau + d\gamma_n$.

As such we arrive to our immediate goal,

III.1.1 Fact *Let $d\mu$ be a closed positive (p,p) current, f a smooth (or for that matter sufficiently regular) function, and $\mathcal{U}$ an open substack whose boundary has zero $d\mu \wedge \omega^p$ measure, then $d\mu|_{\partial\mathcal{U}}$ is well defined, and Stoke's holds, i.e.*

$$\int_{\mathcal{U}} df d\mu = \int_{\partial\mathcal{U}} f d\mu$$

All applications will of course resolve around an essentially random choice of $\mathcal{U}$ amongst a Lebesgue measurable set of possibilities. Such a choice will, unsurprisingly, be irrelevant and whence will be omitted from the notations and/or discussion without comment.

III.2 Remarks on Kählerianity

All applications of the current chapter will revolve around dimension $(1,1)$ currents with measure coefficients, and, in this situation, there is never any need to know whether the stack $\mathcal{X}$ is Kähler or not. Whence the Kähler free nature of the discussion in III.1. Nevertheless it seems worth a brief detour to give some simple conditions under which this holds. The essentially necessary and sufficient condition is that the moduli X should itself be Kähler, but, rather than get into a long winded discussion about what this should mean let's just say that X embeds into a Kähler manifold with ω the restriction of the Kähler form.

With this hypothesis we'll prove by induction on the dimension of substacks that $\mathcal{X}$ is Kähler. Whence call a reduced substack $\mathcal{Y}$ in $\mathcal{X}$ Kähler if there is a non-negative *appropriate* smooth $(1,1)$ form on $\mathcal{X}$ strictly positive on a neighbourhood of $\mathcal{Y}$. The technical definition of appropriate is as follows: for every substack $\mathcal{Z}$ of $\mathcal{X}$ there is a smooth function (infact many) $\|\mathbb{I}_{\mathcal{Z}}\|$ which after blowing up in $\mathcal{Z}$ (or if one prefers resolve $\mathcal{I}_{\mathcal{Z}}$ by blow ups in smooth centres) looks like, i.e. up to a smooth unit, $|f|$ for $f=0$ a local equation of the total exceptional divisor. A $(1,1)$ form is appropriate if it looks like,

$$N\omega + \sum_i N_i dd^c \|\mathbb{I}_{\mathcal{Z}_i}\|^2$$

for N, $N_i > 0$, and $\mathcal{Z}_i$ some finite bunch of substacks. Plainly our induction is on the dimension of $\mathcal{Y}$, and the existence of appropriate smooth $(1,1)$ forms around the same.

Beginning the induction in dimension zero with $x_1, \ldots, x_n$ coordinates at a point, we take a bump function ρ on a small neighbourhood of its image in X, and note that if G is the stabiliser, then,

$$\|\mathbb{I}_{\mathcal{Z}}\|^2 = \sum_{\sigma\in G} \rho \|x\|^{2\sigma}$$

is a perfectly good example of what we mean with $\|x\|^2 = \sum |x_i|^2$. As such,

$$N\omega + dd^c\|\mathbb{I}_{\mathcal{Z}}\|^2$$

for N sufficiently large gets us off and running since ρ itself is defined on X so $dd^c\rho$, $d\rho d^c\rho$ are dominated by multiples of ω.

Now suppose a not necessarily pure dimensional substack $\mathcal{Y}$ is given, then at its generic points it's étale over its own moduli with $\mathcal{Z} \subset \mathcal{Y}$ a proper substack of every component the locus where this is false. In particular there is a non-negative appropriate $(1,1)$ form $\omega_{\mathcal{Z}}$ strictly positive on a neighbourhood of $\mathcal{Z}$, and for M, N sufficently large,

$$M\omega + N\omega_{\mathcal{Z}} + dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2$$

is strictly positive on a neighbourhood of $\mathcal{Y}$, while being globally non-negative. As such we conclude,

III.2.1 Fact *Suppose that the moduli X of a smooth separated stack $\mathcal{X}/\mathbb{C}$ embeds in a Kähler manifold (e.g. X is projective) then $\mathcal{X}$ is Kähler.*

It therefore goes without saying that all the results of harmonic theory go through verbatim for proper Kähler stacks. It therefore seems appropriate to briefly remark on how algebraic stacks put the essentials of vanishing in rather clear light, particularly the role of $\mathbb{Q}$ divisors, i.e.

III.2.2 Divertimento *Let $X/\mathbb{C}$ be a projective variety with at worst quotient singularities, and $\pi_0 : \mathcal{X}_0 \to X$ the minimal smooth stack structure on X of [V] 2.5 with L a line bundle on X such that as $\mathbb{Q}$ divisors,*

$$\pi_0^* L = H + \sum a_i D_i$$

for $\bigcup D_i$ simple normal crossing on $\mathcal{X}_0$, H big and nef., and $0 \leq a_i < 0$, then,

$$H^q(X, L^\vee) = 0 \ , \text{ for, } q < \dim X$$

proof By the exactness of $(\pi_0)_*$, cf. [AV], it suffices to prove $H^q(\mathcal{X}_0, \pi_0^* L^\vee) = 0$. We first do the case of H ample. Plainly for any smooth stack $\pi : \mathcal{X} \to X$ with X as moduli, either of the standard proofs of Kodaira vanishing work. The analytic one gives that the ℓ_p-norm for $p < 2$ of any harmonic (n, q), $q \geq 1$ form with values in H is zero whenever H is an ample bundle by III.2.1, and the algebraic one by the exactness of π_*, which under the same hypothesis easily implies that H is 'co-homologically ample', [M1] I.3.5. As such if we put $a_i = p_i/q_i$, $(p_i, q_i) = 1$, and form a stack $\pi : \mathcal{X} \to \mathcal{X}_0$ by taking successively q_ith roots of the D_i in some order we can suppose that,

$$H^q(\mathcal{X}, K_{\mathcal{X}} \otimes \pi^* H) = 0 \, , \, q > 0$$

which immediately gives the case where $a_i = 1 - 1/q_i$, $\forall i$ by the ubiquitous exactness of π_*. The next easiest case occurs when no integer between p_i and $q_i - 1$ has a common factor with q_i. When this happens, one looks at the long exact sequence associated to the short exact sequence,

$$0 \longrightarrow K_{\mathcal{X}}(H - \frac{(n_1+1)}{q_1} D_1) \longrightarrow K_{\mathcal{X}}(H - \frac{n_1}{q_1} D_1) \longrightarrow K_{\frac{1}{q_1} D_1}(H - \frac{(n_1+1)}{q_1} D_1) \longrightarrow 0$$

where we suppose that D_1 is the last divisor of which we extract a root $p_1 \neq q_1 - 1$, and $0 \leq n_1 \leq (q_1 - 1) - p_1$, $n_1 \in \mathbb{N}$. In particular we write $\pi_1 : \mathcal{X} \to \mathcal{X}_1$, for the map between $\mathcal{X}$ and the stack $\mathcal{X}_1$ where we extracted the q_ith roots for $i \geq 2$. If we view $\frac{1}{q_1} D_1$ as a $B_{\mathbb{Z}/q_1\mathbb{Z}}$ bundle over its image D_1 in $\mathcal{X}_1$ then,

$$H^q(\frac{1}{q_1} D_1, K_{\frac{1}{q_1} D_1}(H - \frac{(n_1+1)}{q_1} D_1)) = H^q(D_1, K_{D_1}(\pi_1)_*(H - \frac{(n_1+1)}{q_1} D_1))$$

Furthermore the class of H in the relative Picard group of $\mathcal{X}$ over $\mathcal{X}_1$ is, by the definition of H as a bundle on $\mathcal{X}$, $-\frac{a_1}{q_1} D_1$, whence we're reduced to computing,

$$H^q(\mathbb{Z}/q_1, \mathbb{C}(\chi))$$

where χ is the $\mathbb{Z}/q_1$ character, $1/q_1 \mapsto \zeta^{p_1+n_1+1}$, for ζ a primitive q_1th root of unity. By hypothesis, however, this character is fully faithful, so the said groups are zero for all q.

In general, everything reduces to this case by a double induction, i.e. at the D_1 level one may ask for the smallest $m_1 > p_1$ such that m_1 and q_1 have a common factor, and one assumes the result for $\frac{m_1}{q_1} D_1$, while the other induction is simply on the number of D_i's.

To get from H ample to H big and nef., one simply takes an honest ample A, produces an $\epsilon \in \mathbb{Q}_+$ such that $H - \epsilon A = B$ is effective, and for very small δ writes,

$$H = (1 - \delta)(H + \frac{\epsilon\delta}{1-\delta} A) + \delta B$$

Certainly B may be very far from normal crossing, so it's necessary to use a resolution of singularities, but since we can take δ as small as we please, there's no problem. More generally the L_2-vanishing theorem reduces to this one since algebraically this is the right technical condition to permit the use of a resolution of singularities. □

There is, of course, really only one vanishing theorem & even one proof if one understood better the relation between complex conjugation & conjugation under Frobenius. As to how one then goes from the basic theorem to its various refinements isn't really that important.

III.3 Segre Classes

Our goal is to associate to a closed (p,p) current $d\mu$ with measure coefficients and a closed substack $\mathcal{Y}$ of a proper stack $\mathcal{X}/\mathbb{C}$ a positive residual $(p-1,p-1)$ current $s_{\mathcal{Y},d\mu}$ with measure coefficients supported on the projective cone $P(C_{\mathcal{Y}/\mathcal{X}})$ generalising the standard segre class when $d\mu$ is a current of integration. In the course of the discussion, it will emerge that there is a well defined notion of currents with measure coefficients on any proper stack, at least provided that it embeds in a (not necessarily proper) smooth stack. As such for convenience we'll start with $\mathcal{X}$ smooth, and indeed Kắhler if $p>1$, but in reality only the properness of $\mathcal{Y}$ is really important. Before getting underway let's establish,

II.3.1 Notation/Definition *For $\mathcal{Y} \hookrightarrow \mathcal{X}$ as above let $\|\mathbb{I}_{\mathcal{Y}}\|$ be a function on X as described in* III.2, *or indeed of the form $|f|$ up to a bounded unit for f a local equation of the total exceptional divisor obtained by blowing up in $\mathcal{Y}$. Marginally confusingly we'll also employ the notation $\mathbb{I}_*$ for the characteristic function of Lebesgue measurabe sets.*

The key point is to verify,

III.3.2 Claim *Let ω be Kắhler in a neighbourhood of $\mathcal{Y}$, then for any non-negative integers i, j with $i = j = p$,*

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|\le\epsilon}\Big(\frac{dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2}{\epsilon^2}\Big)^i\omega^j d\mu$$

is a well defined, non-infinite, number.

proof The argument is, of course, nothing other than a regurgitation of the standard argument for the existence of Le-long numbers, i.e. the case where $\mathcal{Y}$ is a point, which depends not on the fact that $\mathcal{Y}$ is a point, but that it has no boundary. Evidently the case $i=0$ is trivial, and $\omega^j d\mu \ge 0$ so we're reduced, for a possibly different measure, to the case $i=p$. As such if we put,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} := \int_{\|\mathbb{I}_{\mathcal{Y}}\|\le\epsilon}\Big(\frac{dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2}{\epsilon^2}\Big)^p d\mu$$

then for $\epsilon>\delta$, Stokes implies,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} - \sigma^{\delta}_{\mathcal{Y},d\mu} = \int_{\delta<\|\mathbb{I}_{\mathcal{Y}}\|\le\epsilon}(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu$$

However $dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2$ is as good as positive on $\mathcal{X}\setminus\mathcal{Y}$, i.e. bounded below by the restriction of a smooth form, so indeed $s_{\mathcal{Y},d\mu}(1) := \lim_{\epsilon\to 0}\sigma^{\epsilon}_{\mathcal{Y},d\mu}$ exists, and in fact,

$$s_{\mathcal{Y},d\mu}(1) = \lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon} d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^{p-1}d\mu$$

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} = s_{\mathcal{Y},d\mu}(1) + \int_{0<\|\mathbb{I}_{\mathcal{Y}}\|\le\epsilon}(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu$$

We can also check, as the notation suggests, that there's no dependence on the choice of $\|\mathbb{I}_{\mathcal{Y}}\|$, so say $\|\tilde{\mathbb{I}_{\mathcal{Y}}}\|$ is another such, with, without loss of generality $\|\tilde{\mathbb{I}_{\mathcal{Y}}}\| \ge \|\mathbb{I}_{\mathcal{Y}}\|$, then, in the obvious notation,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} - \sigma_{\tilde{\mathcal{Y}},d\mu}{}^{\epsilon} = \int_{\|\tilde{\mathbb{I}_{\mathcal{Y}}}\|\le\epsilon}(dd^c u)^p d\mu + O\Big(\int_{c\epsilon<\|\tilde{\mathbb{I}_{\mathcal{Y}}}\|\le\epsilon}(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu\Big)$$

for some $c>0$, and u smooth, or even just bounded. As such we conclude to the said lack of dependence by way of the essential positivity of $dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2$, and the Skoda El-Mir theorem, albeit that we'll review the latter theorem from our current standpoint momentarily. □

Several important technical points emerge from the proof, of which the first is,

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|\leq\epsilon} dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2\omega^{p-1}d\mu = 0$$

This rather benign looking observation implies a result that would otherwise be false without the properness of $\mathcal{Y}$, i.e.

III.3.3 Fact *Suppose $\mathcal{Y}$ has a smooth neighbourhood (Kähler if $p>1$) then the notion of current on $\mathcal{Y}$ is well defined (independently of the choice of neighbourhood) and if $d\mu = \mathbb{I}_{\mathcal{Y}}d\mu$, then $d\mu = i_*d\nu$ for $d\nu$ positive (p,p) on $\mathcal{Y}$, and i the embedding on $\mathcal{Y}$ in $\mathcal{X}$.*

proof We first look at the locus $\mathcal{Y}_{reg}$ where $\mathcal{Y}$ is smooth, and profiting from,

$$\mathbb{I}_{\mathcal{Y}}d\mu = \mathbb{I}_{\mathcal{Y}_{reg}}d\mu + \mathbb{I}_{\mathcal{Y}\setminus\mathcal{Y}_{reg}}d\mu$$

aim to do $\mathbb{I}_{\mathcal{Y}_{reg}}d\mu$ first. On $\mathcal{Y}_{reg}$ there is an unambiguous notion of current, and what must be verified (to which I'm indebted to M. Paun for bringing to my attention) is that,

$$\int d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|\tau\mathbb{I}_{\mathcal{Y}_{reg}}d\mu = 0$$

for τ smooth on the neighbouhood. This is, however, clear, since up to an un-important error, for ϵ small,

$$d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|\mathbb{I}_{\mathcal{Y}}d\mu << dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2\mathbb{I}_{\mathcal{Y}}d\mu$$

so there's nothing to do. As such on $\mathcal{Y}_{reg}$, $\mathbb{I}_{\mathcal{Y}_{reg}}d\mu$ really is an honest (p,p) current on $\mathcal{Y}_{reg}$, which has a unique extension by zero over $\mathcal{Y}\setminus\mathcal{Y}_{reg}$ independent of the embedding, and we conclude by a Noetherian induction. $\square$

Of course, as we'll discuss later, $s_{\mathcal{Y},d\mu}(1)$ only depends on $\mathbb{I}_{\mathcal{X}\setminus\mathcal{Y}}d\mu$, but equally the above discussion works in somewhat greater generality, i.e.

III.3.4 Fact *For any $n\in\mathbb{N}$, $\frac{d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|^{2(1-1/n)}}$ is integrable with respect to $\mathbb{I}_{\mathcal{X}\setminus\mathcal{Y}}d\mu$ and indeed,*

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|\leq\epsilon}\frac{d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|^{2(1-1/n)}}\tau\mathbb{I}_{\mathcal{X}\setminus\mathcal{Y}}d\mu = 0$$

for any bounded τ. Idem but for $\frac{d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|^2\log^2\|\mathbb{I}_{\mathcal{Y}}\|}$.

proof Let's do the latter. For ϵ small it's sufficient to look at,

$$-\int_{\|\mathbb{I}_{\mathcal{Y}}\|\leq\epsilon} dd^c\log|\log\|\mathbb{I}_{\mathcal{Y}}\||\omega^{p-1}d\mu$$

as ever ω Kähler if $p>1$. This is, however, equal to,

$$\frac{1}{|\log\epsilon|}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon}\frac{d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|}\omega^{p-1}d\mu - \lim_{\delta\to 0}\frac{1}{|\log\delta|}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\delta}\frac{d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|}\omega^{p-1}d\mu$$

with the interior residue plainly going to zero. $\square$

Which, in itself, reproves the Skoda El-Mir theorem in this particular case (or more generally for almost psh functions whose polar set has no boundary), i.e.

III.3.5 Fact *Let α be a $2p-1$ form whose coefficients are L_2 with respect to $\omega^p d\mu$ then,*

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon}\alpha d\mu = 0$$

proof The measure $\frac{d\epsilon}{\epsilon|\log \epsilon|}$ on $\mathbb{R}_+$ is unbounded on neighbourhoods of zero, so integrate the integral under the limit with respect to it, apply III.3.4 and the Cauchy-Schwarz inequality. $\square$

This not unuseful miscellany dealt with, let's re-interpret the key claim III.3.2 by way of,

III.3.6 Fact *Let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up of $\mathcal{X}$ in $\mathcal{Y}$ then $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\mu$ has a well defined proper transform $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\tilde{\mu}$, i.e. the extension of $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\mu$ by zero across the exceptional divisor, which itself is closed (p,p).*

For $\mathcal{Y} \hookrightarrow \mathcal{X}$ smooth, this is exactly the content of the previous claim, i.e. what has to be verified is that $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\tilde{\mu}$ has finite mass. Now what one might quibble about is exactly what the definition is if $\mathcal{Y}$ is arbitrary. We can, however, resolve $\mathcal{I}_{\mathcal{Y}}$ by a sequence of blow ups in smooth centres to get some $\rho_1 : \mathcal{X}_1 \to \mathcal{X}$, and certainly there is therefore a well defined $d\mu^1$ extending $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\mu$ by zero. Furthermore if $\rho_2 : \mathcal{X}_2 \to \mathcal{X}$ is another such, with $d\mu^2$ the obvious thing, there is a modification $\mathcal{X}^{\#}$, as ever by blow ups in smooth centres, with projections $\rho_i : \mathcal{X}^{\#} \to \mathcal{X}$ dominating both, together with a $d\mu^{\#}$ such that, $d\mu^i = (\rho_i)_* d\mu^{\#}$. In this sense the discussion is wholly unambiguous, but it's also unambiguous no matter how, even locally, we choose to embed $\tilde{\mathcal{X}}$ in a smooth. The key point is that,

$$\lim_{\epsilon\to 0} \int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon} d^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^{i-1} \omega^j d\mu \ , \ i+j=p$$

exists, whence anything of the form,

$$\lim_{\epsilon\to 0} \int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon} f d^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^{i-1} \omega^j d\mu$$

exists for f bounded. Whence, no matter how one chooses to define it $(\rho_i)_* d\mu^i = \mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}} d\tilde{\mu}$, and the final re-interpretation of III.3.2 presents itself,

III.3.7 Fact/Definition *There is a well defined closed positive $(p-1,p-1)$ current $s_{\mathcal{Y},d\mu}$, or $d\sigma_{\mathcal{Y},d\mu}$ in infinitesimal form, on $P(C_{\mathcal{Y}/\mathcal{X}})$ with total mass $s_{\mathcal{Y},d\mu}(1)$. The said measure will be called the segre class and/or residual measure, while, sometimes we'll write $\delta_{\mathcal{Y}} \wedge d\mu$, especially if $\mathcal{Y}$ is Cartier in $\mathcal{X}$.*

Indeed we've already seen in III.3.3 that there's no problem, even locally, talking about measures on singular things provided that we have some knowledge of the behaviour in the normal direction in a smooth embedding, which is indeed the case here.

III.4 Intermission/Explanation

It seems reasonable to pause and to ask what's going on, since as everyone knows it's not possible to define a wedge of δ-functions. The key, however, to observing that what's gone before isn't nonsense is, as we've said, to notice that there's no dependence on the part of $d\mu$ supported on $\mathcal{Y}$. Indeed, manifestly, $\delta_{\mathcal{Y}} \wedge \mathbb{I}_{\mathcal{Y}} d\mu = 0$, so things don't descend to co-homology, although, on the plus side, there exists the possibility of defining a cap product of currents modulo rational equivalence (i.e. limits of δ-functions of rational functions on sub-stacks). Specifically in the case of $(1,1)$ measures $d\mu$, and divisors $\mathcal{D}$ we'd be looking at,

$$\delta_{\mathcal{D}} \cap d\mu = (i_{\mathcal{D}})_* (\delta_{\mathcal{D}'} \cap \mathbb{I}_{\mathcal{D}} d\mu) + \delta_{\mathcal{D}} \wedge d\mu$$

for $\mathcal{D}' \hookrightarrow \mathcal{D}$ a not necessarily effective Cartier divisor, rationally equivalent to $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$. Granted this only works for capping divisors with currents, but by way of the main regularisation theorem of [D1] this can actually be extended to arbitrary positive co-dimension $(1,1)$ currents, at least in the wholly scheme like situation. In any case, this is certainly deserving of attention, albeit that the need for a suitable regularisation theorem in all dimensions means that we'll postpone it for another time, and content ourselves by noting the agreement in dimension $(1,1)$ with standard intersection theory, i.e.

II.4.1 Divertimento *Say everything scheme like to avoid notational complication, with D an irreducible Cartier divisor not containing an irreducible curve C with $D \cap C$ their intersection as a positive zero cycle in the sense of algebraic intersection theory, then we have an identity of measures,*

$$\delta_D \wedge \delta_C = \delta_{D\cap C}$$

proof From what we've already seen it's clear that $\delta_D \wedge \delta_C$ is a positive measure supported on the set theoretic intersection of D and C, so all that's at stake is making sure that the local multiplicities are correct. To compute these, observe, quite generally, that if $Z \subset X$ is a closed subscheme of pure co-dimension $e+1$ defined locally by an ideal $(z_1, \ldots, z_d)$ with $\|z\|^2 = \sum |z_i|^2$ then, locally,

$$\delta_Z = dd^c(\log \|z\|^2 (dd^c \log \|z\|^2)^e)$$

To see this, just blow up in Z to get the exceptional divisor $\pi : P(C_{Z/X}) \to Z$ again of pure dimension, with $\|z\|^2$ yielding a Fubini-Study type metricisation $\bar{L}$ of the tautological bundle on the projective cone, according to which, for f a compact test form on X,

$$\int_X \log \|z\|^2 (dd^c \log \|z\|^2)^e dd^c f = \int_{P(C_{Z/X})} \pi^* i_Z^* f c_1(\bar{L})^e$$

while for a generic point z of Z,

$$\int_{P(C_{Z/X})\otimes\mathbb{C}(z)} c_1(\bar{L})^e = 1$$

Applying these considerations to the computation of $\delta_D \wedge \delta_C$ on a compactly supported test function f with $g = 0$ a local equation for D we find,

$$\delta_D \wedge \delta_C(f) = \lim_{\epsilon\to 0} \int_{P(C_{C/X})\cap(|g|=\epsilon}) f c_1(\bar{L})^{dimX-2} d^c \log |g|^2$$

so that the mass of $\delta_D \wedge \delta_C$ around a point of $D \cap C$ is infact the tautological degree of $P(C_{D\cap C/D})$ at the same, as required. □

Of course the said discussion is much more general than the assertion, so let's observe that modulo notational complication we've even deduced,

III.4.2 Sub-divertimento *Let $\mathcal{C}$ be a pure 1-dimensional closed substack of $\mathcal{X}/\mathbb{C}$ with $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed substack not containing any generic point of $\mathcal{C}$, then as measures,*

$$\delta_{\mathcal{Y}} \wedge \delta_{\mathcal{C}} = \sum_{c\in\mathcal{Y}\cap\mathcal{C}} s(c)\delta_c$$

where $s(c)$ is the standard segre class of $\mathcal{Y}$ at geometric points c of the intersection computed, for example, as per [M1] I.8.6

III.5 Deformation to the Normal Cone

We retake verbatim the notations of III.3, so that $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a closed proper stack and $d\mu$ a positive closed (p,p) current. As one might imagine the deformation of $d\mu$ to the cone $C_{\mathcal{Y}/\mathcal{X}} \to \mathcal{Y}$ is slightly more informative than the segre class. The deformation construction is, of course, the standard one of citef §5, i.e. blow up $\mathcal{X} \times \mathbb{P}^1$ in $\mathcal{Y} \times 0$ with say $\mathcal{W}$ the total space, and $\mathcal{E}$ the total exceptional divisor $\xrightarrow{\sim} P(C_{\mathcal{Y}/\mathcal{X}} \oplus \mathbb{I})$.

In particular if ρ is the projection of $\mathcal{X} \times \mathbb{P}^1$ to $\mathcal{X}$, then $\rho^* d\mu$ is positive $(p+1, p+1)$ and we have a well defined positive closed (p,p) current $d\sigma_{\mathcal{E},d\mu}$ as per III.3.7. The deformation of $d\mu$ to the normal cone is then,

III.5.1 Definition *Let things be as above, and identify $C_{\mathcal{Y}/\mathcal{X}} \hookrightarrow P(C_{\mathcal{Y}/\mathcal{X}} \oplus \mathbb{I})$ with the complement of the hyperplane at infinity, then the deformation, or better specialisation, of $d\mu$ to the normal cone is the positive closed (p,p) current,*

$$d\mu^! := \mathbb{I}_{C_{\mathcal{Y}/\mathcal{X}}} d\sigma_{\mathcal{E},d\mu}$$

Superficially this looks like we need to suppose $\mathcal{X}$ Kähler even if $p = 1$, but, in reality, we don't, i.e.

III.5.2 Fact *This is well defined for $p = 1$ even if $\mathcal{X}$ isn't Kăhler.*

proof For $t = 0$ a local equation for 0 on $\mathbb{P}^1$ we need to consider the integral,

$$\int_{\|\mathbb{I}_{\mathcal{Y}}\|^2+|t|^2\le\epsilon} \frac{dd^c(\|\mathbb{I}_{\mathcal{Y}}\|^2 + |t|^2)}{\epsilon^2}\Psi\rho^* d\mu$$

for Ψ on $\mathcal{X}\times\mathbb{P}^1$ strictly positive in a neighbourhood of $\mathcal{Y}\times 0$. Plainly we can take Ψ to be of the form $\rho^*\psi + dd^c|t|^2$ for $\psi > 0$ on $\mathcal{X}$. Consequently,

$$\Psi\rho^* d\mu = \rho^*(\psi d\mu) + dd^c|t|^2 d\mu$$

is still closed positive $(1,1)$, which is what we needed to define $d\sigma_{\mathcal{E},d\mu}$. $\square$

In addition the computation of the segre class commutes with specialisation to the normal cone, i.e.

III.5.3 Fact *let $[0] \hookrightarrow C_{\mathcal{Y}/\mathcal{X}}$ be the zero section, then,*

$$s_{[0],d\mu^!} = s_{\mathcal{Y},d\mu}$$

proof We'll only do the $(1,1)$ case, since apart from being the only one that we require, it's notationally less fastidious. As such for f a test function on a neighbourhood of $\mathcal{Y}$, by definition,

$$\begin{aligned} s_{[0],d\mu^!}(f) &= \lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{[0]}\|=\epsilon} f d^c \log\|\mathbb{I}_{[0]}\|^2 d\mu^! \\ &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_{|t|=\delta} d^c\log|t|^2 \int_{\|\mathbb{I}_{[0]}\|=\epsilon} f d^c\log\|\mathbb{I}_{[0]}\|^2 d\mu \end{aligned}$$

where we profit from from the fact that $t = 0$ is a local equation for $\mathcal{E}$ around $[0]$. Now the function $\|\mathbb{I}_{[0]}\|$ is exactly $\|\mathbb{I}_{\mathcal{Y}}\|(\|\mathbb{I}_{\mathcal{Y}}\| + |t|)^{-1}$ so identifying the fibre over $t \neq 0$ of the deformation with $\mathcal{X}$ the integral can be re-written as,

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\frac{\epsilon\delta}{(1-\delta)}} f(1-\epsilon) d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2 d\mu$$

whence the assertion. $\square$.

The specialised class $d\mu^!$ has other agreeable properties which reflect the purely algebraic fact that for $\mathcal{V}\hookrightarrow\mathcal{X}$ closed, $C_{\mathcal{Y}\cap\mathcal{V}/\mathcal{V}}$ is pure dimensional. For instance if $\nu : C_{\mathcal{Y}/\mathcal{X}} \to \mathcal{Y}$ is the projection,

III.5.4 Fact *Say $d\mu$ is dimension $(1,1)$ for the sake of a clean statement, then $(\mathbb{I}_{\mathcal{Y}} d\mu)^!$ is just the inclusion by zero, and $\nu_*(\mathbb{I}_{\mathcal{X}\setminus\mathcal{Y}} d\mu)^! = 0$.*

proof The first part just amounts to pulling back $\rho^*\mathbb{I}_{\mathcal{Y}} d\mu$ along $\mathcal{Y}\times 0$ in $\mathcal{Y}\times\mathbb{P}^1$, so this is clear. For the second one takes ω to be a $(1,1)$ form on $\mathcal{X}$ and writes,

$$\int_{\|\mathbb{I}_{[0]}\|\le\epsilon} \nu^*(\omega|_{\mathcal{Y}}) d\mu^! = \lim_{\delta\to 0}\int_{|t|=\delta} d^c\log|t|^2 \int_{\|\mathbb{I}_{\mathcal{Y}}\|\le\frac{\epsilon\delta}{(1-\delta)}} \omega d\mu = 0$$

from which we conclude. $\square$

III.6 Transverse Invariant Measures

It's opportune to remark to what extent the previous discussion improves in the presence of a foliation, i.e. $(\mathcal{X}, \mathcal{F})$ is a smooth analytic stack $/\mathbb{C}$ foliated by curves and $d\mu_{\mathcal{X}/\mathcal{F}}$ is an invariant measure. For example,

III.6.1 Fact *let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a proper invariant sub-stack, then $s_{\mathcal{Y},d\mu}$ is supported on $\mathrm{sing}(\mathcal{F}) \cap \mathcal{Y}$.*

This is clear from the global existence of $s_{\mathcal{Y},d\mu}$ combined with the existence of local foliation coordinates. The said coordinates also imply that $s_{\mathcal{Y},d\mu}$ can be defined without $\mathcal{Y}$ being proper, e.g.

III.6.2 Fact *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally closed and (to fix ideas) irreducible divisor with generic point transverse to $\mathcal{F}$ with $\partial\mathcal{Y}$ contained in the smooth locus, then $s_{\mathcal{Y},d\mu}$ is well defined.*

proof The hypothesis imply that around $\partial\mathcal{Y}$ we're looking at a polydisc Δ with $x = 0$ a local equation for $\mathcal{Y}$, and the other coordinates z_i defining the foliation by way of a fibration. We need only consider over Δ an integral of the form,

$$\lim_{\epsilon \to 0} (\sigma_\epsilon := \int_{|x|=\epsilon} d^c \log |x|^2 d\mu_{\mathcal{X}/\mathcal{F}})$$

so that for $\epsilon > \delta$, Stokes gives that $\sigma_\epsilon - \sigma_\delta$ is an integral of $d^c \log |x|^2$ over $|x| \le \epsilon$, but $\max|z_i|$ =constant. The latter real hypersurface is invariant so $d\mu_{\mathcal{X}/\mathcal{F}}$ restricted to it is zero. $\square$

The same sort of idea works in a lot of other situations, e.g.

III.6.3 Fact *Suppose $\mathrm{sing}(\mathcal{F})$ is compact and $\mathcal{W}$ is a formal subscheme of $\mathcal{X}$ completed in $\mathrm{sing}(\mathcal{F})$ admitting an asymptotic expansion (i.e. it converges after a real blow up in $\mathrm{sing}(\mathcal{F})$) then $s_{\mathcal{W},d\mu_{\mathcal{X}/\mathcal{F}}}$ is well defined.*

Unfortunately as we've seen with the untame beast I.5.9 this condition doesn't always hold when one might wish, so a proof will be omitted. The condition can, however, under hypothesis on $s_{sing(\mathcal{F}),d\mu_{\mathcal{X}/\mathcal{F}}}$ (e.g. equals zero) be relaxed to a somewhat softer notion of asymptotic expansion which permits logarithms, so in the case of the beast $x\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y}$, $\log y$, and in this form could be generally applicable.

IV. Foliated Residues

IV.1 Filtrations on co-homology

Sticking to the convention that a foliation is a saturated sub-sheaf of $\mathcal{T}_{\mathcal{X}}$ we necessarily have a short exact sequence of the form,

$$0 \longrightarrow \Omega^1_{\mathcal{X}/\mathcal{F}} \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}} \longrightarrow 0$$

where $\mathcal{Z}$ is the singular sub-stack of $\mathcal{F}$, and the slightly abusive notation $\Omega^1_{\mathcal{X}/\mathcal{F}}$ is used to denote the necessarily reflexive subsheaf of $\Omega^1_{\mathcal{X}}$ which forms the kernel of the penultimate map on the right. Manifestly, however, we have similar short exact sequences for every $1 \leq m \leq n = \dim X$, which take the form,

$$0 \longrightarrow \Omega^m_{\mathcal{X}/\mathcal{F}} \longrightarrow \Omega^m_{\mathcal{X}} \longrightarrow Q_m \longrightarrow 0$$

where generically $\Omega^m_{\mathcal{X}/\mathcal{F}}$ is $\bigwedge^m \Omega^1_{\mathcal{X}/\mathcal{F}}$, but if needs be, we saturate it to guarantee that Q_m is torsion free. In particular associated to the standard DeRham complex,

$$\Omega^\bullet_{\mathcal{X}} : 0 \longrightarrow \mathcal{O}_{\mathcal{X}} \underset{d}{\longrightarrow} \Omega^1_{\mathcal{X}} \underset{d}{\longrightarrow} \dots \underset{d}{\longrightarrow} \Omega^n_{\mathcal{X}} \underset{d}{\longrightarrow} 0$$

we have a filtration by sub-complexes defined by,

IV.1.1 Definition

$$F^p\Omega^m_{\mathcal{X}} = \begin{cases} 0 & \text{if } p > m \\ \Omega^p_{\mathcal{X}/\mathcal{F}} & \text{if } p = m \\ \Omega^m_{\mathcal{X}/\mathcal{F}} & \text{if } p < m \end{cases}$$

where, of course, $0 \leq p \leq n$, and the one term that we haven't defined $\Omega^0_{\mathcal{X}/\mathcal{F}}$ is nothing other than $\mathcal{O}_{\mathcal{X}}$. Consequently in the usual way we get spectral sequences,

$$A_1^{p,q} := \mathbb{H}^{p+q}(\mathcal{X}, \mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}) \Longrightarrow \mathbb{H}^{p+q}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}}) = H^{p+q}(\mathcal{X}, \mathbb{C})$$
$$A_{1,c}^{p,q} := \mathbb{H}_c^{p+q}(\mathcal{X}, \mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}) \Longrightarrow \mathbb{H}_c^{p+q}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}}) = H_c^{p+q}(\mathcal{X}, \mathbb{C})$$

Fortunately our interest is not in the spectral sequences themselves, but in their E_1 terms, so that it's worth noting,

IV.1.2 Notation $\mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}[p] = \{\Omega^p_{\mathcal{X}/\mathcal{F}} \underset{\nabla}{\longrightarrow} Q_{p+1}\}$

where the reflexive hull of the torsion free sheaf Q_{p+1} is simply $\Omega^p_{\mathcal{X}/\mathcal{F}} \otimes K_{\mathcal{F}}$, and whence the notation ∇ for the differential, whereby it should be thought of as a connection along the leaves.

Implicit in this description is the C_∞ preparation theorem, or, more correctly, the corollary that the ring $\mathcal{A}_{\mathcal{X}}$ of smooth functions is flat over $\mathcal{O}_{\mathcal{X}}$. In particular, therefore, if we consider the canonical $\bar\partial$ resolution of $\mathcal{O}_{\mathcal{X}}$, i.e.

$$\mathcal{A}^{0,*}_{\mathcal{X}} : \mathcal{A}^{0,0}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,1}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \dots\dots \underset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,n}_{\mathcal{X}}$$

then the graded complex associated to the total complexes $F^p\Omega^\bullet_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}^{0,*}_{\mathcal{X}}$ is simply the total complex of $\mathrm{gr}^p\Omega^\bullet_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}^{0,*}_{\mathcal{X}}$, which is acyclic with respect to taking global sections, and, of course, similarly for compact support. If, however, we consider resolving $\mathcal{O}_{\mathcal{X}}$ by distributions, i.e.

$$\mathcal{D}^{0,*}_{\mathcal{X}} : \mathcal{D}^{0,0}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \mathcal{D}^{0,1}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \dots\dots \underset{\bar\partial}{\longrightarrow} \mathcal{D}^{0,n}_{\mathcal{X}}$$

then this is no longer flat, and we get a rather different answer. Indeed with the usual notation that $D_X^{\bullet,*}$ is global distributions, then by virtue of the injectivity of $\mathcal{D}$ the graded complex associated to the total complex becomes,

$$\mathrm{gr}^p(D_X^{\bullet,*})[p]_q = \mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(Q_{n-p}, \mathcal{D}_\mathcal{X}^{n,q}) \oplus \mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\Omega^{n-p-1}_{\mathcal{X}/\mathcal{F}}, \mathcal{D}_\mathcal{X}^{n,q-1})$$

and the differential is $\nabla^\vee \oplus \bar\partial$. To identify the co-homology of this complex, observe that it is simply homotopy classes of maps between the complexes,

$$\mathrm{gr}^{n-p-1}\Omega^\bullet_\mathcal{X}[n-p] \text{ and } \mathcal{D}^{n,*}_\mathcal{X}[q]$$

and since $\mathcal{D}^{n,*}_\mathcal{X}$ is an injective resolution of $K_\mathcal{X}$ this is just $\mathbb{E}\mathrm{xt}^q_{\mathcal{O}_\mathcal{X}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_\mathcal{X}[n-p], K_\mathcal{X})$. Consequently, we obtain rather different spectral sequences,

$$D_1^{p,q} = \mathbb{E}\mathrm{xt}^q_{\mathcal{O}_\mathcal{X}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_\mathcal{X}[n-p], K_\mathcal{X}) \Longrightarrow H^{p+q}(\mathcal{X}, \mathbb{C})$$
$$D_{1,c}^{p,q} = \mathbb{E}\mathrm{xt}^q_{c,\mathcal{O}_\mathcal{X}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_\mathcal{X}[n-p], K_\mathcal{X}) \Longrightarrow H_c^{p+q}(\mathcal{X}, \mathbb{C})$$

Again our interest is in the initial term rather than the spectral sequences themselves, and we may note that the usual pairings between ext's and co-homologies imply the same at the hyper level, so that we have,

IV.1.3 Pairings

$$< \quad >: \begin{array}{ll} A_{1,c}^{p,q} \times D_1^{n-1-p,n+1-q} & \longrightarrow \mathbb{C} \\ A_1^{p,q} \times D_{1,c}^{n-1-p,n+1-q} & \longrightarrow \mathbb{C} \end{array}$$

Better still forms always map to distributions, and in the particular case that $p = n-1$ or 0, it's rather easy to determine the relation. Indeed, we have,

$$A_1^{n-1,q} = \mathbb{H}^q(\mathcal{X}, \mathrm{gr}^{n-1}\Omega^\bullet_\mathcal{X}[n-1])$$

and both the terms in $\mathrm{gr}^{n-1}\Omega^\bullet_\mathcal{X}$ are bundles, so this is just,

$$\mathbb{E}\mathrm{xt}^q(\mathcal{O} \to K_\mathcal{F}[1], K_\mathcal{X})$$

from which we obtain a long exact sequence,

$$\ldots \longrightarrow \mathrm{Ext}^q(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X}) \longrightarrow A_1^{n-1,q} \longrightarrow D_1^{n-1,q} \longrightarrow \mathrm{Ext}^{q+1}(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X}) \longrightarrow \ldots$$

where we can, and will, think of the $\mathrm{Ext}^q(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X})$'s as controlling the obstruction to co-homological smoothing of invariant distributions. Unfortunately, however, the smoothing involves a boundary which may well behave in a less than desirable way if $\mathcal{Z}$ is non-compact. As such the particularly important case of $q = n-1$ deserves special mention by way of,

IV.1.4 Definition *Suppose $\mathcal{Z}$ is compact, then we have a diagram* **RES** *with exact rows,*

$$\begin{array}{ccccc} A_{1,c}^{n-1,n-1} & \longrightarrow & D_{1,c}^{n-1,n-1} & \longrightarrow & \mathrm{Ext}^n_c(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X}) \\ \downarrow & & \downarrow & & \| \\ A_1^{n-1,n-1} & \longrightarrow & D_1^{n-1,n-1} & \xrightarrow{Res} & \mathrm{Ext}^n(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X}) \end{array}$$

If one wants to describe the map *Res* explicitly, then there are a couple of possibilities, particularly if we focus on classes in $D_1^{n-1,n-1}$ defined by transverse invariant measures $d\mu_{\mathcal{X}/\mathcal{F}}$. The first one is to observe that this defines a class in $\mathrm{Hom}(\mathcal{I}_\mathcal{Z}, \Omega^{n-1}_{\mathcal{X}/\mathcal{F}} \otimes \mathcal{D}^{0,n-1})$, so a fortiori in $\mathrm{Ext}^{n-1}(\mathcal{I}_\mathcal{Z}, \Omega^{n-1}_{\mathcal{X}/\mathcal{F}})$, and certainly therefore in $\mathrm{Ext}^n(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z}, K_\mathcal{X})$. Alternatively, as the name *Res* suggests, we can use duality, and starting from a section w of $H^0(K_\mathcal{F} \otimes \mathcal{O}_\mathcal{Z})$ lift to a smooth $(1,0)$ form ω over $\mathcal{X} \setminus \mathcal{Z}$ so that,

$$Res(d\mu_{\mathcal{X}/\mathcal{F}})(\omega) = -\lim_{\epsilon\to 0}\int_{\|1\!\!\mathrm{I}_{\mathcal{Z}}\|=\epsilon} \omega d\mu_{\mathcal{X}/\mathcal{F}}$$

Regretably the above analysis fails to address the basic issue of whether $Res(d\mu_{\mathcal{X}/\mathcal{F}})$ can be a priori defined in $\mathrm{Ext}^n_c(K_{\mathcal{F}}\otimes\mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}})$ without supposing that $\mathcal{Z}$ is compact. There are also various possible refinements that we can introduce by way of the log-complex $\Omega^{\bullet}_{\mathcal{X}}(\log\mathcal{D})$ for $(\mathcal{X},\mathcal{D},\mathcal{F})$ a foliated log-stack, but this still fails to grasp the above nettle, so we'll simply consider such things as we need them.

IV.2 Invariant Bundles

We may aswell immediately specialise the previous discussion to what, after blowing up, becomes the critical case, viz: our stack $\mathcal{X}$ is a neighbourhood of a simple normal crossing divisor $\mathcal{D}$ containing the support of $\mathcal{Z}$. Unsurprisingly the discussion is most elegant when the triple $(\mathcal{X},\mathcal{D},\mathcal{F})$ has log-canonical singularities, but for the moment this isn't immediately relevant. In any case we can basically replace at this juncture $H_c(\mathcal{X})$ by local co-homology groups $H_{\mathcal{D}}(\mathcal{X})$. If we understand the latter meromorphically, then strictly speaking this isn't true, but since we're primarily interested in pairings that depend only on things of the form $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n_{\mathcal{D}}$, $n\in\mathbb{N}$, this makes no essential difference. Regardless we wish to consider what it should mean to have a $\mathcal{F}$ invariant bundle on $\mathcal{X}\setminus\mathcal{D}$. Certainly such a bundle, indeed let's say line bundle to fix ideas, should be equipped with a connection along the leaves, but this is only to be defined off $\mathcal{D}$ so we should admit poles, and whence,

IV.2.1 Definition *A line bundle L on $\mathcal{X}$ is said to be invariant on $\mathcal{X}\setminus\mathcal{D}$ with meromorphic poles, if there is a connection,*

$$\nabla : L \longrightarrow L\otimes K_{\mathcal{F}}(a_j D_j)$$

with $a_j\in\mathbb{N}\cup\{0\}$, D_j the components of $\mathcal{D}$, and, of course, the rule: $\nabla(fs) = f\nabla(s) + \partial_{\mathcal{F}}(f)s$ must be satisfied for $\partial_{\mathcal{F}}$ the composition of holomorphic d with restriction to $K_{\mathcal{F}}$, f a function, and s a section.

Before progressing notice that bundles with meromorphic connections arise from bundles with holomorphic connections in a couple of natural ways, viz:

(a) If we start from a foliation $(\mathcal{X}_0,\mathcal{F}_0)$ with arbitrary singularities, and suppose $\rho:(\mathcal{X},\mathcal{F})\to(\mathcal{X}_0,\mathcal{F}_0)$ is a resolution with canonical singularities, then, in general, a bundle with holomorphic connection along $\mathcal{F}_0$ becomes after pull-back a bundle with meromorphic connection along $\mathcal{F}$.

(b) This time we start with a space $(\mathcal{X},\mathcal{F})$, even with canonical singularities, and let W be an invariant subvariety, then if $\mathrm{sing}(\mathcal{F})\cap\mathrm{W}$ is a divisor, a $\mathcal{F}$ invariant bundle on $\mathcal{X}$ restricts, in general, to a bundle with meromorphic connection for the induced foliation on W.

Apart from these being our motivating examples, the immediate upshot of our definition is that we have a well defined class,

$$c_1(L,\nabla)\in H^1_{\mathcal{D}}(K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}})$$

Indeed for s_α some local generator of L on a scheme like open U_α the ∇s_α patch to a section of $K_{\mathcal{F}}(a_jD_j)\otimes\mathcal{O}_{\mathcal{Z}}$, which maps to a class in $H^1(X, K_{\mathcal{F}}(a_jD_j\mathcal{I}_{\mathcal{Z}})$, and the local co-homology is just the direct limit of these over all divisors supported on cD. in particular for $d\mu_{\mathcal{X}/\mathcal{F}}$ an invariant measure, IV.1.3 guarantees that we have a pairing,

$$< c_1(L,\nabla), d\mu_{\mathcal{X}/\mathcal{F}} >\in\mathbb{C}$$

and what we require is to try and factor this through Res. Manifestly if $\mathcal{D}$ is empty, then by what we've said there is actually a class $\nabla\log s\in H^0(K_{\mathcal{F}}\otimes\mathcal{O}_{\mathcal{Z}})$ affording chern, and by the $p=0$ version of IV.1.4 an exact sequence,

$$H^0(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}) \longrightarrow A_{1,c}^{0,2} \longrightarrow D_{1,c}^{0,2}$$

together with a map $H^1_{\mathcal{D}}(K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}}) \to A_{1,c}^{0,2}$ arising from the degeneration at E_2 (for trivial dimension reasons) of the troncation bête spectral sequence all commuting with the formation of $c_1(L, \nabla)$, and so,

$$< c_1(L, \nabla), d\mu_{\mathcal{X}/\mathcal{F}} >= Res(d\mu_{\mathcal{X}/\mathcal{F}})(\nabla \log s)$$

The general situation is, however, rather more delicate, and the first thing to do is consider the situation around a component D_0 which is not invariant. Here we should suppose that $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ has log-canonical singularities so that D_0 is smooth and everywhere transverse to $\mathcal{F}$, and we require to adapt the co-homological smoothing of $d\mu_{\mathcal{X}/\mathcal{F}}$ to the pole order a_0 of the connection around D_0. As such we have to filter $\Omega^\bullet_{\mathcal{X}/\mathcal{F}}$ according to,

$$\tilde{F}^{n-1}\Omega^{n-1}_{\mathcal{X}} = \Omega_{\mathcal{X}/\mathcal{F}}(-D')$$

where, a little more generally, D' is some divisor of high order of vanishing along the non-invariant components of our supposed log-canonical triple $(\mathcal{X}, \mathcal{D}, \mathcal{F})$. Unsurprisingly, therefore, we get a slightly different group $\tilde{A}^{n-1,q}$ at the E_1 level of the spectral sequence abutting to algebraic DeRham co-homology, which may be written in terms of hyperext by way of,

$$\tilde{A}^{n-1,q} = \mathbb{E}\mathrm{xt}^q(\mathcal{O}_{\mathcal{X}} \to K_{\mathcal{F}}(D')[1], K_{\mathcal{X}})$$

and from there an exact sequence,

$$\tilde{A}^{n-1,n-1} \longrightarrow D^{n-1,n-1} \underset{\widetilde{Res}}{\longrightarrow} \mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \oplus \mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_{D'}(D'), K_{\mathcal{X}})$$

where the direct sum decomposition on the right follows from the smoothness of $\mathcal{F}$ around D'. In order to reduce $\widetilde{Res}$ to Res we appeal to,

IV.2.2 Fact *Suppose things are as above, but in addition the measure $d\mu_{\mathcal{X}/\mathcal{F}}$ has zero segre class around the support of D' then $\widetilde{Res}$ factors through Res.*

proof Just as post IV.1.4 we can use duality to reduce the calculation of the projection of $\widetilde{Res}$ to $\mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_{D'}(D'), K_{\mathcal{X}})$ to that of an integral of the form,

$$\lim_{\epsilon \to 0} \int_{\|1\!\!1_{|D'|}\| = \epsilon} \tau d\mu_{\mathcal{X}/\mathcal{F}}$$

where $|D'|$ is the support of D', and if $x = 0$ is a local equation for the necessarily smooth (if not connected) divisor D' then τ has the form $\rho\frac{dx}{x^n}$, for ρ smooth, and $n \in \mathbb{N} \cup \{0\}$. Manifestly if we can calculate this for ρ compactly supported then a fortiori we're done, while the cases $n \leq 1$ follow by III.3.7. On the other hand everything is smooth, so we may even suppose that x, z_i are a local coordinate system with the foliation given by $\frac{\partial}{\partial x}$, so this follows immediately by integration by parts. □

Clearly with the hypothesis of IV.2.2 there's going to be no problem in factoring the calculation of $< c_1(L, \nabla), d\mu_{\mathcal{X}/\mathcal{F}} >$ through Res as a result of meromorphic poles around the non-invariant part, since no matter how high their order, we can smooth to a still higher order. In order to deal with the invariant terms we proceed rather differently, and independently of any foliation hypothesis, appeal to,

IV.2.3 Fact [Ko] *Let ϕ be a smooth $(n-1, n-1)$ form on a n dimensional polydisc with $x_1 \ldots x_k$ an equation of a simple normal crossing divisor, and τ a $(1,0)$ form such that for $p_i \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq k$, $x_1^{p_1} \ldots x_k^{p_k}\tau$ is smooth, then for each $1 \leq i \leq k$ the limit,*

$$\lim_{\epsilon \to 0} \int_{\substack{|x_i| = \epsilon \; |x_j| \geq \epsilon \\ j \neq i}} \tau \wedge \phi$$

is well defined, and independent of the choice of coordinates.

Notice, critically, that the support of τ need not be compact, and in fact something slightly stronger is even true. Namely for each i there is a smooth function τ_i, independent of the coordinate defining the divisor $x_i = 0$ such that,

$$\lim_{\epsilon\to 0}\int_{\substack{|x_i|=\epsilon\ |x_j|\geq\epsilon \\ j\neq i}} \tau\wedge\phi = \lim_{\epsilon\to 0}\int_{\substack{|x_i|=\epsilon\ |x_j|\geq\epsilon \\ j\neq i}} \tau_i \bigwedge_{1\leq j\neq i\leq k} \frac{dx_j d\bar{x}_j}{x_j^{p_j}} \bigwedge_{a>k} dx_a d\bar{x}_a$$

In particular if for a specific choice of coordinate system τ_i is zero almost everywhere, then all of these residues are zero. This has a manifest application to computing the pairing of any class α in $A_1^{n-1,n-1}$ with $c_1(L,\nabla)$ since as post IV.1.4 the pairing may be written in the form,

$$-\lim_{\epsilon\to 0}\int_{\|1\!\!1_{|\mathcal{D}|}\|=\epsilon} \tau\wedge\alpha$$

of course we should a priori insist that α vanishes to sufficiently high order on non-invariant components, and since τ satisfies precisely the hypothesis of IV.2.3, the calculation around the invariant components when performed in coordinates adapted to the foliation over the smooth locus of the same is zero by the above discussion, and the obvious type reason that a is $(n-1,n-1)\oplus(n,n-2)$, whence it's zero everywhere. Consequently, we deduce,

IV.2.4 Fact *Let $d\mu_{\mathcal{X}/\mathcal{F}}$ be a transverse invariant measure, and (L,∇) a line bundle with leafwise meromorphic connection, then if we have log-canonical foliation singularities around the polar locus and $d\mu_{\mathcal{X}/\mathcal{F}}$ has zero segre class on any non-invariant poles, the pairing, $< c_1(L,\nabla), d\mu_{\mathcal{X}/\mathcal{F}} >$ depends only on $Res(d\mu_{\mathcal{X}/\mathcal{F}})$.*

Notice that this is and isn't the algebraic De-Rham theorem in the relative context $\pi : \mathcal{X} \to [\mathcal{X}/\mathcal{F}]$. It plainly isn't stricta dictum by virtue of the fact that we supposed zero segre class along non-invariant divisors. This is, however, trivially rectified since such divisors must be everywhere transverse to $\mathcal{F}$ when the singularities are log-canonical, so one can smooth the residual measure on the divisor, propagated along the foliation, and arrive to pure dependence on $Res(d\mu_{\mathcal{X}/\mathcal{F}})$ by IV.2.3. If one doesn't do this, as Kontsevich pointed out to me, the answer could be infinite in terms of integrals such as those post IV.1.4. On the other hand this is a slightly irrelevant nicety since the structure here is no more or less complicated than a punctured one dimensional disc. The real issue is that except for very special cases π, or more correctly $[\mathcal{X}/\mathcal{F}]$ doesn't exist as a stack in analytic spaces around the singularities. Consequently the words 'invariant bundle' are ambiguous, depending on whether one asks for connections with values in $K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}}$ or just $K_{\mathcal{F}}$. Both possibilities have enough injectives, and both give a sensible notion of π_*, manifestations of which are the different spectral sequences of IV.1. The more important part, therefore, of IV.2.4, which functorially speaking cannot be considered a relative algebraic De-Rham theorem, is that this is the unique possible ambiguity.

IV.3 Isolated Residues

The purpose of this chapter is wholly exemplary. It's objects are two fold: to indicate the difficulties involved in calculating the residue symbol, and to avoid people bombarding me with e-mails claiming that they cannot understand the proof for surfaces in [M3]. Whence let $(\mathcal{X},\mathcal{F})$ be a foliated smooth stack with not just canonical singularities but satisfying the embedded resolution property I.6.7, with I.3.1 in force. As such an isolated singularity where the residue symbol cannot be immediately related to the segre class $s_{\mathcal{E},d\mu}$ along the exceptional divisor has semi-simple part of rank $\dim\mathcal{X}-1$ with, say, eigenfunctions $x_1,\ldots,x_{n-1}$. Consequently if y is a local equation for $\mathcal{E}$, which, by the way, is necessarily smooth at such a singularity, with ∂ a local generator for the foliation, then $\partial y = y^{p+1}u(y) \bmod (x_1,\ldots,x_{n-1})$, $p\in\mathbb{N}$ with u a unit. Plainly the extra dependence of ∂y on the x_i's is pretty irrelevant, e.g. we can just blow up till we get log-flat or $y^{p+1}|\partial y$. Whence we can normalise things so that,

$$\partial = y^{p+1}\frac{\partial}{\partial y} + \lambda_i x_i \frac{\partial}{\partial x_i} + \text{stuff}$$

where $\lambda_i \in \mathbb{C}^\times$, and stuff will prove to be on the irrelevant side. As ever this formula may not be analytically convergent, but a sufficiently good approximation to it modulo the maximal ideal of the point will be more than adequate. Indeed ignoring this subtlety for the moment consider what might be termed the *basic trick*, i.e. using $s_{\mathcal{E},d\mu} = 0$ gives,

$$\int_{\substack{|y|=\epsilon \\ |x_i|\leq\epsilon}} d^c \log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}} = -\sum_i \int_{F_i} d^c \log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}} = -\sum_i \frac{1}{2\pi}\int_{F_i} \Im(y^p \frac{dx_i}{\partial x_i}) d\mu_{\mathcal{X}/\mathcal{F}}$$

where F_i is the face $|x_i| = \epsilon$, $|y| \leq \epsilon$, $|x_j| \leq \epsilon$, $i \neq j$, and what we require is to ensure that on F_i, $|\partial x_i|$ is bounded below by ϵ. The only obstruction to this is the first order terms in the Jordan decomposition, which can easily be killed by sequential weighted blowing up determined by Jordan blocks. Furthermore the conclusion only requires a very good approximation modulo the maximal ideal, and so we obtain,

$$0 \leq \int_{|y|=\epsilon\,,\,|x_i|\leq\epsilon} d^c \log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}} \leq \epsilon^p o(\epsilon)$$

where the $o(\epsilon)$ is an unknown function, necessarily going to zero with ϵ by virtue of the hypothesis $s_{\mathcal{E},d\mu} = 0$. Consequently we deduce,

IV.3.1 Fact *Suppose $(\mathcal{X},\mathcal{F})$ is a foliated stack with the embedded resolution property and $z \in \mathrm{sing}(\mathcal{F})$ an isolated singularity then for $\lambda_z(d\mu_{\mathcal{X}/\mathcal{F}})$ just the standard Le-long number of a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ we have,*

$$\lambda_z(d\mu_{\mathcal{X}/\mathcal{F}}) = 0 \Longrightarrow Res_z(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$$

Notice that this argument does not easily generalise to non-isolated singularities, i.e.

IV.3.2 Difficulties *The non-isolated problems are at least,*

(a) *Unlike standard residue theory one cannot just reduce from non-isolated to isolated by a generic cutting argument, since this would amount to finding invariant functions at each point of the singularity. Even ignoring convergence issues, this cannot always be done. Indeed in each dimension there are new 'non-cuttable' singularities proper to the dimension in the form of higher saddles, e.g. $x\frac{\partial}{\partial x} + y^p(z\frac{\partial}{\partial z} + y^{q+1}\frac{\partial}{\partial y})$, $p, q \in \mathbb{N}$ which don't even have this property formally, while untame beasts cannot posses it analytically. Whence,*

(b) *The basic trick is in general obstructed by the curvature of $\mathcal{E}$ around $\mathcal{Z}$.*

(c) *Already in dimension 3 a purely local study of the non-isolated case looks very hard. Indeed it's not even clear, cf. post IV.1.4, that the residue symbol is even defined locally. A possible approach appears to be a combination of the holonomy methodology of [B1] with the basic trick. In dimension 2 the latter is an upside down, and 1st order, version of the former, but this 'equivalence' no longer holds in dimension 3, where at least both seem to be required. One should note, however, that strict justification of the use of holonomy tends to be quite involved, e.g. for 2-dimensional saddles one needs [H&], which is false at beasts, together with large chunks of [MR].*

IV.4 The Birational Groupoid

Given a foliated stack $(\mathcal{X},\mathcal{F})$ there is constructed in [M1] II.1 an infinitesimal birational groupoid $\tilde{\mathfrak{F}} \rightrightarrows \mathcal{X}$ which in an appropriate sense completes the standard groupoid across the singularities. The essential of the construction is as follows: firstly consider the infinitesimal jet groupoid, $\mathfrak{P}_{\mathcal{X}} \rightrightarrows \mathcal{X}$, which unlike $\mathcal{X} \times \mathcal{X}$ has the property that $\mathcal{X}$ actually embeds diagonally in $\mathfrak{P}_{\mathcal{X}}$. Now away from sing($\mathcal{F}$) the standard infinitesimal groupoid is constructed by adding an infinitesimal germ in the foliation direction to every point of the diagonal Δ so as to obtain a formal sub-stack of $\mathfrak{P}_{\mathcal{X}}$. In general the Zariski closure over the singularities is all of $\mathfrak{P}_{\mathcal{X}}$, so what we do is blow up $\mathfrak{P}_{\mathcal{X}}$ in the diagonal embedding of sing($\mathcal{F}$) with implied nilpotent structure, or, if one likes smooth things, resolve this by a sequence of blow ups in smooth centres, and then to the proper transform $\tilde{\Delta}$ of Δ add curves in the required direction. Taking the source and sink to be the projections to $\mathcal{X}$, we therefore get our groupoid, $\tilde{\mathfrak{F}} \rightrightarrows \mathcal{X}$, albeit with the caveat that the identity is only a bi-rational map.

We wish to extend this to an analytic germ, and since this is exactly the kind of area where stacks and spaces differ we spell it out. To begin with, consider the situation at the level of the moduli, say, $\hat{F} \rightrightarrows X$. The convergent Frobenius theorem applied on étale neighbourhoods of $\tilde{\mathfrak{F}}$ followed by the taking of invariants shows that every point of $\hat{F}$ posseses an analytic extension, and these inturn, by the unicity of analytic continuation, glue to some analytic space F. Now go back up to a point $\xi \in \tilde{\mathfrak{F}}$ with $\hat{U}_\xi$ an etale neighbourhood of the same, such that $\tilde{\mathfrak{F}}$ is locally a formal classifying stack of the form $[\hat{U}_\xi/G_\xi]$ for some finite group G_ξ. Again the Frobenius theorem gives us an analytic extension U_ξ of $\hat{U}_\xi$, and we can consider the groupoid,

$$\coprod_{\xi,\eta} \widetilde{U_\xi \times_F U_\eta} \rightrightarrows \coprod_\xi U_\xi$$

where ˜ denotes normalisation. Completed in $\tilde{\Delta}$ the classifying stack of this groupoid is just $\tilde{\mathfrak{F}}$, so indeed we have our desired analytic extension $\tilde{\mathfrak{F}}_{an}$ (or just $\tilde{\mathfrak{F}}$ if there is no danger of confusion), which a postiori can be locally expressed as $[U_\xi/G_\xi]$ for possibly smaller U_ξ.

IV.5 Harmonic theory on foliated varieties

We continue to let $(\mathcal{X}, \mathcal{F})$ be a foliated smooth (Kăhler) stack, but we fix a (supposed existent) transverse invariant measure $d\mu$. We could, of course, avoid this and proceed à la Connes, [Co], to do things independent of a particular realisation. Nevertheless we have specific measures with specific properties in mind so we'll proceed somewhat more simplistically.

As ever one builds up from $C^\infty(\mathcal{X})$. Nevertheless distributions and so forth should all be understood relative to $d\mu$. In particular we're only interested in the initial terms derived from the filtration of IV.1 on smooth forms, i.e.

$$\mathrm{gr}^{i,\bar{j}} = \frac{F^{i,\bar{j}} A^*_{\mathcal{X}}}{F^{i,\bar{j}-1} A^*_{\mathcal{X}} + F^{i-1,\bar{j}} A^*_{\mathcal{X}}} \quad i, j \in \{0, 1\}$$

where $A^*_{\mathcal{X}}$ is global smooth forms on $\mathcal{X}$. Consequently when we talk about a distribution $\mathrm{dr}^{i,\bar{j}}$ this will mean an element of $\mathrm{gr}^{1-i,1-\bar{j}}$ calculated along $d\mu$, i.e.

IV.5.1 Definition *A distribution of type $(i, \bar{j})$ of regularity class (p, q) is an element of the dual of $gr^{1-i,1-\bar{j}}$ in the sobolev (p, q) norm along $d\mu$, i.e. the qth derivatives are $d\mu$ ℓ_p. In general all statements about sobolev spaces, ℓ_1 spaces, etc. are to be understood in this sense.*

The restriction of the standard differential operators gives, therefore, a Hodge diamond,

$$\begin{array}{ccc} & \mathrm{gr}^{0,\bar{0}} & \\ \partial \swarrow & & \searrow \bar{\partial} \\ \mathrm{gr}^{1,0} & & \mathrm{gr}^{0,\bar{1}} \\ \bar{\partial} \searrow & & \swarrow \partial \\ & \mathrm{gr}^{1,\bar{1}} & \end{array}$$

so that if we think of $\mathrm{gr}^{i,\bar{j}}$ as embedded in smooth sections of $K^i_{\mathcal{F}} \otimes \bar{K_{\mathcal{F}}}^j$, with $\partial_{\mathcal{F}}$ a local generator of the foliation these can be written as $\partial_{\mathcal{F}} \otimes \partial^{\vee}_{\mathcal{F}}$ etc. Now the $*$ operator from $\mathrm{gr}^{1,0}$ to $\mathrm{gr}^{0,\bar{1}}$ is plainly just complex conjugation. Whence if H is the class of a Kähler metric, and we desire the standard formula,

$$\int \omega \wedge *\tau d\mu = \int < \omega, \tau > H d\mu$$

we see that the right choice of metric on $K_{\mathcal{F}}$ is a norm of the form smooth$\times 1/\|\mathbb{I}_{\mathcal{Z}}\|$ where $\mathcal{Z}$ is the singular substack with nilpotent structure, and $\|\mathbb{I}_{\mathcal{Z}}\|$ is as per III.3.1. As a particular consequence, in terms of the local generator $\partial_{\mathcal{F}}$ the Laplacian on $\mathrm{gr}^{0,0}$ has the form,

$$\Delta f = \frac{1}{\|\mathbb{I}_{\mathcal{Z}}\|^2} \partial_{\mathcal{F}} \bar{\partial}_{\mathcal{F}}(f)$$

and not as one might suspect $\partial_{\mathcal{F}} \bar{\partial}_{\mathcal{F}}(f)$. More generally, on identifying an element of $\mathrm{gr}^{i,\bar{j}}$ with its image in $K^i_{\mathcal{F}} \otimes \bar{K_{\mathcal{F}}}^j$ the other Laplacians look like,

$$\Delta^{1,0} f = \partial_{\mathcal{F}} \Big(\frac{1}{\|\mathbb{I}_{\mathcal{Z}}\|^2} \bar{\partial}_{\mathcal{F}}(f) \Big) \ , \ \ \Delta^{1,1} f = \partial_{\mathcal{F}} \bar{\partial}_{\mathcal{F}} \Big(\frac{1}{\|\mathbb{I}_{\mathcal{Z}}\|^2} f \Big)$$

where we've been deliberately vague about which Laplacian we're using, since, infact, the Hodge-Riemann bilinear relations continue to hold, so they're all the same.

Certain things are, therefore, pure formalism. For example for $s \in \mathbb{N}$ we have sobolev spaces,

$$\mathrm{gr}^{i,\bar{j}}_{2,s} := \{ f : \|\Delta^p f\|^2 < \infty \, , \, 0 \leq p \leq s \}$$

and their duals $\mathrm{gr}^{i,\bar{j}}_{2,-s}$ with an inverse limit $\mathrm{gr}^{i,\bar{j}}_{2,\infty}$ and a direct limit $\mathrm{gr}^{i,\bar{j}}_{2,-\infty}$. Whence, essentially by definition, if $\Delta f = g$ for $g \in \mathrm{gr}^*_{2,s}$, then $f \in \mathrm{gr}^*_{2,s+2}$. What, however, is not evident is that the inclusions,

$$\mathrm{gr}^*_{2,s} \longrightarrow \mathrm{gr}^*_{2,t}$$

for $s > t$ are compact. Plainly there isn't an obvious way to relate this to standard theory around the singularities, and whence the lack of a general nonsense proof in the spirit of [Co] for the existence of the Green's operator. Fortunately, classical formulae for the Green's operator are well known, and these will prove better adapted to our present circumstances.

IV.6 The Green's Operator

We continue to suppose that $(\mathcal{X}, \mathcal{F})$ is a smooth proper Kähler stack over $\mathbb{C}$ foliated by curves. The birational groupoid $\tilde{\mathfrak{F}}$ may equally well be observed to be the saturation of $\tilde{\Delta}$ in either of the foliation directions obtained by pulling back a generator of $\mathcal{F}$ via the standard projections of the modified jet groupoid. In particular if $d\mu$ is a transverse invariant measure and $d\tilde{\mu}$ is the extension over $\tilde{\Delta}$ by zero, then there is a well defined saturation on $\tilde{\mathfrak{F}}$ which agrees with $s^* d\mu = t^* d\mu$ over the smooth locus. Due to the lack of ambiguity we therefore have,

IV.6.1 Notation/Definition *The measure $d\mu_{\tilde{\mathfrak{F}}}$ so constructed will, lest there be cause for confusion, be denoted as $d\mu$.*

Now we can construct a Green's operator in the standard way. Specifically denote by $\|\mathbb{I}_{\tilde{\Delta}}\|$ a function as per III.3.1 which is eventually zero on $\tilde{\mathfrak{F}}$, and put,

$$G : \mathrm{gr}^{0,0} \longrightarrow \mathrm{gr}^{0,0}_{?} : f \longmapsto \int_{\tilde{\mathfrak{F}}} s^* f t^* \omega \log \|\mathbb{I}_{\tilde{\Delta}}\|^2 s^* H d\mu$$

where the ? denotes a deliberate vagueness about where the Green's operator takes its values. Let's proceed, therefore, to a standard sort of computation in order to investigate how far this formula is from being correct, i.e.

$$G(f)(dd^c\phi) = \int_{\tilde{\mathfrak{F}}} d(d^c(t^*\phi)\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 s^*(fH)d\mu) - d(t^*\phi)d^c\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 s^*(fH)d\mu$$

Plainly, therefore, we need to know that the first of these terms cannot develop a residue along $\tilde{\Delta}$. At smooth points, however, this is clear since the term in question is bounded on $\|\mathrm{I\!I}_{\tilde{\Delta}}\| = \epsilon$ by,

$$(s^*H)\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 \log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 d^c\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 d\mu$$

for ϕ smooth. Equally at singular points the situation is no more complicated, since part from the above term there is another of the form,

$$\|\mathrm{I\!I}_{\tilde{\Delta}}\|\log\|\mathrm{I\!I}_{\tilde{\Delta}}\| \times \{\text{smooth}\} \times d\mu$$

so, again, our generalities on residual measure oblige this to go to zero. Proceeding with the calculation, we therefore have,

$$G(f)(dd^c\phi) = \lim_{\epsilon\to 0}\int_{\|\mathrm{I\!I}_{\tilde{\Delta}}\|=\epsilon} t^*\phi s^*(fH)d^c\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 d\mu + \int_{\tilde{\mathfrak{F}}} t^*\phi s^*(fH)dd^c\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 d\mu$$

where here and elsewhere we're notationally a bit loose about the difference between dd^c and c_1, and we need to know that the residue on the left is as it should be. Again at smooth points this is clear, while at singular points the extra terms once more give things like $\|\mathrm{I\!I}_{\tilde{\Delta}}\|d^c\log\|\mathrm{I\!I}_{\tilde{\Delta}}\|^2 \times \{\text{smooth}\} \times d\mu$. Indeed even if f isn't smooth as per III.3.5, L_2 is more than adequate to kill either this term or the previous one, and for that matter to deduce that the Green's operator takes L_2 forms to L_2 forms. Whence,

IV.6.2 Fact *Let $L_2^{0,0}$ be the space of L_2 forms deduced from $gr_2^{0,0}$, and suppose* $\mathrm{sing}(\mathcal{F})$ *is compact, then there is a Green's operator $G : L_2^{0,0} \to L_2^{0,0}$ defined as above satisfying,*

$$\Delta G = \mathrm{id} - *t_*(s^*H \wedge c_1(\tilde{\Delta}))$$

Plainly the situation for $L_2^{1,1}$ is just as clean. Indeed if to clarify the notation $G^{0,0}$ etc. is the Green's operator in the appropriate dimension, $*G^{0,0}*$ is a perfectly good operator on $L_2^{1,1}$, and $\Delta^{1,1}G^{1,1} = \mathrm{id} - t_*(s^*H \wedge c_1(\tilde{\Delta}))*$. The situation with respect to the $L_2^{1,0}$ and $L_2^{0,1}$ is that just for a Riemann surface it's on the tricky side to write down an explicit formula since $K_{\mathcal{F}}$ may not have a global smooth generator. Nevertheless locally this isn't a problem, so that combined with standard stuff about conjugate operators we conclude,

IV.6.3 Fact/Summary *Let $(\mathcal{X},\mathcal{F})$ be a foliated smooth stack with projective moduli, or more generally a germ around a compact singular locus in a complete Kǎhler metric H, then there are operators $*$, Λ, G satisfying the usual sort of relations, and the latter possibly a little different from the above, together with a decomposition,*

$$L_{2,d\mu}^{\bullet,\bullet} = H_{2,d\mu}^{\bullet,\bullet} \oplus \Delta G L_{2,d\mu}^{\bullet,\bullet}$$

where $L_{2,d\mu}^{\bullet,\bullet}$ are the L_2 forms with respect to a transverse invariant measure $d\mu$, $H_{2,d\mu}^{0,0} (= *^{-1}H_{2,d\mu}^{1,1})$ is the μ holomorphic functions along the leaves; and $H_{2,d\mu}^{1,0} (= *^{-1}H_{2,d\mu}^{0,1})$ is the μ holomorphic forms.

IV.7 Almost psh functions

The basic moral of the previous discussion is that notwithstanding the complication of the singularities harmonic theory in the foliation direction in the presence of a transverse invariant measure $d\mu$ just looks like that on a Riemann surface. As such, and with rather greater pertinence, we wish to examine to what extent the theory of psh functions still holds. Our interest is rather specific, so we'll confine ourselves to what is directly relevant, i.e. closed substacks $\mathcal{Y}$ of $\mathcal{X}$ where the residual measure $s_{\mathcal{Y},d\mu}$ has been shown to exist. As ever let $\|\mathbb{I}_{\mathcal{Y}}\|$ be a function as per III.3.1 so that by hypothesis $dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 d\mu$ is also known to exist as a $d\mu$ bounded $(1,1)$ form - i.e. the chern class for an actual divisor. Whence for ψ a smooth test $(1,1)$ form, consider,

$$\int_0^\rho \frac{dr}{r} \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq r} s^* dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 t^* \psi d\mu$$

In particular if ψ is $d\mu$ ℓ_p then the first of these integrals can be bounded by something of order,

$$\left(\int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq r} s^* H t^* H d\mu\right)^{\frac{1}{q}} \|\psi\|_p r^{2/p}$$

where as ever $1/p + 1/q = 1$. Whence the total integral certainly exists, and for any smooth ψ is equal to,

$$-\int_0^\rho \frac{dr}{r} s^* s_{\mathcal{Y},d\mu}(t^*\psi \mathbb{I}_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq r}) + \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq \rho} t^* \psi s^* d \log \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 d\mu$$

Now let's consider the first of these terms. As with the discussion of the Green's operator it is comfortably bounded by terms which eliminate the pole in r at the singularities plus a main term which is the same everywhere of order,

$$\int_0^\rho \frac{dr}{r} \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq r} d\|\mathbb{I}_{\mathcal{Y}}\| d^c \|\mathbb{I}_{\mathcal{Y}}\| s^* ds_{\mathcal{Y},d\mu}$$

where $s_{\mathcal{Y},d\mu}$ is the infinitesimal form of the residual measure $s_{\mathcal{Y},d\mu}$. Again the general theory of residual measure assures us that the first integral is of order r^2, so the whole thing exists, whence the 2nd term occuring in the re-writing of our original integral exists too. Indeed, $t_*(s^* d \log \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2)$ is absolutely integrable. Applying Stokes to the said term, gives, in turn,

$$\int_{\|\mathbb{I}_{\tilde{\Delta}}\| = \rho} t^* \psi s^* \log \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2 d\mu - \lim_{\epsilon \to 0} \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \|\|\mathbb{I}_{\mathcal{Y}}\| = \epsilon} t^* \psi s^* \log \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2 d\mu$$
$$+ \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq \rho} t^* \psi s^* \log \|\mathbb{I}_{\mathcal{Y}}\| c_1(\tilde{\Delta}) d\mu$$

There are several things to check. To begin with we view our neighbourhood of $\tilde{\Delta}$ as projecting to $\tilde{\Delta}$, at least locally, via t and seek to integrate terms such as $s^* \log \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2$, or $s^* \log \|\mathbb{I}_{\mathcal{Y}}\| c_1(\tilde{\Delta})$ in the ordinary Lebesgue sense over the fibres. The latter is trivially $O(1)$, and the former over the boundary $\|\mathbb{I}_{\tilde{\Delta}}\| = \rho$ is no worse than $\log \rho$. Consequently if we a priori take ψ to be sufficiently zero on $\|\mathbb{I}_{\mathcal{Y}}\| = 0$, then the middle residue not only exists but is bounded by the sup-norm of ψ. Whence $\log \|\mathbb{I}_{\mathcal{Y}}\|$ is actually ℓ_1 with respect to $d\mu$ - a fact we could have already concluded from our previous considerations on the Green's operator- and, moreover,

$$\int_{\mathcal{X}} \psi \log \|\mathbb{I}_{\mathcal{Y}}\| d\mu = -\int_0^\rho \frac{dr}{r} \int_{\|\mathbb{I}_{\tilde{\Delta}}\| \leq r} s^* (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2) t^* \psi d\mu + \int_{\|\mathbb{I}_{\tilde{\Delta}}\| = \rho} t^* \psi s^* \|\mathbb{I}_{\mathcal{Y}}\| d^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2 d\mu$$

$$+\int_{\|\mathbb{I}_{\tilde\Delta}\|\le\rho} t^*\psi s^*\log\|\mathbb{I}_{\mathcal{Y}}\|c_1(\tilde\Delta)d\mu+\int_0^\rho\frac{dr}{r}\int_{\|\mathbb{I}_{\tilde\Delta}\|\le r} t^*\psi s^*\sigma_{\mathcal{Y},d\mu}$$

Now everything on the right is comfortably bounded by the $d\mu$ ℓ_p norm of ψ for any p, except possibly the residual measure term. For any ℓ_1 ψ, however, we can view the integral over $\|\mathbb{I}_{\tilde\Delta}\|\le r$ against the residual measure as a limit of some bounded function,

$$y\longmapsto\int_{s^{-1}(y)\cap\|\mathbb{I}_{\tilde\Delta}\|\le r} t^*(\psi d\mu)$$

for y on $\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon$, averaged over a standard circle measure, so that the whole integral over $\|\mathbb{I}_{\tilde\Delta}\|\le r$ is bounded by the ℓ_1 norm of ψ. Consequently if ψ is actually ℓ_p we can achieve,

$$\int_{\|\mathbb{I}_{\tilde\Delta}\|\le r} t^*\psi s^*\sigma_{\mathcal{Y},d\mu}<<|\psi|_p\Big(\int_{\|\mathbb{I}_{\tilde\Delta}\|\le r} d\|\mathbb{I}_{\tilde\Delta}\|d^c\|\mathbb{I}_{\tilde\Delta}\|s^*\sigma_{\mathcal{Y},d\mu})^{1/q}<<\psi|_p r^{2/q}$$

where as ever $1/p+1/q=1$, and $<<$ is up to some irrelevant constants. Consequently we deduce,

IV.7.1 Fact *Let $\mathcal{Y}\hookrightarrow\mathcal{X}$ be a closed substack of a proper stack then $\log\|\mathbb{I}_{\mathcal{Y}}\|$ is $(\mathbb{I}_{\mathcal{X}\setminus\mathcal{Y}})d\mu-\ell_p$ for any p, so in particular $\partial\|\mathbb{I}_{\mathcal{Y}}\|/\|\mathbb{I}_{\mathcal{Y}}\|$ is ℓ_1.*

proof Just apply Cauchy-Schwarz using III.3.4. $\square$

We next concentrate on a special case, viz:

IV.7.2 Hypothesis *Suppose $\mathcal{X}$ itself is obtained by way of a resolution in smooth invariant centres of the singular substack of the foliation on some $\mathcal{X}_0$, so that $\mathcal{Y}$ is some component of the total exceptional divisor, and, moreover the segre class of the measure around the total exceptional divisor is zero. In particular $d\mu$ has no support on the exceptional divisor.*

As such, if, rather more correctly we write $dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2$ as $-c_1(\mathcal{Y})$ then one observes that the bound for the ℓ_p norm of $\log\|\mathbb{I}_{\mathcal{Y}}\|$ is almost uniform in p. Indeed the only problem ($c_1(\mathcal{Y})$ is after all smooth) occurs on the integral over $\|\mathbb{I}_{\tilde\Delta}\|=\rho$, where it's possible to get terms of the form,

$$t^*(\Psi|\log|\eta||dd^c|\eta|^2)d^c\log\|\mathbb{I}_{\tilde\Delta}\|^2d\mu$$

for Ψ some linear function of the coefficients of ψ, and $\eta=0$ a local equation for a component of the exceptional divisor in $\tilde\Delta$ ($\xrightarrow{\sim}$ resolution of sing($\mathcal{F}$) by a sequence of blow ups in smooth invariant centres). Equally there's a similar term in the integral over $\|\mathbb{I}_{\tilde\Delta}\|\le\rho$, but this can be absorbed on the left when appropriate by taking ρ sufficiently small. Consequently if we hadn't a priori known from the theory of residual measure, III.3.4, that $\log|\eta|$ was ℓ_p for all p with respect to $d\eta d\bar\eta d\mu$ we wouldn't have got anywhere. Thus the Green's operator isn't precisely smoothing, but rather, works like : smoothing + a priori knowledge, whence, given the stability of IV.7.2 under blowing up in sing($\mathcal{F}$) we're going to be able to apply,

IV.7.3 Bootstrapping Procedure *Given* IV.7.2 *we assert that $\|\mathbb{I}_{\mathcal{Y}}\|^{-1}$ is ℓ_p for all p.*

step 1 We know that $\log\|\mathbb{I}_{\mathcal{Y}}\|$ is ℓ_1, so we have,

$$\begin{aligned}\infty &> -\int_{\mathcal{X}}|\log\|\mathbb{I}_{\mathcal{Y}}\||c_1(\mathcal{Y})d\mu\\ &=\lim_{t\to-\infty}-\int_{\mathcal{X}}\log\max\{\|\mathbb{I}_{\mathcal{Y}}\|,t\}dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2d\mu\\ &=\lim_{t\to-\infty}\lim_{\epsilon\to0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon}\log\max\{\|\mathbb{I}_{\mathcal{Y}}\|,t\}d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2d\mu+\lim_{t\to-\infty}2\int_{\log\|\mathbb{I}_{\mathcal{Y}}\|\ge t}\frac{d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|^2}d\mu\end{aligned}$$

Now the hypothesis of the nullity of residual measure plainly obliges the first term to be zero, so infact $\|\mathbb{I}_{\mathcal{Y}}\|^{-1}$ is ℓ_2 with respect to $d\|\mathbb{I}_{\mathcal{Y}}\|d^c\|\mathbb{I}_{\mathcal{Y}}\|d\mu$.

step 2 Step 1 is equally valid on $\tilde{\Delta}$, so we can use this to get a respectable bound for the ℓ_p norm of $\log \|\mathbb{I}_{\mathcal{Y}}\|$ on $\mathcal{X}$.

Indeed the maximum of $x|\log x|^p$ in the interval $[0,1]$ is $e^{-p}p^p$ so no worse than $p!$, whence a power series computation gives $\|\mathbb{I}_{\mathcal{Y}}\|^{-1}$ is $\ell_{2-\epsilon}$ for every $\epsilon > 0$

step 3 Suppose more generally $\|\mathbb{I}_{\mathcal{Y}}\|^{-2}$ is ℓ_q and as before, consider,

$$\infty > -\int_{\mathcal{X}} \frac{1}{\|\mathbb{I}_{\mathcal{Y}}\|^{2q}} c_1(\mathcal{Y}) d\mu = \lim_{t\to-\infty} q \int_{\log\|\mathbb{I}_{\mathcal{Y}}\|\geq t} \frac{d\|\mathbb{I}_{\mathcal{Y}}\| d^c \|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|^{2(q+1)}} d\mu$$

so that $\|\mathbb{I}_{\mathcal{Y}}\|^{-2}$ is ℓ_{q+1} with respect to $d\|\mathbb{I}_{\mathcal{Y}}\| d^c \|\mathbb{I}_{\mathcal{Y}}\| d\mu$

step 4 Apply step 3 on $\tilde{\Delta}$, to conclude that on $\mathcal{X}$ we can improve our bound for the ℓ_1 norm of $|\log\|\mathbb{I}_{\mathcal{Y}}\||^{2p}$ to $p!(q+1)^{-p}$, whence conclude that $\|\mathbb{I}_{\mathcal{Y}}\|^{-2}$ is $\ell_{q+1-\epsilon}$ for every $\epsilon > 0$.

step 5 Bootstrap stricta dictum, i.e. we can get from $\|\mathbb{I}_{\mathcal{Y}}\|^{-2}$ ℓ_q to say $\ell_{q+1/2}$ by step 4, so indeed $\|\mathbb{I}_{\mathcal{Y}}\|^{-2}$ is ℓ_q for all $q < \infty$.

From which we obtain,

IV.7.4 corollary *Suppose $(\mathcal{X}, \mathcal{F})$ is the germ of an analytic stack with proper singular locus, $d\mu$ the extension by zero of a measure on $\mathcal{X}\backslash\mathrm{sing}(\mathcal{F})$ with finite total mass and zero segre class along* $\mathrm{sing}(\mathcal{F})$ *then $Res(d\mu) = 0$.*

proof (No hypothesis on $\mathrm{sing}(\mathcal{F})$) Let $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ be a resolution of the ideal of $\mathrm{sing}(\mathcal{F})$ by a sequence of blow ups in smooth invariant centres, then at every geometric point ζ of $\tilde{\mathcal{X}}$ there is a 1-form ω_ζ with at worst meromorphic poles along the total exceptional divisor which becomes holomorphic in $K_{\tilde{\mathcal{F}}}$ and generates the latter around ζ. In particular any residue that we may wish to compute may be expressed in the form,

$$\lim_{\epsilon\to 0} \int_{\|\mathbb{I}_{\mathcal{E}}\|=\epsilon} \rho_\zeta \omega_\zeta d\mu$$

for $\mathcal{E}$ the total exceptional divisor, and ρ_ζ suitable C^∞-functions. On the other hand the poles of ω_ζ are ℓ_p for every p, so ω_ζ itself is ℓ_2. Whence by III.3.5 this limit is zero. □

IV.8. Motivic Remarks

Given a foliated stack $(\mathcal{X}, \mathcal{F})$ one would like to form the classifying stack $[\mathcal{X}/\mathcal{F}]$. In an ideal world this would be done as follows: the smooth infinitesimal groupoid $\mathfrak{F} \rightrightarrows \mathcal{X} \setminus \mathcal{F}$ would have a well defined Zariski closure as a flat infinitesimal groupoid, which could be extended to a small analytic space, and from there to a domain of holomorphy for either the source or the sink. Indeed when $\mathcal{F}$ is smooth, and $\mathcal{X}$ is a space this is just the construction of the holonomy groupoid, and it's already verified in [M1] VI.2 that for $\mathcal{X}$ a Deligne-Mumford stack the said groupoid is representable in analytic spaces so infact there are no set of all sets kind of problems and $[\mathcal{X}/\mathcal{F}]$ is well defined. The situation when there are effectively arbitrary, even canonical singularities, is less clear. We could try to do this at the level of rational maps, but this is problematic since we may not be able to define composition. We are, however, just making some remarks, so, let's suppose,

IV.8.1 Hypothesis *Suppose that we have an intelligible sense for the symbol $[\mathcal{X}/\mathcal{F}]$, based around the bi-rational groupoid, or more likely a pro-finite limit of such, or whatever, with, equally, an intelligible sense to the statement $\mathcal{Y} \to [\mathcal{X}/\mathcal{F}]$ is flat of pure relative dimension - plainly, $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$ for $(\mathcal{X}, \mathcal{F})$ with canonical singularities is the model example.*

Given these caveats and/or hypothesis we can consider the 2-category of algebraic Deligne-Mumford stacks $\mathcal{Y} \to [\mathcal{X}/\mathcal{F}]$ which are flat of pure relative dimension such that the associated groupoid $\mathcal{Y} \times_{[\mathcal{X}/\mathcal{F}]} \mathcal{Y} \rightrightarrows \mathcal{Y}$ is not only 'representable' in analytic spaces but is infinitesimally so in formal algebraic stacks. Rather abusively we'll call this $\mathbf{sch}/[\mathcal{X}/\mathcal{F}]$, i.e. we want to think of such objects as schemes over $[\mathcal{X}/\mathcal{F}]$. Having got this far we can introduce some motives, i.e. a category $\mathbf{Msch}/[\mathcal{X}/\mathcal{F}]$ whose objects are $[\mathcal{X}/\mathcal{F}]$ schemes $\mathcal{Y}$ but written $M_{\mathcal{Y}}$, and if $\mathcal{Y}$ has relative dimension n the morphisms to another $[\mathcal{X}/\mathcal{F}]$ scheme $\mathcal{Z}$ are,

$$\mathrm{Hom}(M_{\mathcal{Y}}, M_{\mathcal{Z}}) = A^{n,n}(\mathcal{Y} \times_{[\mathcal{X}/\mathcal{F}]} \mathcal{Z})$$

where the right hand side is defined as follows: $\mathcal{Y} \times_{[\mathcal{X}/\mathcal{F}]} \mathcal{Z}$ is almost a space. Specifically, it's an analytic Deligne-Mumford stack, so it has moduli, whence a perfectly sensible space $A^{n;n}_{\infty}(\mathcal{Y} \times_{[\mathcal{X}/\mathcal{F}]} \mathcal{Z})$ of smooth compactly supported (n,n) forms, which can be completed in some norm that we're deliberately vague about. One notes, however, that when X is a space, $\mathcal{F}$ is smooth, and we complete in the sup-norm that,

$$\mathrm{End}(M_X) = A^{n,n}(X \times_{[X/\mathcal{F}]} X)$$

is infact the C^* algebra of a foliation in the sense of [Co], and, of course, there is no need for hypothesis to ensure the existence of $[\mathcal{X}/\mathcal{F}]$ in this case.

Now while we've only discussed Hodge decomposition for that which is strictly necessary, it's plain that the entire discussion continues to work under very mild hypothesis, e.g. say $\mathcal{Y}$ smooth with projective moduli, and some technical conditions on the relevant tangent sheaf such as $[\beta\mu]$. Whence for a given transverse measure with now $\tilde{\Delta}$ some birational identity on a representable groupoid with the same objects as $\mathcal{Y}$, the only change in the definition of the Green's operator ought to be for reasons of the relative dimension n, i.e.

$$G(f)(\beta) = \int s^*(fH)t^*\beta \log \|\mathbb{I}_{\tilde{\Delta}}\|^2 (dd^c \log \|\mathbb{I}_{\tilde{\Delta}}\|^2)^{n-1} d\mu$$

with, of course, β a (n,n) form, and H a hyperplane. In particular for a given transverse measure $d\mu$ the situation is no more complicated than the situation over $\mathbb{C}$. Notice, we did not in IV.6.3, nor here, make any precise re-interpretation of L_2 harmonic spaces in terms of some more intrinsic derived functors due to technical issues associated to the singularities, with, for example, as post IV.2.4, 'natural definitions' of cohomology and ext disagreeing for 'invariant' bundles. We can, however, simply view the resulting harmonic groups $H^*_{2,d\mu}$ as a realisation of the motive, which themselves don't truly depend on $d\mu$, i.e. there is a map,

$$H^*_2 : \text{Measures on } [\mathcal{X}/\mathcal{F}] \longrightarrow \text{Realisations of } \mathbf{Msch}/[\mathcal{X}/\mathcal{F}] : d\mu \mapsto H^*_{2,d\mu}$$

As such we can really talk about H^*_2, or perhaps notationally better, for $\pi : \mathcal{Y} \to [\mathcal{X}/\mathcal{F}]$ in $\mathbf{sch}/[\mathcal{X}/\mathcal{F}]$, $R^{\bullet}_2\pi_*\mathbb{C}$. Better still, considerations of our initial model example $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$ for E any vector bundle on $\mathcal{X}$ and $\nabla : E \to K_{\mathcal{F}} \otimes \mathcal{I}_{\mathcal{Z}}$ a C^∞ connection, $\mathcal{Z} = \mathrm{sing}(\mathcal{F})$, suggest we can construct groups $H_{2,d\mu}(\bar{\partial})$, $H_{2,d\mu}(\nabla)$, $H_{2,d\mu}(\nabla + \bar{\partial})$ by way of appropriate Laplacians which have formally identical properties to the groups so defined over $\mathbb{C}$. One should note, however, that if we take ∇ to be a connection with values in $K_{\mathcal{F}}$ the situation is a lot more delicate, since there will be residues. Regardless, and again without trying to define it intrinsically we can define $R^{\bullet}_2\pi_*E$ using the $\bar{\partial}$-Laplacian, or a more subtle object $R^{\bullet}\pi_*(E,\nabla)$ using the operator $[\bar{\partial}, [\Lambda, \nabla]]$ where Λ is the adjoint of a hyperplane, and ∇ is allowed to take arbitrary bounded values in the relative co-tangent sheaf of $\pi : \mathcal{Y} \to [\mathcal{X}/\mathcal{F}]$.

To ensure the existence of a motivic chern character it suffices to work with bundles with connection rather than just bundles, or we can divide by an appropriate equivalence relation. Consequently up to some suitable refinement of the discussion we would anticipate a relative Riemann-Roch theorem of the form,

$$\sum (-1)^i R^i\pi_*(E,\nabla) = \int ch(E,\nabla) td(\pi) + \text{Fudge}$$

where the fudge factor is rather big, i.e. it contains residues for the non-scheme like points of $\mathcal{Y}$, for the extent to which ∇ doesn't factor through the relative co-tangent bundle, and both of these at the same time. Regardless, we move to a much more satisfactory description of the residue symbol IV.1.4. Notice also that the topological Euler characteristic also involves a residue, for example in our model example $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$ it looks much more like $1/2\{c_1(K_{\mathcal{F}}) - s_{\mathcal{Z}}\}$, where $s_{\mathcal{Z}}$ is the segre class of $\mathrm{sing}(\mathcal{F})$ viewed as an operator on measures, rather than its 'arithmetic' counterpart $1/2c_1(K_{\mathcal{F}})$.

V. Hyperbolicity (Emphasis on 3-D)

V.1 Parabolic Measures

Consider the following statements for maps to a foliated $\mathbb{Q}$-gorenstein projective triple $(X, B, \mathcal{F})$ with canonical singularities and conventions on the weights of the necessarily non-invariant boundary B as per [M1] I.6.1.,

V.1.1 Finiteness *There exists a proper closed subvariety Z of X such that the subspace of* $\mathrm{Hom}(\Delta, X)$ *consisting of invariant maps not factoring through B, with $f^{-1}(B_i)$ a divisor of weight divisible by that of B_i, for every component B_i of B, and not arbitrarily close to Z in the compact open sense is compact.*

Now suppose this statement is false, then by op. cit. V.6,

V.1.2 Summary *We can find a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ with the following properties,*

(a) $(K_{\mathcal{F}} + B).d\mu_{X/\mathcal{F}} \leq 0$

(b) $D.d\mu_{X/\mathcal{F}} \geq 0$, $\forall$ *effective Cartier divisors D.*

(c) *For any proper generically finite map $\rho : \mathcal{X} \to X$ in the 2-category of algebraic stacks there is a closed positive transverse invariant measure $d\nu_{\mathcal{X}}$ satisfying (b) on $\mathcal{X}$ and $\rho_* d\nu_X = d\mu_{X/\mathcal{F}}$. Furthermore if we have a foliated triple such that $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ is unramified in the leaf direction, then we have (a) on $\mathcal{X}$ too.*

In particular for the stack $\rho : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ with its divisorial ramification over B of [M1] I.7.4, we get a transverse invariant measure satisfying $K_{\tilde{\mathcal{F}}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, and whence equality if $K_{\tilde{\mathcal{F}}}$ is nef. Consequently we introduce,

V.1.3 Definition *Call an invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on a minimal model $(\mathcal{X}, \mathcal{F})$ of a foliated gorenstein stack with projective moduli parabolic if $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} = 0$, and we have conditions (b) & (c) of* V.1.2.

A useful initial remark is to clarify the condition (c) for bi-rational modification, so let's say $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ a blow up in an irreducible centre $\mathcal{Y}$ of co-dimension p with $\mathcal{E}$ the exceptional divisor. As such we have,

$$\text{multiplicity } := m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) = \sup\{\mathcal{E}.d\nu_{\mathcal{X}} \ : \ \rho_* d\nu_{\mathcal{X}} = d\mu_{\mathcal{X}/\mathcal{F}},\ d\nu_{\mathcal{X}} \text{ satisfies (b)}\}$$

and we assert,

V.1.4 Claim *If* $\dim \mathcal{Y} <$ *numerical Kodaira dimension of $K_{\mathcal{F}}$, then $m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$.*

proof The proposition is absolutely nothing to do with most of the hypothesis, and is infact a simple counting argument. Indeed let $\epsilon \in \mathbb{Q}_{>0}$, and $d \in \mathbb{N}$ be sufficiently large and divisible, then for ν the numerical Kodaira dimension of a nef K on the n-dimensional moduli X, with H ample,

$$h^0(X, K^d \otimes H^{d\epsilon}) \backsimeq \frac{d^n \epsilon^{n-\nu} k^\nu H^{n-\nu}}{\nu!}(1 + O(\epsilon))$$

On the other hand to vanish to order m on a sub-scheme Y of co-dimension p is determined inductively by exact sequences,

$$0 \longrightarrow H^0(X, K^d \otimes H^{d\epsilon} \mathcal{I}_Y^{i+1}) \longrightarrow H^0(X, K^d \otimes H^{d\epsilon} \mathcal{I}_Y^i) \longrightarrow H^0(K^d \otimes H^{d\epsilon} \otimes \mathcal{I}_Y^i / \mathcal{I}_Y^{i+1})$$

for $0 \leq i < m$, from which we deduce,

$$h^0(X, K^d \otimes H^{d\epsilon} \mathcal{I}_Y^m) \gtrsim h^0(X, K^d \otimes H^{d\epsilon}) - \text{const} \frac{m^p d^{n-p}}{p!(n-p)!}$$

where the implied constant depends on Y. Whence if we're aiming for multiplicity $d\delta$, then we need,

$$\delta^p << \epsilon^{n-\nu}$$

and by hypothesis, $n-\nu \leq p-1$, so if we take, as we may δ of order $\epsilon^{n-\nu/p}$ then δ is at least of order $\epsilon^{1-1/p}$, so we find,

$$\rho^*(K+\epsilon H) >> \epsilon^{1-1/p}\mathcal{E}$$

on $\mathcal{X}$, from which,

$$\epsilon^{1/p}H > d\mu_{\mathcal{X}/\mathcal{F}} >> m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) \geq 0$$

for every $\epsilon > 0$. $\square$.

As such, and particularly given that our primary interest centres on Kodaira dimension $n-1$, it's invariably the case that we may suppose that the multiplicity is zero. In such a scenario, it's often the case that V.1.2 (c) is a bit empty. Specifically suppose $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ is a blow up in a lci substack $\mathcal{Y}$, so that the exceptional divisor $\rho : \mathcal{E} \to \mathcal{Y}$ is relatively smooth. Consequently for any remotely sensible metricisation of $\mathcal{O}_{\mathcal{E}}(1) := \mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ the class $\mathcal{O}_{\mathcal{E}}(1)^{p-2}.\rho^* s_{\mathcal{Y},d\mu}$ is a positive measure, and for $\mathbb{I}_{\mathcal{Y}} d\mu_{\mathcal{X}/\mathcal{F}} = 0$ we get a perfectly sensible 'proper transform',

$$\mathbb{I}_{\mathcal{X}\backslash\mathcal{E}} d\mu_{\mathcal{X}/\mathcal{F}} + (i_{\mathcal{E}})_* \mathcal{O}_{\mathcal{E}}(1)^{p-2}.\rho^* s_{\mathcal{Y},d\mu}$$

which incidentally is just $\rho^* d\mu_{\mathcal{X}/\mathcal{F}}$ as an 'analytic cycle' if one were to use the theory of residual measure to adapt [F] 6.7 to measures. Nevertheless when $\mathbb{I}_{\mathcal{Y}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0$ the existence of a proper transform is a highly non-trivial hypothesis, whose relation to residual measure merits clarification. To fix ideas and/or consider a case that captures all the essentials , suppose that $d\mu_{\mathcal{X}/\mathcal{F}}$ arises from a countable Zariski dense set of curves $\mathcal{C}_i \hookrightarrow \mathcal{X}$ of unbounded degree, i.e. after subsequencing,

$$d\mu_{\mathcal{X}/\mathcal{F}} = \lim_i \frac{1}{H.\mathcal{C}_i}\mathcal{C}_i$$

where $\mathcal{C}_i$ is equally used to denote its current of integration, or if one prefers over C_i, where the latter is smooth, scheme like, and dominates $\mathcal{C}_i$ by way of f_i. Plainly therefore we have an a priori notion of proper transform for, say, $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ the blow up in $\mathcal{Y}$ as before, given by,

$$d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} = \lim_i \frac{1}{H.\mathcal{C}_i}\tilde{\mathcal{C}}_i$$

where $\tilde{\mathcal{C}}_i$ is the proper transform of the still Zariski dense set of $\mathcal{C}_i$'s not factoring through $\mathcal{Y}$, with extra subsequencing as necessary. In order to relate this to the theory of residual measure consider for $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed substack with $\|\mathbb{I}_{\mathcal{Y}}\|$, $\|\mathbb{I}_{\mathcal{Z}}\|$ as per III.3.1, and $\delta > 0$, the various integrals over a given $\mathcal{C}_i$,

$$\begin{aligned}\int_{1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||<\|\mathbb{I}_{\mathcal{Y}}\|<\delta} dd^c \log\|\mathbb{I}_{\mathcal{Y}}\|^2 &= \int_{\delta=\|\mathbb{I}_{\mathcal{Y}}\|>1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||} (d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2 + d^c \log\log^2|\log\|\mathbb{I}_{\mathcal{Z}}\||) \\ &\quad - \int_{\delta\geq\|\mathbb{I}_{\mathcal{Y}}\|>1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||} dd^c\log\log^2|\log\|\mathbb{I}_{\mathcal{Z}}\|| - s_{\mathcal{Y},\mathcal{C}_i}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})\end{aligned}$$

Now divide out by the degree of the $\mathcal{C}_i$ take the limit over i, then let δ go to zero to get,

$$\begin{aligned}-\mathcal{E}.\mathbb{I}_{\mathcal{E}\cap\rho^{-1}(\mathcal{Z})} d\mu_{\mathcal{X}/\mathcal{F}} &= \lim_{\delta\to 0} s^{\delta}_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|<\exp(-\exp(1/\delta))}) - m_{\mathcal{Y},d\mu_{\mathcal{X}/\mathcal{F}}}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) \\ &\leq s_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) - m_{\mathcal{Y},d\mu_{\mathcal{X}/\mathcal{F}}}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})\end{aligned}$$

where s^δ is the segre type class, but calculated at a fixed radius $\|\mathbb{I}_{\mathcal{Y}}\| = \delta$, while m, evidently for multiplicity, is the multiplicity type measure arising from the normalised limits of the $s_{\mathcal{Y},\mathcal{C}_i}$, and the terms involving $\log\log|\log\|\mathbb{I}_{\mathcal{Z}}\||$ all disappear because $dd^c \log|\log\|\mathbb{I}_{\mathcal{Z}}\||$ is absolutely integrable with respect to $d\mu$ irrespective of any choices. Obviously an equality would have been preferable, but we'll invariably be able to get by by via the generally valid formula,

$$\mathcal{O}_{\mathcal{E}}(1).\mathbb{I}_{\mathcal{E}} d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} = s_{\mathcal{Y},d\mu}(1) - m_{\mathcal{Y},d\mu}(1)$$

Consequently, one should note that the possibility exists to refine our definition of parabolic measure to include,

V.1.5 (V.1.2 bis.) Summary/Definition *A parabolic measure could, for all practical purposes, be asked to satisfy the following further naturality condition for blow ups $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ in lci centres $\mathcal{Y}$ with exceptional divisor $\mathcal{E}$,*

$$\text{(d)} \qquad -\mathcal{E}.\mathbb{I}_{\mathcal{E}\cap\rho^{-1}(\mathcal{Z})} d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} \leq s_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) - m_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})$$

for $\mathcal{Z} \hookrightarrow \mathcal{X}$ any closed substack, and $\|\mathbb{I}_{\mathcal{Z}}\|$ as per III.3.1.

V.2 Parabolic Decomposition

Suppose we're presented with a parabolic measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on $(\mathcal{X},\mathcal{F})$ with, say, $\mathcal{X}$ smooth of dimension 3 to fix ideas. Notice that by definition the singularities of $\mathcal{F}$ are canonical so any invariant divisor $\mathcal{D}$ that moves in a pencil defines not just a rational map, but an honest 1st integral, $\lambda : \mathcal{X} \to \mathcal{C}$, so if $\mathcal{D}_t$ is generic in the pencil $\mathcal{O}_{\mathcal{D}_t}(\mathcal{D}_t)$ is at worst torsion. Consider, therefore, the condition,

$$\mathbb{I}_{\mathcal{D}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0\,, \text{ and } \mathcal{D}.\mathbb{I}_{\mathcal{D}} d\mu_{\mathcal{X}/\mathcal{F}} < 0$$

Now certainly $d\mu_{\mathcal{X}/\mathcal{F}}$ may have support on curves, which are necessarily invariant, and countably many, so write,

$$d\mu_{\mathcal{X}/\mathcal{F}} = d\nu + \sum_i \lambda_i L_i\,,\ \mathbb{I}_{\mathcal{C}} d\nu = 0 \text{ for all curves } \mathcal{C}$$

where L_i are the classes of integration of the said curves. Consequently, if we still have $\mathcal{D}.d\nu < 0$ then $\mathcal{D}$ is the only effective Cartier divisor that meets $\mathbb{I}_{\mathcal{D}} d\nu$ negatively, and indeed, $\mathcal{C}.\mathbb{I}_{\mathcal{D}} d\nu \geq 0$ for all effective Cartier divisors $\mathcal{C}$ on $\mathcal{D}$. Now let's proceed to an understanding of $\mathbb{I}_{\mathcal{D}} d\nu$ in the usual way, i.e. as in II.1 consider the diagram,

$$\begin{tikzcd}
(\mathcal{Y}_0,\mathcal{G}_0) & (\mathcal{Y},\mathcal{G}) \arrow[l, "\pi"'] \arrow[d, "\rho"] & \\
& (\tilde{\mathcal{D}},\tilde{\mathcal{F}}) \arrow[d] & \\
& (\mathcal{D},\mathcal{F}) \arrow[r] & (\mathcal{X},\mathcal{F})
\end{tikzcd}$$

where $\tilde{\mathcal{D}} \to \mathcal{D}$ is the normalisation, $\rho : \mathcal{Y} \to \tilde{\mathcal{D}}$ a relatively canonical model contracting curves E_i, and $\pi : \mathcal{Y} \to \mathcal{Y}_0$ the actual canonical model contracting curves C_j. As such we get a formula of the form,

$$\rho^* K_{\mathcal{F}} = \pi^* K_{\mathcal{G}_0} + Z + \sum_i a_i E_i + \sum_j b_j C_j$$

$a_i \in \mathbb{Q}_{>0}$, $b_j \in \mathbb{Q}_{\geq 0}$, Z supported on $\mathrm{sing}(\mathcal{F})$ with parts Z', Z'' invariant and non-invariant by $\tilde{\mathcal{F}}$. In particular it's perhaps better to think in terms of relatively canonical & canonical models for $(\tilde{\mathcal{D}}, Z'', \tilde{\mathcal{F}})$. Regardless if $(\mathcal{Y}_0, \mathcal{G}_0)$ or more generally $(\mathcal{Y}_0, Z'', \mathcal{G}_0)$ is fibred in rational curves or weighted projective stacks then $\mathbb{I}_{\mathcal{D}} d\nu$ has a fairly obvious shape, and we make no further comment. At the other end of the spectrum if we have general type then $\mathbb{I}_{\mathcal{D}} d\nu$ doesn't exist by the index theorem, while rather more delicately the classification theorem in dimension 2, [M2], tells us that that it cannot exist for numerical Kodaira dimension 1 either without being supported on curves, or arising in the obvious way from an elliptic fibration. If the Kodaira dimension in zero then the cases where $\mathcal{G}_0$ is a smooth foliation is distinguished. Here there are invariant measures unique in homology, say $d\nu_0$, with the property that $d\nu_0^2 = 0$, so that $\pi^* d\nu_0$ can be identified with its extension by zero across $\bigcup C_j$ when it's not supported on curves, or by the pull-back of a curve when it is. Either way, since $d\nu$ itself isn't supported on curves $\rho^* \mathbb{I}_{\mathcal{D}} d\nu$ may be identified with the extension by zero across $\bigcup E_i$ plus some curves, which in turn is homologous to $\pi^* d\nu_0$ by the index theorem. Furthermore the log-Kodaira dimension must also be zero, so Z'' is empty, and $\rho^* \mathcal{O}_{\mathcal{D}}(\mathcal{D})$ admits a connection along the leaves with meromorphic poles on Z'. On the other hand $Res(\pi^* d\nu_0)$ by construction, so $\mathcal{D}.\mathbb{I}_{\mathcal{D}} d\nu = 0$, contrary to hypothesis. This leaves the possibility that $Z'' = \emptyset$, and $(\mathcal{Y}_0, \mathcal{D}_0)$ is defined by a $\mathbb{G}_m \times \mathbb{G}_m$ or $\mathbb{G}_a \times \mathbb{G}_m$ action, or $Z'' \neq \emptyset$ and we have an isotrivial family in $\mathbb{G}_m$'s, nearly all of which can occur. More precisely consider a $\mathbb{G}_a \times \mathbb{G}_m$ action modeled on $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose $d\nu_0$ is a measure on it not supported on curves, then by [B1], $Res(d\nu_0) = 0$, so by the index theorem every cycle C supported on invariant curves satisfies $C^2 \leq 0$, which is nonsense, so this case is inexistent. Similarly by op. cit. $Res(d\nu_0)$ would also be zero on $\mathbb{P}^1 \times \mathbb{P}^1$ for a $\mathbb{G}_m \times \mathbb{G}_m$ action given by a field of the form $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \notin \mathbb{R}$, the case of $\lambda \in \mathbb{R}$ may, though, a priori occur. It's therefore worth noting that apart from the real eigenvalue, if this example did occur then the index theorem tells us that $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ has dimension zero, while $(\mathcal{Y}, \mathcal{G})$ and $(\tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ coincide, and are modifications of $(\mathcal{Y}_0, \mathcal{G}_0)$ by a sequence of blow ups in invariant centres. In particular $\mathbb{I}_{\mathcal{D}} d\nu$ is not only homologous to a positive sum of invariant curves, but is so in the strong sense that the difference intersects any invariant object in zero, whence, to all intents and purposes we'll be able to assume that they are equal. Consequently, modulo bearing in mind the need for a little care in this case, let us simply write,

$$d\mu_{\mathcal{X}/\mathcal{F}} = d\nu + \sum_i \lambda_i L_i$$

where now $\mathcal{D}.d\nu \geq 0$ for every effective Cartier divisor $\mathcal{D}$, and the L_i are allowed to be not only invariant curves, but invariant measures on any divisor on which the induced foliation is a conic pencil. Whence if $K_{\mathcal{F}}^2 \neq 0$,

V.2.1 Fact *Let things be as above with $K_{\mathcal{F}}^2 \neq 0$, then $d\mu_{\mathcal{X}/\mathcal{F}}$, $d\nu$, and $\sum L_i$ are all parallel when projected into* $\mathrm{NE}_1(\mathcal{X})$.

This is, of course, just an application of III.2.1, but $d\nu$ has further structure, such as,

V.2.2 Further Fact *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be an invariant substack of dimension at most the numerical Kodaira dimension of $\mathcal{F}$ then the residual measure $s_{\mathcal{Y},d\nu}$ is zero.*

proof By hypothesis $\mathbb{I}_{\mathcal{Y}} d\nu = 0$, so if $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ is the blow up in $\mathcal{Y}$ with $\mathcal{E}$ the exceptional divisor then,

$$s_{\mathcal{Y},d\nu}(1) = \mathcal{E}.d\tilde{\nu}$$

where $d\tilde{\nu}$ is the extension of $d\nu$ across $\mathcal{E}$ by zero. Arguing as per V.1.4, we're plainly going to be done if we can show that $d\tilde{\nu}$ intersects every Cartier divisor on $\mathcal{X}$ non-negatively. Replacing $\tilde{\mathcal{X}}$ by a resolution via a sequence of blow ups in invariant centres we deduce the existence of an irreducible Cartier divisor $\mathcal{D}$ such that $\mathcal{D}.d\tilde{\nu} < 0$, so $\mathcal{D}.\mathbb{I}_{\mathcal{D}} d\tilde{\nu} < 0$. On the other hand ρ is bi-rational at the generic point of $\mathcal{D}$, and we've already removed everything that might have this property. □.

From which we proceed to,

V.2.3 Main Fact *Either the number of possible L_i's covered by a rational curve is 'finite' - i.e. something appearing in the sum $\sum \lambda_i L_i$ is: one of a finite list, moves in one of finitely many families, or moves in a family covering $\mathcal{X}$.*

proof Most of this has already been done in §III, so that in the notation of III.9.2 we may assume that $\mathcal{X} = \mathcal{X}_{pr}$, and L_i is a rigid curve, étale covered by a generalised weighted projective stack $\mathbb{P}^1(p,q)$, where p, q are the orders of the generic monodromy at either end modulo the gcd of the same. What's critical to know is that $\mathbb{I}_{\mathcal{X}\backslash L_i} d\mu$ has no residual measure around L_i. For $K_{\mathcal{F}}^2 \neq 0$ this follows from V.2.2 & the construction of $\mathcal{X}_{pr}$. For $K_{\mathcal{F}}$ numerically nil, there is never an issue since the eigenvalues at any end are constant. The case $K_{\mathcal{F}} \neq 0$, but $K_{\mathcal{F}}^2 = 0$ is rather subtle. One has to first prove by specialisation ad nauseam to the normal cone of L_i that the only way that this could happen is if $d\nu$ were infinitely tangent to the formal schemes D_0 and D_∞ of II.6 at the $\mathcal{X}_{pr}$ level, then go back up to $\mathcal{X}$ and proceed as per II.8.3 to extract a contradiction. We don't, however, need this for our main 3-D application, so we omit further discussion. Now we can just construct the flap $\tilde{\mathcal{X}}_{pr}$ of L_i, i.e. the weighted blow up mapping down to the flop, let $\mathcal{E}$ be the exceptional divisor and conclude,

$$\mathcal{E}.\mathbb{I}_{\mathcal{E}} d\tilde{\mu} \geq 0$$

which is nonsense, since $\mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ is ample on $\mathcal{E}$. □

V.3 The Diffuse Part

Plainly at this juncture our knowledge of the 'atomic' part $\sum L_i$ that may appear in the parabolic decomposition of a parabolic measure $d\mu_{\mathcal{X}/\mathcal{F}}$ is rather good. Indeed,

V.3.1 Summary *We have the following alternatives,*

(i) *Some rational L_i moves in a family covering $\mathcal{X}$, so, in particular, if $K_{\mathcal{F}}^2 \neq 0$, there is a foliated surface $(\mathcal{S}, \mathcal{G})$ and a rational map $(\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{S}, \mathcal{G})$ compatible with the foliations, whence, 'much' of $d\mu_{\mathcal{X}/\mathcal{F}}$ descends to $\mathcal{S}$, and in any case $K_{\mathcal{X}}.K_{\mathcal{F}}^2 < 0$.*

(ii) *Some L_i is actually elliptic, and wholly contained in the smooth locus, so its normal bundle is flat. Whence, again if $K_{\mathcal{F}}^2 \neq 0$ then such a curve is parallel to $K_{\mathcal{F}}^2$ and $K_{\mathcal{X}}.K_{\mathcal{F}}^2 = 0$.*

(iii) *The number of L_i's is 'finite' in the sense of V.2.3, although one should bear in mind that there may be curves wholly contained in the* $\mathrm{sing}(\mathcal{F})$*, so it's important to remember that the definition of invariant measure supposes the vanishing of differentials in the co-normal bundle along $d\mu$, so this can only happen for curves that are singular solutions in the sense of* I.4.10 *i.e. contained in the weak branch and invariant by the induced foliation.*

Otherwise we can further enhance our understanding of $d\mu_{\mathcal{X}/\mathcal{F}}$ under some additional hypothesis, i.e.

V.3.2 Fact *If $K_{\mathcal{F}}^2 \neq 0$, then either the 'diffuse' part $d\nu$ doesn't exist or $K_{\mathcal{X}}.K_{\mathcal{F}}^2 = 0$.*

proof Suppose $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$, then $d\nu$ is parallel to $K_{\mathcal{F}}^2$, so $K_{\mathcal{X}}.d\nu \neq 0$. The co-normal bundle is, however, invariant so by IV.7.4 $s_{\mathcal{Y},d\nu} \neq 0$ for $\mathcal{Y} \hookrightarrow \mathcal{X}$ the entire singular substack, contradicting V.2.2. □.

In particular,

V.3.3 corollary *If $K_{\mathcal{X}}.K_{\mathcal{F}}^2 > 0$ then $d\mu_{\mathcal{X}/\mathcal{F}}$ is a (possibly countable) sum of 'finitely many' curves in the sense of* V.2.3 *which are either rational invariant not factoring through* $\mathrm{sing}(\mathcal{F})$ *or contained in the weak branch, and invariant by the induced foliation on the same.*

V.4 The Weak Branch

For this section we'll suppose $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$, and that we have the alternative V.3.1 (iii). As such there could be a number of curves L_i whose completion around sing($\mathcal{F}$) factors through the weak branching scheme. Necessarily we can suppose that this happens at a single irreducible, even smooth, 2-dimensional component $\mathfrak{W}$ of the weak branch which is non-algebraic, so that by I.4.11 the number of L_i involved is finite stricta dictum, and, to re-iterate, may contain some components of sing($\mathcal{F}$). In any case denote, slightly abusively, by $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ the sum $\sum_{i\in I'} \lambda_i L_i$, where I' is the set of L_i in question. Our basic assertion is,

V.4.1 Claim $\mathfrak{W}$ *extends as a formal scheme with trace* $\bigcup L_i$, $i \in I'$, *and for any* $i \in I'$, $L_i.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.

proof For $L_i \subset$ sing($\mathcal{F}$) we've already done this. Otherwise as per I.2.2, it's harmless to blow up in components of sing($\mathcal{F}$) sufficiently to guarantee that the L_i in question is contractible, so that we can find $\mathfrak{W}$ around L_i by way of Jordan decomposition on the contraction.

For the next part we proceed as follows: extract a dth root of $\mathfrak{W}$ around the completion in the trace so extended to get a formal stack $\hat{\mathcal{X}}_d$ which is pseudo reflecting around $\mathfrak{W}$, then blow up in the pre-image of the given L_i. Having done this, kill all the pseudo-reflections and sew the thing back into the ambient $\mathcal{X}$ to get a weighted blow up $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ with exceptional divisor $\mathcal{E}$. By construction $\tilde{\mathcal{X}}$ has no more monodromy around the proper transform $\tilde{\mathfrak{W}}$ than $\mathcal{X}$ has around $\mathfrak{W}$, and has extra non-scheme like structure with monodromy $\mathbb{Z}/d$ concentrated on some curve L_i^∞ wholly disjoint from $\tilde{\mathfrak{W}}$. Furthermore any part of the $d\tilde{\mu}$ guaranteed by V.1.2 which is neither on $\mathcal{E}$ or factors through $\tilde{\mathfrak{W}}$ meets $\mathcal{E}$ in L_i^∞ by I.4.11, whence,

$$L_i.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} + \mathcal{E}.\mathbb{I}_{\mathcal{E}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq -\frac{1}{d} s_{L_i, \mathbb{I}_{\mathcal{X}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}}(1)$$

Within $\mathcal{E}$, we've a priori taken $\mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ to be ample for curves $L_i \subsetneqq$ sing($\mathcal{F}$), while otherwise the foliation in $\mathcal{E}$ is of conic type with integral projection to L_i, so, $-\mathcal{E}.\mathbb{I}_{\mathcal{X}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, and we let $d \to \infty$. $\square$

The critical use of this proposition is as follows: call the induced foliation in $\mathfrak{W}$ $\mathcal{G}$, then $K_{\mathcal{G}} = K_{\mathcal{F}}|_{\mathfrak{W}} - Z$, where Z is some bunch of curves in sing($\mathcal{F}$) that may perfectly well contain some of the L_i's . Nevertheless thanks to V.4.1, we know $K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, so we can examine the structure of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ by doing minimal model theory on $\mathfrak{W}$. Even though $\mathfrak{W}$ is formal, there's no problem constructing a contraction $\pi : \mathfrak{W} \to \mathfrak{W}_0$ which contracts finitely many $\mathcal{G}$-invariant components of the trace such that $K_{\mathcal{G}_0}.C \geq 0$ for every curve C in the trace of $\mathfrak{W}_0$ unless, of course, $\mathfrak{W}$ were a pencil of conics in the induced foliation, which would contradict its non-algebraicity. As a result we can apply the 'index theorem' (more precisely elementary considerations about quadratic forms) to conclude that $K_{\mathcal{G}_0} \neq 0$ unless every component of sing($\mathcal{F}$) $\cap\, \mathfrak{W}$ is $\mathcal{G}$ invariant, and deduce in any case $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, $K_{\mathcal{F}}|_{\mathfrak{W}}$ are parallel. Furthermore the inequality $K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$ becomes an identity, and every $L_i \subset$ sing($\mathcal{F}$) must be rational if there is more than one component, with any intersections being simple normal crossing and at most one for any point of sing$\mathcal{G}$. As such we have two possibilities: $\pi_* \mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ is supported on at least one rational curve disjoint from all the others, which would then have to move, contradicting the non-algebraicity of $\mathfrak{W}$, or $\pi_* \mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ is elliptic -equivalently the curves $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ form one of the Dynkin diagrams of [M3] IV.4.3. In addition, $D.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$ for every Cartier divisor D. If one wishes, it's possible to proceed as per the discussion of II.9 to flop in $\mathcal{X}$ rather than just contract in $\mathfrak{W}$, albeit that we may loose projectivity of the moduli if we proceed as per op. cit., and whence replace $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ by a single irreducible smooth elliptic or irreducible nodal substack. In principle one then wishes to deduce that the curve moves. This is not, indeed cannot be (just consider an arbitrary representation of $\mathbb{Z}^2$ in Diff($\mathbb{C}$)), a purely formal question, and, unfortunately, neither of the proofs in the case of $\mathfrak{W}$ an algebraic surface generalises to the current situation. Indeed [M3] IV.5.1 is highly global, and, although [B2] is a little better it needs meromorphic forms with poles along the curve in the co-normal bundle of $\mathcal{G}$. Nevertheless we're happy enough to conclude,

V.4.2 Fact *If* $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0$, *and* $\mathfrak{W}$ *is non-algebraic then infact* $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ *defines a divisor of elliptic fibre type within* $\mathfrak{W}$, *and* $\mathcal{D}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$ *for every effective Cartier divisor on* $\mathcal{X}$. *In particular* $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ *is parallel to* $K_{\mathcal{F}}^2$.

VI Algebraic Surfaces

VI.1 Jets

Although jets are a well documented subject, we'll need some notation, so we may aswell establish this by doing some definitions into the bargain. Our ultimate interest is in stacks over $\mathbb{C}$, and since the semi-stability arguments to be employed aren't quite valid arbitrarily we may aswell suppose for simplicity that this our base. We begin with,

VI.1.1 Definition [GG] *For $n \in \mathbb{N} \cup \{0\}$, let $\Delta_n = \operatorname{Spec} \mathbb{C}[z]/(z^{n+1})$, and for $\mathcal{X}/\mathbb{C}$ an algebraic stack define a fibred category $V_{\mathcal{X},n}$ over $\mathbb{C}$-affine schemes T by way of,*

$$V_{\mathcal{X},n}|_T = \operatorname{Hom}(\Delta_n|_T, \mathcal{X}|_T)$$

Plainly $V_{\mathcal{X},n}$ is representable as a locally closed substack of $\operatorname{Hilb}(\Delta_n \times_{\mathbb{C}} \mathcal{X})$, or at least it would be if the latter were known to exist. What's at stake, however, in this particular case is rather easy, and one just checks it by hand - indeed, $V_{\mathcal{X},n}$ is a substack of the cone $\mathfrak{P}_{\mathcal{X}} \to \mathcal{X}$ by way of the forgetful map $\pi_{0,n}$, where $\pi_{i,j}$, or just π if there is no likelyhood of confusion, goes from $V_{\mathcal{X},j}$ to $V_{\mathcal{X},i}$ for $i \leq j$. More importantly $V_{\mathcal{X},n}$ is a principal homogeneous space under $\operatorname{Aut}(\Delta_n)$, and essentially what we'd like to do is construct the quotient. Unfortunately the quotient includes the special case of $\mathbb{G}_m$ acting on $T_{\mathcal{X}}$, and as is well known, one only gets a sensible answer after excluding the zero section. Ultimately, however, this is the sum total of complication even in the general setting, so let us first note the compactified answer by way of,

VI.1.2 Further Definition ([A&] or [D2]) *Suppose in addition that $\mathcal{X}/\mathbb{C}$ is smooth, and define inductively a tower P_n of $\mathbb{P}^{dim \mathcal{X}-1}$ bundles together with embeddings $P_n \hookrightarrow \mathbb{P}(\Omega_{P_{n-1}})$, $n \in \mathbb{N}$ according to,*

(1) $P_0 = \mathcal{X}$

(2) $P_1 = \mathbb{P}(\Omega_{\mathcal{X}})$

(3) $P_{n+1} = \mathbb{P}(J_{\mathcal{X},n+1})$, *where $J_{\mathcal{X},n+1}$ is the quotient of Ω_{P_n} defined by the restriction to P_n of the universal sub-bundle on $\mathbb{P}(\Omega_{P_{n-1}})$.*

In particular we have projections, also denoted $\pi_{i,j}$, with the same possibility of omission, from P_j to P_i, $i \leq j$, tautological bundles L_n, along with short exact sequences,

$$0 \longrightarrow L_n \longrightarrow J_{\mathcal{X},n+1} \longrightarrow \Omega_{P_n/P_{n-1}} \longrightarrow 0 \ , \ n \in \mathbb{N}$$

giving rise to distinguished divisors $\infty_{n+1} \hookrightarrow P_{n+1}$, linearly equivalent to $L_{n+1} - L_n$. Notation established, up to dropping $\mathcal{X}$ which will be fixed, we assert,

VI.1.3 Claim *Let V_n^* be V_n complemented by the pull-back of the zero section from V_1 then the groupoid $Aut(\Delta_n) \times V_n^* \rightrightarrows Aut(\Delta_n)$ is representable and $[V_n^*/Aut(\Delta_n)]$ exists as an algebraic stack and is isomorphic to $P_n \backslash \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p)$.*

proof Consider first the local question, i.e. $\mathcal{X}$ an affine, then $\operatorname{Aut}(\Delta_n)$ acts on V_n^* without stabilisers, so a fine quotient exists by [KM]. Better still the lack of stabilisers automatically means everything globalises so $[V_n^*/\operatorname{Aut}(\Delta_n)]$ exists as an algebraic stack, and the projection from V_n^* is flat, so the quotient is even smooth if V_n is, i.e. if $\mathcal{X}$ is smooth. Furthermore the inductive definition of the P_n gives a map by way of nth derivation $V_n^* \to P_n$, and one easily checks that this induces an isomorphism on closed points,

$$[V_n^*/\operatorname{Aut}(\Delta_n)](\mathbb{C}) \xrightarrow{\sim} P_n \backslash \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p)(\mathbb{C})$$

and since everything is smooth, we conclude. $\square$

We wish to make this somewhat more explicit, so to this end observe that $\mathrm{Aut}(\Delta_n)$ maps naturally to $\mathrm{Aut}(\Delta_{n-1})$ with kernel $\mathbb{G}_a$ for $n \geq 2$, and of course $\mathrm{Aut}(\Delta_1) \xrightarrow{\sim} \mathbb{G}_m$. There's also an embedding of $\mathbb{G}_m$ in $\mathrm{Aut}(\Delta_n)$ by way of $z \mapsto \lambda z$, but this isn't a normal subgroup, so we have to be a little cautious as to how we use it. Nevertheless, we'll frequently identify $\mathbb{G}_m$ with its image in $\mathrm{Aut}(\Delta_n)$, and use the embedding of $\mathcal{O}_{V_n}$ that it provides, i.e.

$$\mathcal{O}_{V_n}(m) := \{f \in \mathcal{O}_{V_n} \,:\, f^\lambda = \lambda^m f,\, \lambda \in \mathbb{G}_m\}$$

In any case, we can use this to construct the quotient inductively, so that we get diagrams of the form,

$$\begin{tikzcd} V_n^* \arrow[r] & V_n^*/\mathbb{G}_a \arrow[r] \arrow[d] & V_n^*/\mathbb{G}_a/\mathrm{Aut}(\Delta_{n-1}) = V_n^*/\mathrm{Aut}(\Delta_n) \arrow[r] \arrow[d] & P_n \backslash \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p) \arrow[d] \\ & V_{n-1}^* \arrow[r] & V_{n-1}^*/\mathrm{Aut}(\Delta_{n-1}) \arrow[r] & P_{n-1} \backslash \bigcup_{n-1 \geq p \geq 2} \pi_{pn}^{-1}(\infty_p) \end{tikzcd}$$

where the last square is fibred by virtue of flat descent. Now let's simplify the notation a little by putting ∞_p equal to $\pi_{ip}^{-1}(\infty_p)$, irrespective of $i > p$, A_n the kernel of the natural map from $\mathrm{Aut}(\Delta_n)$ to $\mathrm{Aut}(\Delta_1)$, and note the isomorphism,

$$\mathcal{O}_{V_n^*}^{A_n} \xrightarrow{\sim} \mathcal{O}_{V_1^*} \otimes_{\mathcal{O}_{P_1}} \varinjlim_{\mathbf{t}} \mathcal{O}_{P_1}(t_2\infty_2 + \ldots + t_n\infty_n)$$

where, just to clear up notation , $\mathcal{O}_{P_1}(t_2\infty_2 + \ldots + t_n\infty_n)$ is just the push forward under π_{n1} of $\mathcal{O}_{P_n}(t_2\infty_2 + \ldots + t_n\infty_n)$. At this point $\mathbb{G}_m$ acts on the whole thing, so we get a map,

$$\mathcal{O}_{V_n^*}^{A_n} \cap \mathcal{O}_{V_n}(t_1) \longrightarrow \pi_* L_1^{t_1}(\varinjlim_{\mathbf{t}} t_2\infty_2 + \ldots + t_n\infty_n)$$

Now unfortunately A_n doesn't act on the individual $\mathcal{O}_{V_n}(t)$'s, but it acts on the suns of such, so that finally we get a map,

$$(\bigoplus_{t_1 \leq m} H^0(\mathcal{X}, \pi_*\mathcal{O}_{V_n}(t_1)))^{A_n} \longrightarrow \bigoplus_{t_1 \leq m} \varinjlim_{(t_2,\ldots,t_n)} H^0(\mathcal{X}, \pi_* L_1^{t_1}(t_2\infty_2 + \ldots t_n\infty_n))$$

where the directed limit structure comes from tensoring with various canonical sections $\mathbb{1}_{\infty_p}$, so certainly all the implied maps on the right are injective. Better still we don't need the full range of $(t_2, \ldots, t_n)$ going all the way to infinity in order to describe where an element on the left may end up. Indeed if $x_1, \ldots, x_l$ are local coordinates on $\mathcal{X}$, then, in an obvious notation we get coordinates $d^k x_j$, $O \leq k \leq n$ on V_n where $d^k x_j$ of a jet is its kth, or more correctly $1/k!$ times it, derivative of its jth projection. Whence elements of $\pi_*\mathcal{O}_{V_n}(t_1)$ are generated as an $\mathcal{O}_{\mathcal{X}}$ module by elements of the form,

$$(d^1\mathbf{x})^{P_1} \ldots (d^n\mathbf{x})^{P_n}$$

where P_i is some multi-indice, so $(d^1\mathbf{x}) = \prod_{j=1}^l (d^1 x_j)^{p_{1j}}$, etc., with $|P_i| = \sum_j p_{ij}$, and necessarily $|P_1| + 2|P_2| + \ldots + n|P_n| = t_1$. In order to compute the order of pole of such a thing around any ∞_p, we can proceed using appropriate liftings to $\mathrm{Aut}(\Delta_n)$ of the various kernels $\mathrm{Aut}(\Delta_p) \to \mathrm{Aut}(\Delta_{p-1})$, $2 \leq p \leq n$, say, $\xi_p(z) = z + \xi z^p$, and observe that each ξ_p acts on the basis $d^k x_j$ not just in an upper semi-triangular way, but fixing the $d^k x_j$ for $k < p$. Suppose now, and without loss of generality, that we're working on the affine patch $d^1 x_1 \neq 0$, then we require to express the $d^k x_j$, $2 \leq k \leq n$, $2 \leq i \leq l$ in terms of the $d^1 x_j / d^1 x_1$, the eigenfunctions corresponding to the divisors at infinity, and $d^k x_1$'s. We know, however, that the quotient exists and is smooth, so in the maximal ideal of a suitable point, say, $d^k x_j =$ for $k, j \neq 1$, the eigenfunctions cannot be in the square of the maximal ideal, since there's obviously at least one which isn't, and since the action is already linearisable on an affine by way of the ξ_p's we must be able to express the $d^k x_j$'s $2 \leq k \leq n$, $2 \leq i \leq l$ as a linear $\mathcal{O}_{V_1^*}$ sum of the $d^k x_1$'s and the eigenfunctions. Consequently the pole of an element of $\mathcal{O}_{V_n}(t_1)$ which happened to be invariant under A_n around ∞_i would be at worst $p_i + p_{i+1} + \ldots + p_n$, for some integers p_i satisfying $p_1 + 2p_2 + \ldots + np_n = t_1$, so in particular, no bigger than t_1. As such after tensoring with appropriate $\mathbb{I}_{\infty_p}$'s we finally obtain a map,

$$(\bigoplus_{t_1 \leq m} H^0(\mathcal{X}, \pi_* \mathcal{O}_{V_n}(t_1)))^{A_n} \longrightarrow \bigoplus_{t_1 \leq m} H^0(P_n, L_n^{\otimes t_1})$$

The advantage of this over the inductive definition of the L_i is that the left hand side is rather computable. Indeed the Euler characteristic of $\bigoplus_{t_1 \leq m} \pi_* \mathcal{O}_{V_n}(t_1)$ is for m sufficiently large and divisible, asymptotic to,

$$\frac{m^{dim \mathcal{X}(n+1)}}{\{(n+1) dim \mathcal{X}\}! (n!)^{dim \mathcal{X}}} {}^n C_{dim \mathcal{X}} (K_{\mathcal{X}}^{top} + O(\frac{1}{n}))$$

with appropriate corrections as per [T] when things aren't sufficiently divisible. Better still if $\mathcal{X}$ has small dimension, i.e. 2, we can compute the higher cohomology rather easily thanks to the semi-stability of $\Omega_{\mathcal{X}}$ with respect $K_{\mathcal{X}}$. Indeed this is effectively what's done in [GG], and it works in this generality thanks to [TY] or [Me]. Whence,

VI.1.4 Fact (cf. [GG]) *Let $\mathcal{S}$ be a minimal 2-dimensional algebraic stack, then for m sufficiently large, about 9 will do,*

$$\bigoplus_{t_1=0}^{\infty} H^0(\mathcal{S}, \pi_* \mathcal{O}_{V_n}(t_1))$$

is an algebra of transcendence degree $2(n+1)$.

Better still on taking A_n invariants we can profit from its filtration by $\mathbb{G}_a$'s to conclude that we loose precisely n dimensions. Infact we even see that the tautological bundles L_n are big provided $c_1^2 > \frac{2}{n+1} c_2$, which isn't bad given the approximations that we've employed, albeit that with a little fooling around for small n one sees that definition VI.1.2 allows direct computation of the Euler characteristic of sums of the L_i's, which indeed hovers around $2/n + 1$. Notice also that a similar, and rather easier, argument shows with the hypothesis of VI.1.4 that $L_n|_{P_n \otimes \mathcal{K}}$ is big for any n with $\mathcal{K}$ an appropriately general element of a high multiple of the linear system $K_{\mathcal{S}}$, and so we have,

VI.1.5 Fact/Summary *Let $\mathcal{S}$ be a smooth 2 dimensional algebraic stack with projective moduli, which is minimal and of general type, then its nth tautological jet bundle L_n is big for $c_1^2(\mathcal{S}) > \frac{2}{n+1} c_2(\mathcal{S})$. In addition for $\mathcal{K} \hookrightarrow \mathcal{S}$ a generic closed substack defined by a sufficiently large multiple of $K_{\mathcal{S}}$, $L_n|_{P_n \otimes \mathcal{K}}$ is big for any n.*

VI.2 Value Distribution

We wish to study the value distribution of holomorphic lines, or more generally ramified covers of the same around divisors in surfaces. It is, therefore, notationally and conceptually convenient to pretend that the punctured disc is a stack. More precisely rather than talking about log-stacks $(\mathcal{S}, \mathcal{D})$ we'll make,

VI.2.1 Convention/Definition *Unless there is a possibility for confusion we'll often use the word stack to include log-stacks $(\mathcal{S}, \mathcal{D})$, i.e. if $\mathcal{D}$ is a simple normal crossing divisor, we'll think of this as a stack $\tilde{\mathcal{S}}$ with infinite stabiliser around $\mathcal{D}$ in just the same way that the classical theory of 1-dimensional orbifolds thinks of a punctured disc. Technically speaking $\tilde{\mathcal{S}}$ doesn't exists even in the analytic topology, but the only thing that we'll have to be careful about is that for $f : \mathcal{C} \to \mathcal{S}$ a map between stacks in this broad sense, the pre-images of points with infinite stabiliser should have the same property, while the generic stabiliser should never be infinite.*

This is, as we've said, a notational/conceptual issue which allows us to re-phrase value distribution theory in terms of maps between stacks. In this respect let's note that for $\mathcal{C}$ a 'smooth complete curve' in the sense of VI.2.1, and L_n the tautological bundle on any jet space $\pi : P_n \to \mathcal{X}$ over a stack of any dimension, with $f^{(n)} : \mathcal{C} \to P_n$ the derivative of a map $f : \mathcal{C} \to \mathcal{X}$ we have,

$$L_{n \cdot f^{(n)}} \mathcal{C} \leq -\chi_{\mathcal{C}}^{top}$$

where $\chi_{\mathcal{C}}^{top}$ is the topological Euler characteristic of $\mathcal{C}$. The log-case of VI.1.5 is known for $\mathcal{S}$ a minimal log-stack of general type, so for H ample on the moduli,

$$H._f \mathcal{C} \leq \frac{1}{\kappa_{\mathcal{S}}} \chi_{\mathcal{C}}^{top}$$

for some $\kappa_{\mathcal{S}} < 0$, unless $f^{(n)}$ factors through the base locus of some linear system on the form $L_n^N \otimes \pi_n^* H^\vee$, for n, and N sufficiently large, i.e f satisfies a nth order ODE on $\mathcal{S}$.

We can argue similarly not just for a complete curve $\mathcal{C}$ and algebraic maps f, but for ramified covers $p : \mathcal{C} \to \mathcal{S}$. In this context the Euler characteristic is a function of $r \in \mathbb{R}_+$, which for $\mathcal{C}(r)$ the pre-image of the disc of radius r is given by,

$$-\chi_{\mathcal{C}}(r) = -\deg(p) \log r + \sum_{0 < |p(c)| < r} \log |\frac{r}{p(c)}| \mathrm{ord}_c(\mathrm{Ram}_p)$$

where, for simplicity, we suppose p étale over 0, and conventionally $\mathrm{ord}_c(\mathrm{Ram}_p) = 1$ if the stabiliser is infinite. With this in mind, and $f^{(n)}$ the nth derivative we have,

$$\rlap{\,/}\!\!\int_{\mathcal{C}(r)} f^{(n)*} c_1(L_n) \leq_{exc} -\chi_{\mathcal{C}}(r) + (\text{small error})$$

where the integral on the left is the standard Nevanlinna sort of integral, and the subscript exc. means r outside a set of finite Lebesgue measure, cf. [M1] V.3. Whence, again if $f^{(n)}$ doesn't factor through the base locus we get,

$$\rlap{\,/}\!\!\int_{\mathcal{C}(r)} f^* c_1(H) \leq_{exc} \frac{1}{\kappa_{\mathcal{S}}} \chi_{\mathcal{C}}(r) + (\text{small error})$$

There is also a similar statement for maps of discs to $\mathcal{S}$, but unfortunately it's only valid for discs that are not arbitrarily close in the compact open sense to the base locus. In particular, and rather unfortunately, one doesn't have the situation that one finds above whereby discs failing to satisfy an analogue to the latter inequality would have to factor through at least one irreducible component of the base, nor indeed do big discs even need to accumulate on an irreducible component, but only a connected one. This suggests that we make,

VI.2.2 Definition *Say that there is an isoperemtric inequality in Nevanlinna's sense for $\mathcal{S}$ if there is a proper closed substack $\mathcal{Z}$, and a constant $\kappa_{\mathcal{S}} < 0$ such that for every map $f : \mathcal{C}/\mathbb{C} \to \mathcal{S}$ from a ramified covering of the line not factoring through $\mathcal{Z}$ we have,*

$$\rlap{\,/}\!\!\int_{\mathcal{C}(r)} f^* c_1(H) \leq_{exc} \frac{1}{\kappa_{\mathcal{S}}} \chi_{\mathcal{C}}(r)$$

Notice that the given formulation implies the boundedness in moduli of maps from honest complete curves. Either way if the inequality is false, there is some sequence of maps f_m such that as $m \to \infty$,

$$\chi_{f_m} := \inf_{\substack{E \subset \mathbb{R}_+ \\ \lambda(E) < \infty}} \sup_{r \notin E} \chi_{\mathcal{C}_m}(r) \{ \rlap{\,/}\!\!\int_{\mathcal{C}_m(r)} f_m^* c_1(H) \}^{-1}$$

where λ is ordinary Lebesgue measure, without loss of generality, increases and satisfies $\lim_m \chi_{f_m} \geq 0$. Now quite generally given a sequence of maps g_m from appropriate $\mathcal{C}_m$'s to a stack $\mathcal{X}$ we can for $N \in \mathbb{N}$ define the Zariski closure $\mathcal{V}_N$ of g_m, $m \geq N$, to get a decreasing chain,

$$\cdots \supseteq \mathcal{V}_N \supseteq \mathcal{V}_{N+1} \supseteq \cdots$$

of reduced stacks, which must eventually stabilise at some $\mathcal{V}_\infty$, which we call the Zariski closure of the sequence. In the particular case that the isoperemetric inequality in Nevanlinna's sense is supposed to fail on a surface $\mathcal{S}$, we can find a divisor $\mathcal{D}$ in some jet space P_n containing the Zariski closure $f_m^{(n)}$. Either $\mathcal{D}$ is the closure of this sequence, or it is not, if it's not we take a component $\mathcal{D}^1$ of the closure which contains an infinite subsequence. Notice that $\mathcal{D}^1$ itself dominates some P_i for $i \leq n-2$, or maps to an algebraic curve in $\mathcal{S}$. Either way we can replace $\mathcal{D}^1$ by the divisor in the corresponding P_{i+1}, respectively an algebraic curve. There is some minor possibility that the $f_m^{(i+1)}$'s aren't Zariski dense in this replacement, but we'll just continue until we eventually do get some ODE of order $i+1$ (including the case $i=-1$) such that an infinite subsequence of the f_m^{i+1}'s factor through, and are Zariski dense, in the ODE considered as a divisor in P_{i+1}. Plainly by the same subterfuge be it for $\mathcal{D}$, or something smaller, we can even insist that every subsequence of the appropriate derivatives of the f_m's is also Zariski dense. As such, cleaning up the notations by supposing that the $f_m^{(n)}$'s and all subsequences thereof are dense in the nth order ODE $\mathcal{D}$, we first discuss the order 0 case. One of two things can happen: either the 0th order ODE is a curve of genus $g \geq 2$, more generally has normalisation étale covered by a complete 1-dimensional hyperbolic orbifold, and we immediately get a contradiction by the so called second main theorem of Nevanlinna theory, or it's rational or elliptic. In the latter case all such curves have to be included in $\mathcal{Z}$ anyway, so modulo the set of all such being finite we have a contradiction again. Whence supposing the said finiteness, our ODE is of order at least 1, and we have a foliation by curves of the form $(\mathcal{D}, L_n|_{\mathcal{D}})$. Supposing LCR, I.6.1, we can clean this up to a model $\pi : (\mathcal{X}, \mathcal{F}) \to (\mathcal{D}, L_n|_{\mathcal{D}})$ with log-canonical singularities, and no radial singularities, then pass to a minimal model $\rho : (\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{X}_0, \mathcal{F}_0)$. The semi-stability of $\Omega_{\mathcal{S}}$ with respect to $K_{\mathcal{S}}$ continues to imply that $K_{\mathcal{F}}$ is big on a generic cut by some divisor in a high multiple of the $K_{\mathcal{S}}$ linear system, so $K_{\mathcal{F}_0}$ has Kodaira dimension at least $n = \dim \mathcal{X} - 1$. Furthermore $K_{\mathcal{F}}$ and $K_{\mathcal{X}}$ contain in the cone they span a big divisor, so the same is true for $K_{\mathcal{F}_0}$ and $K_{\mathcal{X}_0}$. Indeed we can suppose that $\mathcal{F}_0$ is not of general type, and harmlessly replace $\mathcal{X}$ by a modification that actually maps to $\mathcal{X}_0$, so we must have $K_{\mathcal{X}_0}.K_{\mathcal{F}_0}^{n-1} > 0$, which by II.2.1 is enough to guarantee that for some $\epsilon > 0$, $K_{\mathcal{F}_0} + \epsilon K_{\mathcal{X}_0}$ is big. Now in carrying out these various modifications in the 2-category of stacks we may have to introduce some extra monodromy, whence the various maps $f_m^{(n)}$ may only lift, be it on $\mathcal{X}$ or $\mathcal{X}_0$ on some ramified cover $\tilde{\mathcal{C}}_m$ of $\mathcal{C}_m$. At which point we may get some slightly different Euler characteristics $\chi_{\tilde{f}_m}$ in the obvious notation. The difference between the limiting values of χ_{f_m} and $\chi_{\tilde{f}_m}$ is completely dealt with by the various computations of how invariant maps meet singularities in [M1] V.4., and by op. cit. it's still the case that $\lim_m \chi_{\tilde{f}_m} \geq 0$, where, without loss of generality, we sub-sequence so that these increase. Now define harmonic measures on $\mathcal{X}_0$ by the formulae,

$$T_m(r) : A^{(1,1)}(\mathcal{X}_0) \longrightarrow \mathbb{C} : \omega \longmapsto \Big(\fint_{\mathcal{C}_m(r)} f^* c_1(H)\Big)^{-1} \fint_{\mathcal{C}_m(r)} \tilde{f}^* \omega$$

after possible subsequencing to dump any $\tilde{f}_m$'s lying in the loci which is modified during the minimal model algorithm, we can then subsequence in m and r to obtain a closed transverse invariant measure $d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ satisfying,

$$d\mu_{\mathcal{X}_0/\mathcal{F}_0} = \lim T_m(r) \ , \ K_{\mathcal{F}_0}.d\mu_{\mathcal{X}_0/\mathcal{F}_0} = 0$$

where the latter inequality follows from [M1] V.6.1. The measure $d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ is therefore parabolic satisfying all of the hypothesis of V.1.2, including the bis., V.1.5. Similarly there is a measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on $\mathcal{X}$, or indeed any invariant modification of it, satisfying $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, so $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} = 0$ since $d\mu_{\mathcal{X}/\mathcal{F}}$ intersects every Cartier divisor non-negatively. Furthermore any part of $d\mu_{\mathcal{X}/\mathcal{F}}$ in the loci modified by the minimal model algorithm (cf. the next section for a precise definition of 'any part') must push forward by the projection π to $\mathcal{S}$ to zero or one of finitely many rational curves. Consequently it is notationally unambiguous to assert,

VI.2.3 Claim *Suppose LCR* I.6.1 *then $\pi_* d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ is at worst a countable sum of rational or elliptic curves, i.e. $\pi_* d\mu_{\mathcal{X}_0/\mathcal{F}_0} = \sum a_i \mathcal{L}_i$, for $\mathcal{L}_i$ the current of integration over a closed substack of $\mathcal{S}$ whose normalisation has non-negative topological Euler characteristic.*

The proof of the claim will occupy the next several sections. Curiously,

VI.2.4 Remark *If similarly we suppose that solutions of the initial ODE, considered as invariant discs did not converge in the compact open topology provided they weren't arbitrarily close to some closed substack of $\mathcal{D}$, i.e. almost solutions of another ODE, then again by [M1] V.6.1 we'd be able to produce an invariant measure for $\mathcal{F}_0$ with the same properties, and whence the same claim holds. Nevertheless it doesn't seem to follow from this rather strong hyperbolicity of solutions to the ODE that we get VI.2.2 due to the possible degeneracies in the associated Kobayashi type metric.*

VI.3 Inductive Decomposition

Quite generally when presented with a measure $d\mu$, say of type $(1,1)$ to fix ideas on a stack $\mathcal{X}$ with projective moduli $\pi : \mathcal{X} \to X$ we can decompose into smaller pieces. More precisely if we filter the chow scheme of X (bear in mind the existence of the Hilbert scheme of $\mathcal{X}$ isn't actually a theorem, whence the subtrefuge) by dimension d and degree δ, and starting in dimension 1 ask,

VI.3.1 Question *For fixed δ, $\epsilon > 0$ and H ample, how many curves $\mathcal{C}$ of degree δ satisfy $H.\mathbb{I}_{\mathcal{C}} d\mu \geq \epsilon$?*

Plainly the answer is necessarily finite, since if $\mathcal{C}_i, i \in I$ is the collection of such curves then since $\mathbb{I}_{\mathcal{C}_i \cap \mathcal{C}_j} d\mu = 0$, we must have $H.d\mu \geq \#I\epsilon$, and so independently of the degree we find countably many curves $\mathcal{C}_i$ with $\mathbb{I}_{\mathcal{C}_i} d\mu \neq 0$, and we can replace $d\mu$ by $d\mu^1$ defined by,

$$d\mu^1 := d\mu - \sum_i \mathbb{I}_{\mathcal{C}_i} d\mu$$

Now by construction $d\mu^1$ has no support on curves and we can ask,

VI.3.2 Question bis. *For fixed δ, $\epsilon > 0$ how many surfaces $\mathcal{S}$ of degree δ satisfy $H.\mathbb{I}_{\mathcal{S}} d\mu^1 \geq \epsilon$?*

At this juncture if we have two distinct surfaces $\mathcal{S}_i, \mathcal{S}_j$ then $\mathcal{S}_i \cap \mathcal{S}_j$ has dimension at most 1 so $\mathbb{I}_{\mathcal{S}_i \cap \mathcal{S}_j} d\mu^1 = 0$, so that again we find, independently of the degree countably many such surfaces $\mathcal{S}_i$, and we write,

$$d\mu^2 := d\mu^1 - \sum_i \mathbb{I}_{\mathcal{S}_i} d\mu^1$$

and, of course, $d\mu^2$ cannot have support on either curves or surfaces. This process can certainly be continued inductively to obtain,

VI.3.3 Fact *In each dimension d there are countably irreducible substacks $\mathcal{Y}_i^d$ satisfying $\mathbb{I}_{\mathcal{Y}_i^d} d\mu^{d-1} \neq 0$ where $d\mu^e$ is defined inductively by,*

$$d\mu^e := d\mu^{e-1} - \sum_i \mathbb{I}_{\mathcal{Y}_i^e} d\mu^{e-1}$$

In particular by III.3.3, *each $\mathbb{I}_{\mathcal{Y}_i^e} d\mu^{e-1}$ is a closed positive current on $\mathcal{Y}_i^e$, and not just a current on $\mathcal{X}$ with support.*

It's evidently natural enough to refer to the terms $\mathbb{I}_{\mathcal{Y}_i^e} d\mu^{e-1}$ appearing in the implied decomposition of a measure as its *components*. Notice in particular, by construction, a component which we'll denote $\mathbb{I}^0_{\mathcal{Y}_i^e} d\mu$ satisfies $\mathbb{I}_{\mathcal{V}}.\mathbb{I}^0_{\mathcal{Y}_i^e} d\mu = 0$ for any substack $\mathcal{V}$ not containing $\mathcal{Y}_i^e$.

Now if we apply the discussion in the presence of a foliation $\mathcal{F}$ together with transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ then it goes without saying that all the components are supported on invariant subvarieties. If moreover, the foliation arises from an ODE $\mathcal{D}$ of order n on an algebraic surface $\mathcal{S}$ then we have more structure still. Specifically for P_n the nth jet bundle we have, on supposing LCR, a situation of the form,

$$\begin{array}{ccc} & (\mathcal{X},\mathcal{F}) & \\ \sigma \swarrow & \downarrow \rho & \\ & & \\ P_n \hookleftarrow & X_n & (=\mathcal{D}) \end{array}$$

where $\rho : (\mathcal{X},\mathcal{F}) \to X_n$ is a resolution of the induced foliation by curves on X_n. Consequently if $d\nu$ is a component of $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a substack $\mathcal{Y}$, then $\sigma(\mathcal{Y})$ defines a subvariety of P_n. One can then ask for the necessarily unique P_i (including, by the way, P_0, i.e. our original surface) such that the image of $\sigma(\mathcal{Y})$ is a divisor dominating P_{i-1} or, alternatively $\mathcal{Y}$ simply pushes forward to a point. This ultimate degenerate case is plainly agreeable, since it says that our component on $\mathcal{X}$ has nothing to do with $\mathcal{S}$, while the other cases allow us to think of $\mathcal{Y}$ itself as an ODE, $\mathcal{D}'$, say, of smaller order m. Plainly we can use LCR to form diagrams of the form,

$$\begin{array}{ccc} (\mathcal{X},\mathcal{F}) & \xleftarrow[\iota]{} & (\mathcal{Y},\mathcal{F}|_{\mathcal{Y}}) \\ \big\downarrow & & \big\downarrow \\ P_n & \longrightarrow & P_m \leftarrow (\mathcal{Y}_0,\mathcal{G}) \end{array}$$

where ι is an embedding, $(\mathcal{Y}_0,\mathcal{G})$ is a resolution of the foliation singularities of the mth order ODE defined by $\mathcal{Y}$, and $\mathcal{X}$ a possibly bigger modification of our original $\mathcal{X}$ in order to guarantee that $(\mathcal{Y},\mathcal{F}|_{\mathcal{Y}})$ maps to $\mathcal{Y}_0$. In any case the important thing is that the components of a transverse invariant measure associated to such an ODE are themselves naturally what might be termed dense, or better *diffuse* components of another ODE, i.e. of the form $d\mu^t op$, in the notation of VI.3.3. Plainly, however, this discussion is completely useless unless we can control the diffuse component. To this end we have,

VI.3.4 Fact *Suppose LCR & let* $\sigma : (\mathcal{X},\mathcal{F}) \to P_n$ *be a resolution of singularities of an ODE* $\mathcal{D}$ *of order* $n \geq 1$ *on an algebraic surface with* $d\nu$ *the diffuse component of a transverse invariant measure with* $K_{\mathcal{F}}.d\nu \leq 0$, *then infact* $d\nu$ *doesn't exist.*

proof By [M1] IV.7.5 there is a birational map $\rho : (\mathcal{X},\mathcal{F}) \dashrightarrow (\mathcal{X}_0,\mathcal{F}_0)$ by way of contractions and flips to a minimal model, such that $K_{\mathcal{F}} = \rho^* K_{\mathcal{F}_0} + \mathcal{E}_0$ for $\mathcal{E}_0 \subset \mathcal{X}$ a divisor supported on invariant substacks, and on which the induced foliation is even by rational curves. Since $d\nu$ is diffuse, $\mathcal{E}_0.d\nu \geq 0$, so infact $0 = K_{\mathcal{F}_0}.\rho_* d\nu = \mathcal{E}_0.d\nu$. On the other hand, as remarked prior to VI.2.3 the cone generated by $K_{\mathcal{F}}$ & $K_{\mathcal{X}}$, or better $K_{\mathcal{F}_0}$ & $K_{\mathcal{X}_0}$, contains a big divisor, so $K_{\mathcal{X}}.d\nu = (K_{\mathcal{X}} - K_{\mathcal{F}}).d\nu > 0$. Whence if $\mathcal{Z} \hookrightarrow \mathcal{X}$ is the singular substack then by IV.7.4 the residual measure $s_{\mathcal{Z},d\nu} \neq 0$. Now form the blow up $\tilde{\rho} : (\tilde{\mathcal{X}},\tilde{\mathcal{F}}) \to (\mathcal{X},\mathcal{F})$, or if one prefers resolve $\mathcal{I}_{\mathcal{Z}}$ by a sequence of blow ups in smooth invariant centres, and let $\mathcal{E}$ be the total exceptional divisor with $d\tilde{\nu}$ the extension of $d\nu$ across $\mathcal{E}$ by zero, so that $s_{\mathcal{Z},d\nu}(1) = \mathcal{E}.d\tilde{\nu}$. On the other hand $d\tilde{\nu}$ still intersects every Cartier divisor $\mathcal{D}$ on $\tilde{\mathcal{X}}$ non-negatively. Indeed if $\mathcal{D}.d\tilde{\nu} < 0$, then $\mathbb{I}_{\mathcal{D}} d\tilde{\nu} \neq 0$. At the same time $d\tilde{\nu}$ has by construction no support on $\mathcal{E}$, so $\tilde{\rho}$ must be bi-rational at the generic point of $\mathcal{D}$, whence $\mathbb{I}_{\rho(\mathcal{D})} d\nu \neq 0$, contrary to it's construction. Better still by VI.1.5, $K_{\mathcal{F}_0}$ has Kodaira dimension at least $\dim\mathcal{X} - 1$, so arguing as per V.2.2 we deduce an absurdity. $\square$

Which extends to,

VI.3.5 Refined Fact *Suppose LCR & let* $\sigma : (\mathcal{X},\mathcal{F}) \to P_n$ *be a resolution of an ODE* $\mathcal{D}$ *of order* $n \geq 1$ *with* $d\mu_{\mathcal{X}/\mathcal{F}}$ *a parabolic invariant measure, then* $d\mu_{\mathcal{X}/\mathcal{F}}$ *has no components supported in the smooth locus of* $\mathcal{F}$ *other than those defined by ODEs of order at most* 0 *(so say conventionally points have order* -1 *), and those of order* 0 *are rational or elliptic curves on* $\mathcal{S}$.

proof Let $d\nu$ be a component supported on an invariant substack $\mathcal{Y}$ whose generic point does not lie in $\mathrm{sing}(\mathcal{F})$. Furthermore let $\rho : (\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{X}_0, \mathcal{F}_0)$ be a minimal model as before with $K_{\mathcal{F}} = \rho^* K_{\mathcal{F}_0} + \mathcal{E}_0$. Since $d\mu_{\mathcal{X}/\mathcal{F}}$ is parabolic $K_{\mathcal{F}_0}.\rho_* d\mu_{\mathcal{X}/\mathcal{F}} = 0$. Notice, moreover, that at every stage of the minimal model algorithm one either contracts, or flips, an invariant subvariety on which the induced foliation is by rational curves. In particular unless $\mathcal{Y}$ pushes forward to a rational curve or a point on our surface $\mathcal{S}$, it's generic point is unaffected by the algorithm, and we get a component $d\nu_0$ of $\rho_* d\mu_{\mathcal{X}/\mathcal{F}}$ which necessarily satisfies $K_{\mathcal{F}_0}.d\nu_0 = 0$.

Better still if $(\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0})$ is the induced foliation, and $\sigma^\# : (\mathcal{V}, \mathcal{G}) \to P_m$ a resolution of singularities of the induced ODE, supposed of non-negative order, we have a rational map (which may be highly non-birational) of foliated varieties $(\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0}) \dashrightarrow (\mathcal{V}, \mathcal{G})$, so that if $\rho^\# : (\mathcal{V}, \mathcal{G}) \dashrightarrow (\mathcal{V}_0, \mathcal{G}_0)$ is a minimal model, we can appeal to [M1] VI.1.3 to form a diagram,

$$\begin{array}{cccc} & (\tilde{\mathcal{X}}_0, \mathcal{F}_0) \hookleftarrow & (\tilde{\mathcal{Y}}_0, \mathcal{F}_0|_{\tilde{\mathcal{Y}}_0}) \underset{\tilde{\sigma}^\#}{\to} & (\mathcal{V}_0, \mathcal{G}_0) \\ \text{invariant modification} = \tilde{\rho} \downarrow & & \downarrow & \uparrow \\ & (\mathcal{X}_0, \mathcal{F}_0) \hookleftarrow & (\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0}) \dashrightarrow & (\mathcal{V}, \mathcal{G}) \end{array}$$

where $\tilde{\sigma}^\#$ is an honest map. Now for $\tilde{\mathcal{G}}_0$ the induced foliation on $\tilde{\mathcal{Y}}_0$ which we may suppose smooth with canonical singularities, we have,

$$K_{\tilde{\mathcal{G}}_0} = K_{\mathcal{F}_0}|_{\tilde{\mathcal{Y}}_0} - \mathcal{D}$$

for $\mathcal{D}$ some effective Cartier divisor supported on $\tilde{\mathcal{Y}}_0$. Furthermore we have a map from $\tilde{\sigma}^\# K_{\mathcal{G}_0}$ to $K_{\tilde{\mathcal{G}}_0}$ away from $\mathrm{sing}(\mathcal{G}_0)$ so at worst,

$$K_{\tilde{\mathcal{G}}_0} = \tilde{\sigma}^{\#*} K_{\mathcal{G}_0} + \mathcal{D}_+ - \mathcal{D}_-$$

where the Cartier divisor $\mathcal{D}_-$ maps down to $\mathrm{sing}(\mathcal{G}_0)$. Now take an irreducible component D_- of $\mathcal{D}_-$ and inductively blow up in the reduced structure of its image until such times that we get a map of germs,

$$\mathrm{Spec}\ \mathcal{O}_{\tilde{\mathcal{Y}}_0, D_-} \underset{\tilde{\sigma}^\#}{\longrightarrow} \mathrm{Spec}\ \mathcal{O}_{\tilde{\mathcal{V}}_0, E_-}$$

for E_- a Cartier divisor on an appropriate modification of $\mathcal{V}_0$. In terms of the induced canonical bundle $K_{\tilde{\mathcal{G}}_0^\#}$ on $\tilde{\mathcal{V}}_0$ we must have an honest map $\tilde{\sigma}^{\#*} K_{\tilde{\mathcal{G}}_0^\#}$ to $K_{\tilde{\mathcal{G}}_0}$ around D_-, and since the singularities of $(\mathcal{V}_0, \mathcal{G}_0)$ are canonical we conclude that D_- doesn't exist. Consequently,

$$K_{\mathcal{F}_0}|_{\tilde{\mathcal{Y}}_0} = K_{\tilde{\mathcal{G}}_0} + \mathcal{D} = \tilde{\sigma}^{\#*} K_{\mathcal{G}_0} + \mathcal{D} + \mathcal{D}_+$$

Now $d\nu_0$ is diffuse in $\mathcal{Y}_0$, so infact $K_{\mathcal{G}_0}.(\tilde{\sigma}^\#)_* d\nu_0 = 0$, and we can appeal to VI.3.4, or slightly more correctly its proof, to deduce that $d\nu_0$ cannot exist for an ODE of order at least 1, while if the order is zero, the minimal model is just the normalisation of the corresponding 1-dimensional substack of $\mathcal{S}$, so it must be rational or elliptic. $\square$

VI.4 Singular Components

To complement our previous investigation, we have to consider the possibility that a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on a foliated stack $(\mathcal{X}, \mathcal{F})$ develops components inside the singular locus itself, so, say a closed irreducible substack $\mathcal{Z} \hookrightarrow \mathrm{sing}(\mathcal{F})$. Now the definition here is that all smooth 1-forms vanishing in $K_{\mathcal{F}}$ vanish on $\mathcal{Z}$, which is not particularly easy to work with. As such the following is rather useful, i.e.

VI.4.1 Lemma (local on $\mathcal{X}$) *Let $\mathcal{V}$ be a closed invariant irreducible substack containing $\mathcal{Z}$ such that for ∂ a local generator of the foliation $\partial(\mathcal{I}_{\mathcal{V}})$ generates $\mathcal{I}_{\mathcal{V}}$, then $d\mu$ is invariant by the induced foliation on $\mathcal{V}$.*

proof Let ω be a 1-form mapping to zero on the induced canonical of $\mathcal{V}$ with N the kernel of $\Omega^1_{\mathcal{X}} \to K_{\mathcal{F}}$. A priori, since $\mathcal{V}$ is irreducible we have $\partial(\omega) \in \mathcal{I}_{\mathcal{V}}$. However by hypothesis $\partial(\mathcal{I}_{\mathcal{V}})$ generates $\mathcal{I}_{\mathcal{V}}$ so if $f_1, \ldots, f_t$ generate $\mathcal{I}_{\mathcal{V}}$ we can solve,

$$\partial(g_i df_i) = g_i \partial(f_i) = \partial(\omega)$$

for some functions g_i. Whence $\omega - g_i df_i \in N$, and since the component is the honest push-forward of a measure on $\mathcal{Z}$ by III.3.3, we are done. $\square$

We can build on this to discern what happens to the component under blowing up. Specifically we have a weak branching scheme $\mathfrak{W}_{\mathcal{Z}}$, a priori at the generic component, but infact with well defined formal Zariski closure by EmbLCR, albeit that this is initially irrelevant. Now suppose $\mathcal{V}$ is a smooth substack of sing($\mathcal{F}$) containing $\mathcal{Z}$, and let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up in $\mathcal{V}$, with $\mathcal{E}$ the exceptional divisor, then we assert,

VI.4.2 Claim *Any singular component $d\nu$ of the proper transform of $d\mu$ dominating our given component, i.e. $\rho_* d\nu = \mathrm{I\!I}^0_{\mathcal{Z}} d\mu$, factors through the proper transform of $\mathfrak{W}_{\mathcal{Z}}$.*

proof The discussion is generic on $\mathcal{Z}$, so take $z_1, \ldots, z_m$ functions such that $dz_1, \ldots, dz_m$ generate $\Omega_{\mathcal{Z}}$ locally. By hypothesis for ∂ a local generator, $\partial(z_i) \in \mathcal{I}_{\mathcal{V}}$, so after blowing up $\partial(z_i) \in \mathcal{I}_{\mathcal{E}}$. Whence for $\partial_{\mathcal{E}}$ the generator of the induced foliation $\partial_{\mathcal{E}}(z_i) = 0$. As such $dd^c|z_i|^2 d\nu = 0$ by VI.4.1 whenever the induced foliation on $\tilde{\mathcal{X}}$ is log-flat around $\mathcal{E}$ at the generic point of the support. This is, however, necessarily the case at all points outside the proper transform of $\mathfrak{W}_{\mathcal{Z}}$. $\square$

Now while this doesn't quite allow us to assert that every singular component on the modification factors through the proper transform of the weak branch, it does allow us to assert that if we have a singular component on $\mathcal{X}$ then there is a singular component on $\tilde{\mathcal{X}}$, with similar sorts of properties, factoring through the proper transform of $\mathfrak{W}_{\mathcal{Z}}$. The consequence of this is that we may safely employ EmbLCR or more correctly its proof, cf. I.6.7. Specifically (starting from an empty background divisor to avoid further confusion) we arrive to the situation where $\mathfrak{W}_{\mathcal{Z}}$ is smooth with log-canonical singularities. We may however be unlucky and find that $\mathcal{Z}$ (strictly speaking a different $\mathcal{Z}$, but still the support of a singular component) is still singular in $\mathfrak{W}_{\mathcal{Z}}$. In which case we put $\mathfrak{W}_{\mathcal{Z}} = \mathfrak{W}^0_{\mathcal{Z}}$, with $\mathfrak{W}^1_{\mathcal{Z}}$ the weak branch at the generic point of $\mathcal{Z}$ in $\mathfrak{W}^0_{\mathcal{Z}}$. We can, without difficulty, modify the ambient space so that $\mathfrak{W}^1_{\mathcal{Z}}$ is also smooth with induced log-canonical singularities, and, of course, continue until we find ourselves in the situation where our component is generically smooth for the induced foliation in $\mathfrak{W}^p_{\mathcal{Z}}$. This latter statement has perfect sense, since components are always inside the trace of any $\mathfrak{W}^i_{\mathcal{Z}}$, and the trace of this is just some bunch of components of the singular locus of $\mathfrak{W}^{i-1}_{\mathcal{Z}}$.

The unfortunate thing in this discussion is that $\mathfrak{W}^p_{\mathcal{Z}}$, or indeed any of the $\mathfrak{W}^i_{\mathcal{Z}}$ for $i > 0$ may not actually come from the weak branches in the ambient space. Necessarily there is a maximal i for which this is true, so our initial goal is simply to understand '$\mathrm{I\!I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$' for $\mathfrak{W}$ a weak branch in $\mathcal{X}$, which although not immediately good enough to understand our given singular component is plainly very much closer to it than what we know on $\mathcal{X}$. Before proceeding let's pull,

VI.4.3 Rabbit out of the Hat *A numerically relevant component for $\mathfrak{W}$ of $d\mu_{\mathcal{X}/\mathcal{F}}$ (i.e. one that has $\mathcal{Y}.\mathrm{I\!I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0$, for $\mathcal{Y}$ a component of the trace, and, irrespective of how the latter is defined) is supported on a substack $\mathcal{V}$ of $\mathcal{X}$ lying in the pseudo trace of $\mathfrak{W}$, i.e. the completion of $\mathcal{V}$ in the trace of $\mathfrak{W}$ is non-empty, and factors through $\mathfrak{W}$.*

As such $\mathrm{I\!I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ will just really be a short hand for encoding the components of $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on substacks in the pseudo trace of $\mathfrak{W}$, or, if one prefers, infinitely tangent to it. To understand $\mathrm{I\!I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, however, one needs some knowledge of how the eigenvalues degenerate on $\mathfrak{W}$. Fortunately this comes out in the wash from the proof of EmbLCR, or more correctly the above variant. Indeed there are a bunch of smooth weak branches, albeit not necessarily very weak, in the ambient space $\mathcal{X}$ which we can suppose with log-canonical singularities for their induced foliations, which we take to be nodes of a directed graph $\mathbb{W}$. We put a directed arrow $\mathfrak{W}' \to \mathfrak{W}''$, if the latter has bigger dimension than the former, and the completion of $\mathfrak{W}'$ in the trace of $\mathfrak{W}''$ is non-empty, and locally a weak branch in $\mathfrak{W}''$ completed in the said intersection.

The directed graph $\mathbb{W}$ is filtered by co-dimension by directed subgraphs $F^p\mathbb{W}$ in the obvious way, and an important thing to note is that the formal stack $\mathfrak{W}i$ which is the union over all nodes of pure co-dimension i is actually smooth. Plainly given VI.4.3, we can talk about $\mathbb{I}_{F^p\mathbb{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ etc. in the obvious way.

VI.5 Numerical Relevance

We take up where we left off, so that we have not just our foliated stack $(\mathcal{X}, \mathcal{F})$ - for the moment still with projective moduli- but our directed graph $\mathbb{W}$, and various filtrations $F^\bullet\mathbb{W}$. As such for $\mathfrak{W}$ a node we wish to investigate the numerical properties of $\mathbb{I}_{\mathbf{W}} d\nu$ for $d\nu$ a component of our transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a substack $\mathcal{V}$. Our initial aim is to prove VI.4.3, so to begin with we'll forget our previous discussion and just pretend we can define $\mathbb{I}_{\mathbf{W}} d\nu$ as a germ of a closed measure around the trace of $\mathfrak{W}$ by way of asymptotic expansion, cf. III.6.3. A postiori, however, a wholly algebraic definition will emerge. In any case plainly $\mathbb{I}_{\mathbf{W}} d\nu$ should only be considered numerically relevant if $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\nu \neq 0$, for $\mathcal{Y}$ a Cartier divisor on $\mathfrak{W}$ corresponding to some irreducible component of the trace. We therefore have the following possibilities,

VI.5.1 Case I *$\mathcal{V}$ is contained in the pseudo trace of $\mathfrak{W}$.*

Obviously this is purely algebraic, wholly satisfactory, and we move on. In particular since $d\nu$ is a component $\mathfrak{W} \cap \mathcal{V}$ must be a Zariski dense formal substack of $\mathcal{V}$. This is still true no matter how much we blow up, so we may aswell, albeit that it's only for convenience, suppose that $\mathcal{V}$ is smooth, and we distinguish,

VI.5.1 bis Case II *Otherwise*

Now what we'll do next is actually a very general fact with next to no hypothesis beyond projectivity of the moduli, and the strict difference of dimension between $\mathfrak{W} \cap \mathcal{V}$ and $\mathcal{V}$. Specifically, we may aswell suppose that the trace is $\mathcal{Y}$, so that $\mathcal{Y} \cap \mathcal{V}$ is either a Cartier divisor in a given component of $\mathfrak{W} \cap \mathcal{V}$ or all of the component. In the latter case, the component is far from Zariski dense in $\mathcal{V}$, so we may aswell say that $\mathbb{I}_{\mathbf{W}} d\nu$ is a sum $\sum \mathbb{I}_{\mathbf{W}_i} d\nu$ over the components which are not in $\mathcal{Y} \cap \mathcal{V}$, then say $\mathcal{Y}.\mathbb{I}_{\mathbf{W}_i} d\nu \neq 0$, and replace $\mathcal{Y} \cap \mathcal{V}$ by a, possibly non-reduced, but regularly embedded algebraic stack $\tilde{\mathcal{Y}} \hookrightarrow \mathfrak{W}_i$. As such we get ideals I_n defining $\mathfrak{W}_i$ in the completion of $\mathcal{V}$ in $\tilde{\mathcal{Y}}$ together with a standard thickening sequence,

$$0 \longrightarrow I_{n+1} \longrightarrow I_n \longrightarrow N^{-n}_{\tilde{\mathcal{Y}}/\mathfrak{W}_i} \longrightarrow 0$$

To make sections, however, of $H^0(\mathcal{V}, H^m I_n)$, for H ample on the moduli, is cheap, e.g. it's no problem to take n of order m^2, yet if $\mathcal{E}_n$ is the exceptional divisor on the blow up $\mathcal{V}_n \to \mathcal{V}$ in I_n we must have,

$$\frac{m}{n} H.d\nu \geq \frac{1}{n}\mathcal{E}_n.d\nu_n \geq \mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\nu > 0$$

where $d\nu_n$ is the extension by zero of $d\nu$ on $\mathcal{V}_n$, and we profit from the fact that $d\nu$ is a component. This, though, is nonsense so we must be in Case I. Better still the essential of the contradiction is,

VI.5.2 Fact/Definition *In the above notation, the following are equivalent,*

(I) *$d\nu$ is numerically relevant, i,e, the limit of the above intersection numbers $1/n\mathcal{E}_n.d\nu$ which we take to be the definition of $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\nu$ is non-zero.*

(II) *The support $\mathcal{V}$ of $d\nu$ is in the pseudo trace of $\mathfrak{W}$.*

So as promised we have VI.6.3, and we'd ideally like to deduce a nefness statement for $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ from the same for $d\mu_{\mathcal{X}/\mathcal{F}}$, and, perhaps rather surprisingly, this is indeed possible, i.e.

VI.5.3 Fact *Suppose, as it does, $d\mu_{\mathcal{X}/\mathcal{F}}$ satisfies V.1.2, then for any component $\mathcal{Y}$ of its trace, $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ ≥ 0.*

proof This is a reasonably intricate extension of V.4.1, therefore, we'll proceed in stages. Firstly we'll do the case when $\mathfrak{W}$ considered as a node of $\mathbb{W}$ has no directed arrows coming out, i.e. it's at the bottom of our Christmas tree equivalently there is no degeneration in the eigenvalues, before proceeding to the general case. Furthermore, even in the general case, we'll firstly give a slightly heuristic argument in order to convey the idea, which we'll subsequently make rigorous.

So let's get underway with the mildly heuristic argument, according to which we'll hypothesise a well defined residual measure in a neighbourhood associated to $\mathfrak{W}$ in a neighbourhood of the component $\mathcal{Y}\cap\mathfrak{W}$ of $\mathrm{sing}(\mathcal{F})$, so, for example $\mathfrak{W}$ algebraic, but the condition is radically weaker than this, cf. III.6.3. Now take $p\in\mathbb{N}$, and let I_p be the ideal in $\mathcal{X}$ of the pth thickening of $\mathcal{Y}\cap\mathfrak{W}$ by $\mathfrak{W}$, i.e. if x_i define $\mathfrak{W}$ in the completion of $\mathcal{X}$ in the trace of $\mathfrak{W}$, and $y=0$ is the local equation for $\mathcal{Y}\cap\mathfrak{W}$ (which, without loss of generality can be taken as coming from a global Cartier divisor) we're just looking at the ideal (x_i, y^p). This has a rather easy resolution by a sequence of blow ups $\mathcal{X}_{i+1}\to\mathcal{X}_i$ in smooth centres. Indeed if we blow up in $\mathcal{Y}\cap\mathfrak{W}$, with $\mathcal{X}_0=\mathcal{X}$, and the former $\mathcal{X}_1$, then the pull-back of I_p is resolved everywhere, except at the proper transform of $\mathfrak{W}$, $\mathfrak{W}_1$, say, itself isomorphic to $\mathfrak{W}$, where it becomes 'I_{p-1}' for the induced weak branch. Plainly $\mathcal{X}_2\to\mathcal{X}_1$ is just the blow up in the appropriate component of the singular locus, and so forth. Denote the various maps $\mathcal{X}_j\to\mathcal{X}_i$ by ρ_{ij}, $j\geq i$, and $\mathcal{E}_i$ the exceptional divisor for $\mathcal{X}_i\to\mathcal{X}_{i-1}$, $p\geq i\geq 1$, then the resolution of I_p is,

$$\mathcal{E}^p := \sum_{i=1}^{p}\rho_{ip}^*\mathcal{E}_i$$

so that in particular on $\mathfrak{W}_p(\xrightarrow{\sim}\mathfrak{W})$, $\mathcal{O}_{\mathfrak{W}_p}(\mathcal{E}^p)\xrightarrow{\sim}\mathcal{O}_{\mathfrak{W}}(p\mathcal{Y})$. Furthermore by hypothesis we have well defined measures $d\mu^p_{\mathcal{X}/\mathcal{F}}$ with the property that $\mathcal{E}^p.d\mu^p_{\mathcal{X}/\mathcal{F}}\geq 0$. This global fact can be refined by breaking down $d\mu^p_{\mathcal{X}/\mathcal{F}}$ into pieces. Specifically the working hypothesis on the existence of a residual measure associated to $\mathfrak{W}$ implies, as per III.3.3, that there are well defined germs of closed measures $\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$, $\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ in a neighbourhood of $\mathcal{Y}\cap\mathfrak{W}$, and this is good enough to divide up most of the calculation. Identifying $\mathfrak{W}$ with $\mathfrak{W}^p$, and leaving the latter alone, we certainly have,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} = \mathcal{Y}.\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$$

while notationally taking $\mathcal{Y}$ to be the same as some global Cartier divisor with $\|\mathbb{I}_{\mathbf{W}}\|, \|\mathbb{I}_{\mathcal{Y}}\|$ as in III.3.1 (remember we're being heuristic as regards $\mathfrak{W}$) we can consider the extension of $\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ on $\mathcal{X}_p$, and notationally identify it with the same to deduce,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} = \frac{1}{p}s_{\mathcal{Y}\cap\mathbf{W},\mathbb{I}_{\mathcal{Y}\backslash\mathbf{W}}d\mu}(1) + \frac{1}{p}\lim_{\epsilon\to 0}\int_{\substack{\|\mathbb{I}_{\mathbf{W}}\|=\epsilon\\ \|\mathbb{I}_{\mathcal{Y}}\|\leq\epsilon^{\frac{1}{p}}}} d^c\log\|\mathbb{I}_{\mathbf{W}}\|^2\,\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$$

$$+\lim_{\delta\to 0}\int_{\substack{\|\mathbb{I}_{\mathcal{Y}}\|=\delta\\ \|\mathbb{I}_{\mathbf{W}}\|\leq\delta^{\frac{1}{p}}}} d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2\,\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$$

The first of these terms is unimportant, while the quasi-heuristic hypothesis in force for the moment says that the first of these integrals is bounded independently of p, so dividing by p certainly makes it go to zero. The latter we calculate by Stokes. Indeed fix $\eta>0$, then,

$$\int_{\substack{\|\mathbb{I}_{\mathcal{Y}}\|=\delta\\ \|\mathbb{I}_{\mathbf{W}}\|\leq\delta^{1/p}}} \mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} = -\int_{\substack{\|\mathbb{I}_{\mathcal{Y}}\|=\eta\\ \|\mathbb{I}_{\mathbf{W}}\|\leq\delta^{1/p}}} d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2\,\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$$

$$-\int_{\substack{\delta^{1/p}\leq\|\mathbb{I}_{\mathcal{Y}}\|\leq\eta\\ \|\mathbb{I}_{\mathbf{W}}\|=\delta^{1/p}}} d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2\,\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} + o(\eta)$$

where, as ever, $o(\eta)$ goes to zero with η. Plainly for η fixed the first of these integrals goes to zero with δ, while we're at a singularity with weak branching so the second, given the hypothesis is $O(\eta)$, albeit with a constant depending on p. Nevertheless for p fixed, we may take η sufficiently small to conclude,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} = O(\frac{1}{p})$$

This leaves the contribution coming from $\mathbb{I}_{\mathcal{E}^p}d\mu_{\mathcal{X}/\mathcal{F}}$ to take care of. A quick and efficient way to do this is to blow down, in order, the proper transforms of the exceptional divisors $\mathcal{E}_1,\dots,\mathcal{E}_{p-1}$ in the same direction, then add the minimum amount of extra monodromy required to get a smooth stack $\mathcal{X}^p$. As ever this is a weighted blow up resolving I_p, and we can identify the total exceptional divisor $\mathcal{E}^p_\#$ with $\frac{1}{p}\mathcal{E}^p$ by way of the map between the moduli of $\mathcal{X}_p$ and $\mathcal{X}^p$, with the extra $\mathbb{Z}/p$ monodromy concentrating on a 'hyperplane at infinity', $\mathcal{Y}_\infty$ inside $\mathcal{E}^p_\#$. Outside $\mathcal{E}^p$ nothing changes, so infat it suffices to look at $c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}}$. Now the projection from $\mathcal{E}^p_\#$ to $\mathcal{Y}$ is a first integral of the induced foliation, and since VI.4.1 applies to $\mathcal{E}^p_\#$ off the proper transform of $\mathfrak{W}$, $\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ intersects every $(1,1)$ form coming from $\mathcal{Y}$ in zero. Necessarily though $c_1(-\mathcal{E}^p_\#)+\rho^*\omega$ is a positive smooth form for some sufficiently positive ω on $\mathcal{Y}$, so,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{E}^p\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}} = c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}} \le 0$$

From which we deduce,

$$0 \le \frac{1}{p}c_1(\mathcal{E}^p).d\mu^p_{\mathcal{X}/\mathcal{F}} \le c_1(\mathcal{Y}).\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} + O(\frac{1}{p})$$

and we conclude by letting p go to infinity.

Let's consider, therefore, what we need to make this rigorous by way of the definition of $\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ in terms of components of the pseudo trace, or more accurately for $\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ as the non-pseudo trace components. In this context there is nothing non-rigorous in regard to the computation of $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ or $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{E}^p\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}}$, while for $d\nu$ a component,

$$\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p \le c_1(\mathcal{E}^1).d\nu^1 = s_{\mathcal{Y}\cap\mathfrak{W},d\nu}(1)$$

where $d\nu^i$ is extension by zero. Consequently to calculate the limit as p goes to infinity of $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}$ $d\mu_{\mathcal{X}/\mathcal{F}}$ we can divide by components, and apply the dominated convergence theorem, i.e. it suffices to show,

$$\lim_{p\to\infty}\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p = 0$$

for $d\nu$ a component supported on an irreducible substack $\mathcal{V}$ not contained in the pseudo trace. As such take $n\in\mathbb{N}$, and deform everything to the normal cone of $\mathcal{E}^n_\#$ in $\mathcal{X}^n$, III.5, so that the deformation of $\mathfrak{W}$ itself may now be identified with an honest substack $C_{\mathcal{Y}/\mathbf{W}}$ of $C_{\mathcal{E}^n_\#/\mathcal{X}^n}$. In addition there is a specialised measure $d\nu^!$ supported on the pure dimensional cone $C_{\mathcal{E}^n_\#\cap\mathcal{V}/\mathcal{V}}$, without support on the zero section $[0_n]$. Better still, cf. op. cit., for $p\ge n$, the calculation of $\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p$ commutes with specialisation to the cone. A convenient way to re-write the calculation is to blow up in $[0_n]\cap C_{\mathcal{Y}/\mathbf{W}}$, with say C_1 the exceptional divisor, then in the proper transform $C^1_{\mathcal{Y}/\mathbf{W}}(\xrightarrow{\sim} C_{\mathcal{Y}/\mathbf{W}})\cap C_1$, etc., $(p-n)$ times. This gives us a modification $\tilde{C}^p_{\mathcal{E}^n_\#/\mathcal{X}^n}\to C_{\mathcal{E}^n_\#/\mathcal{X}^n}$, with,

$$\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p = \frac{n[0_n]}{p}.d\nu^! + \frac{1}{p-n}\sum_{i=1}^{p-n}C_i.(d\nu^!)^i$$

where $(d\nu^!)^i$ is the extension by zero of $d\nu^!$ to $\tilde{C}^i_{\mathcal{E}^n_\#/\mathcal{X}^n}$. This reformulation has the advantage that although $C_{\mathcal{Y}/\mathbf{W}}$ is a component of the singular locus of the specialised foliation, we can, for n sufficiently large, find a generator $\partial^!$ of the same together with a local equation $y=0$ for $[0_n]$ about $C_{\mathcal{Y}/\mathbf{W}}\cap[0_n]$ such that $\partial^!(y)=0$. Whence if we write,

$$d\nu^! = \mathrm{I\!I}_{C_{\mathcal{Y}/\mathbf{W}}}d\nu^! + \mathrm{I\!I}_{C_{\mathcal{E}^n_{\#}/\mathcal{X}^n}\setminus C_{\mathcal{Y}/\mathbf{W}}}d\nu^!$$

then what were previously heuristic considerations apply rigorously, to conclude,

$$\frac{1}{p-n}\sum_{i=1}^{p-n} C_i.(\mathrm{I\!I}_{C_{\mathcal{E}^n_{\#}/\mathcal{X}^n}\setminus C_{\mathcal{Y}/\mathbf{W}}}d\nu^!)^i = O(\frac{1}{p-n})$$

Furthermore, the intersections $[O_n].\mathrm{I\!I}_{C_{\mathcal{Y}/\mathbf{W}}}d\nu^!$ are no bigger than the $\mathcal{E}_n.d\nu^n$ appearing in VI.5.2(I), so for n a priori sufficiently large the contribution of $\mathrm{I\!I}_{C_{\mathcal{Y}/\mathbf{W}}}d\nu^!$ in the formula for $\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p$ is $o(n)$, so that we deduce,

$$\lim_{p\to\infty}\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p \le o(n)$$

for every n, and whence it's actually zero as required.

More generally we wish to consider the situation of an arbitrary node in the directed graph $\mathbb{W}$, or equivalently its apex, i.e. the node which is not the sink of any directed arrow. Again for $\mathcal{Y}\hookrightarrow\mathfrak{W}$ a component of its trace, and in particular a Cartier divisor, we make appropriate modifications of $\mathcal{X}$. The weighted blow up formulation $\mathcal{X}^p\to\mathcal{X}$ with $\mathcal{E}^p_{\#}$ the exceptional divisor is convenient. The calculation of $c_1(\mathcal{E}^p_{\#}).\mathrm{I\!I}_{\mathcal{X}^p\setminus\mathcal{E}^p_{\#}}d\mu^p_{\mathcal{X}/\mathcal{F}}$ just proceeds as before with no changes, and what requires a little more care is to know that $c_1(\mathcal{E}^p_{\#}).\mathrm{I\!I}_{\mathcal{E}^p_{\#}\setminus\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}}\le 0$. Certainly it's still the case that the projection of $\mathcal{E}^p_{\#}$ to $\mathcal{Y}$ is a first integral for the induced foliation, so it's more than adequate to know that the induced foliation of $\mathcal{E}^p_{\#}$, or appropriate substacks, leaves any component of $\mathrm{I\!I}_{\mathcal{E}^p_{\#}\setminus\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ invariant. To this end observe that the filtration $F_i\mathbb{W}$ by increasing dimension gives a decreasing filtration $F^i\mathcal{E}^p_{\#}$ of $\mathcal{E}^p_{\#}$. Indeed the proper transform of the nodes of a given dimension, say $\dim\mathfrak{W}+i$ intersected with $\mathcal{E}^p_{\#}$ precisely define $F^i\mathcal{E}^p_{\#}$. On $\mathcal{E}^p_{\#}\setminus F^1\mathcal{E}^p_{\#}$ VI.4.1 continues to apply, so there's nothing to do. More generally on $F^p\mathcal{E}^p_{\#}\setminus F^{p+1}\mathcal{E}^p_{\#}$, we can again apply VI.4.1 to conclude that any component with generic support in this open is invariant by the induced foliation on the proper transform $\tilde{\mathfrak{W}}_i$ of some weak branch $\mathfrak{W}_i$. Around here, however, $F^i\mathcal{E}^p_{\#}$ is the exceptional divisor on $\tilde{\mathfrak{W}}_i$, and since we're off the degeneracy locus $F^{i+1}\mathcal{E}^p_{\#}$, VI.4.1 applies again, so that once more the component is invariant by the induced foliation in $F^i\mathcal{E}^p_{\#}$, which continues to have the induced projection as first integral, from which we conclude. □

Notice that the proof works in slightly greater generality. Specifically, suppose $\mathcal{V}\hookrightarrow\mathcal{X}$ is a smooth invariant substack around which we may extend $\mathfrak{W}$, i.e. $\mathcal{V}$ is a piori in the pseudo trace, but, for whatever reason $\mathfrak{W}$ has a well defined extension around $\mathcal{V}$. Suppose further that $\mathcal{V}$ is Cartier in $\mathfrak{W}$, e.g. blow up in it, which we might aswell say is the case. Furthermore, albeit this is unnecessary, to keep ourselves strictly within the previous discussion let's say $\mathfrak{W}$ is smooth around $\mathcal{V}$, then as before we have a well defined smooth weighted blow up $\rho:\mathcal{X}^p\to\mathcal{X}$ with exceptional divisor $\mathcal{E}^p_{\#}$, and,

VI.5.4 Fact *Let things be as above, suppose that $c_1(\mathcal{E}^p_{\#}).\mathrm{I\!I}_{\mathcal{E}^p_{\#}\setminus\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}\le 0$ for p sufficiently large, and all the singularities in the induced foliation on $\mathfrak{W}$ extended around $\mathcal{V}$ are in the components of* $\mathrm{sing}(\mathcal{F})$ *contained in the original $\mathfrak{W}$ then $\mathcal{V}.\mathrm{I\!I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}\ge 0$.*

proof The calculation of the $\mathcal{E}^p_{\#}$ intersection on components outside the pseudo trace of $\mathfrak{W}$ considered as a formal substack extended around $\mathcal{V}$ goes as before, since $\mathcal{V}$ is invariant so the intersection concentrates at the singularities, and is given by a necessarily vanishing residue. One should note, however, that the pseudo trace, to be more precise that of $\tilde{\mathfrak{W}}$ defined to be the extension as a formal stack around $\mathcal{V}$, may not be the same as that of our original $\mathfrak{W}$. Indeed viewed from $\mathcal{V}$ our considerations of numerical relevance VI.5.1/2 says that a component supported on $\mathcal{A}$ should be considered such if its completion in its intersection with

$\mathcal{V}$ is non-empty and factors through $\tilde{\mathfrak{W}}$. Plainly though this intersection may not contain any points of the trace of $\mathfrak{W}$, so in the obvious variant of our notation we have to organise a sub-calculation,

$$\mathcal{V}.\mathbb{I}_{\tilde{\mathbf{W}}} d\mu_{\mathcal{X}/\mathcal{F}} = \mathcal{V}.\mathbb{I}_{\mathbf{W}} \mathbb{I}_{\tilde{\mathbf{W}}} d\mu_{\mathcal{X}/\mathcal{F}} + \mathcal{V}.\mathbb{I}_{\tilde{\mathbf{W}}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$$

In particular if the component of $d\nu$ associated to $\mathcal{A}$ is in the $\mathbb{I}_{\tilde{\mathbf{W}}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ part, then the intersection of $\mathcal{A}$ with the trace of $\mathfrak{W}$ is empty. On the other hand $\mathcal{O}_{\mathcal{A}}(\mathcal{V})$ is an invariant Cartier divisor for which the construction of a leafwise holomorphic connection depends only on the embedding of $\mathcal{V}$ in $\tilde{\mathfrak{W}}$. By hypothesis $\mathcal{A}$ doesn't meet the singularities of $\tilde{\mathfrak{W}}$, so, infact, $\mathcal{V}.d\nu = 0$. The remaining part of the calculation is, of course, $\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq)$, which we're supposing is true. $\square$

VI.6 Weak Flips

For $\mathcal{X}$ a stack the words *rational curve* will be employed rather than the more long winded 1-dimensional irreducible closed substack whose normalisation has positive Euler characteristic. Now consider the situation of the previous section with $\mathfrak{W}$ a node of $\mathbb{W}$, and $\mathcal{G}$ the induced foliation. We know that $K_{\mathcal{G}}$ is related to $K_{\mathcal{F}}$ by a formula of the form $K_{\mathcal{G}} = K_{\mathcal{F}}(-\mathcal{E})$, for $\mathcal{E}$ a Cartier divisor in $\mathfrak{W}$ supported on the trace. Better still, we know that $\mathcal{E}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, so,

$$K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq K_{\mathcal{F}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$$

If, therefore, to fix ideas, our singular component of interest $d\nu$ were generically smooth for $\mathcal{G}$, then to get $K_{\mathcal{G}}.d\nu = 0$ we basically need to know that $K_{\mathcal{G}}$ is nef. Before elaborating let's consider how this applies to the situation where $(\mathcal{X}, \mathcal{F})$ arises from an ODE on algebraic surface. In this situation associated to the substack $\mathcal{V}$ on which $d\nu$ is supported there is a corresponding ODE, which expressed as a foliation by curves has a minimal model of the form $(\mathcal{V}_0, \mathcal{H}_0)$. As soon as we get $K_{\mathcal{G}}.d\nu = 0$ for any sort of induced foliation which is generically smooth, then we can argue exactly as in VI.3.5 to construct a modification $\tilde{\mathcal{V}}$ of $\mathcal{V}$, mapping by π to $\mathcal{V}_0$ with $K_{\mathcal{H}_0}.\pi_* d\nu = 0$, and so conclude that the push forward of $d\nu$ to the original surface is rational or elliptic. Of course it may happen that $(\mathfrak{W}, \mathcal{G})$ is not of 'nef. type', but in this case it's 'minimal model' will be étale covered by a bundle of generalised weighted projective stacks in the radial foliation by lines, so apart from giving us the desired conclusion in the surface case, it also eliminates the possibility of any 'sub-singular components' since anything which is algebraically integrable does not exhibit weak branching.

Plainly, therefore, nef. is good. Unfortunately we'll need a bunch of definitions and remarks to explain what it should mean. As one can imagine there isn't that much to do beyond this, since if $\mathfrak{W}$ were algebraic we'd already be done by [M1] IV.7.5, and what's involved is really just a certain amount of unraveling to check that it works for formal stacks. Even here formal stack must be understood strictly since we'll need to carry with us the pseudo-trace. Whence,

VI.6.1 Definition *An irreducible component $\mathcal{V}$ of a formal stack $\mathfrak{W}$ will be said to be in the pseudo trace of $\mathfrak{W}$ if,*

(a) *$\mathcal{V}$ is a stack, with not necessarily projective moduli.*

(b) *It intersects at least one component which isn't a stack.*

(c) *The completion of $\mathcal{V}$ in this intersection factors through the non-stack like components.*

A connected formal stack will be called pseudo irreducible if exactly one component is an irreducible formal stack which is not itself a stack, and every other component lies in the pseudo trace.

Basically we're planning on proving the minimal model theorem for foliated pseudo irreducible formal stacks, which sounds worse than it actually is, i.e. what we're doing is keeping track of of tangency conditions. As such we'll view $\mathcal{F}$ as a not necessarily saturated foliation on $\mathfrak{W}$, together with a distinguished closed substack of the trace around which $\mathfrak{W}$ is genuinely formal, each component of this distinguished subset is a Cartier divisor in $\mathfrak{W}$, and the induced foliation at this juncture has log-canonical singularities. This leads to,

VI.6.2 Abusive Definition (i.e. not particularly functorial with respect to the ideas) *By a foliated (pseudo irreducible) foliated stack $(\mathfrak{W}, \mathcal{F})$ we mean,*

(a) *A pseudo irreducible formal stack $\mathfrak{W}$ together with a $\mathbb{Q}$-gorenstein, not necessarily saturated, foliation by curves $\mathcal{F}$ all understood functorially with respect to the ideas.*

(b) *A distinguished open formal substack $\mathfrak{V}$, which itself is an irreducible proper formal stack which is not a stack, every component of the trace of which is Cartier in $\mathfrak{V}$ (whence in $\mathfrak{W}$), and at which $\mathcal{F}$ is never saturated.*

The induced saturated foliation on $\mathfrak{V}$ will be written $\mathcal{G}$, and the trace of $\mathfrak{V}$ will be refered to as the strict trace, with trace employed for components where things are strcitly formal. Furthermore,

(c) *We say that $(\mathfrak{W}, \mathcal{F})$ has log-canonical singularities iff $(\mathfrak{V}, \mathcal{G})$ has log-canonical singularities in the functorial sense.*

(d) *We'll write $K_{\mathcal{F},\mathbf{W}}$ for the class $K_{\mathcal{F}}(-\mathcal{Y})$, where $\mathcal{Y}$ is the necessarily effective Cartier divisor on $\mathfrak{W}$ defined by $i_*\mathcal{O}_{\mathfrak{W}}/(K_{\mathcal{F}} - K_{\mathcal{G}})$, with i the inclusion of $\mathfrak{V}$ in $\mathfrak{W}$.*

The model for this monologue is our original node $\mathfrak{W}$ of $\mathbb{W}$ together with all the components in its pseudo trace, except that $\mathfrak{W}$ has become $\mathfrak{V}$ and the union $\mathfrak{W}$. It may happen though that the trace can be extended. From a notational/definition point of view this risks a certain confusion, as such one should note that when this occurs it is not assumed to change the distinguished component $\mathfrak{V}$ in the model example. More importantly the key instance where this occurs is when the induced foliation in a pseudo trace component $\mathcal{V}$ is wholly by rational curves on which $K_{\mathcal{F}}$ is nil. To see why this is possible one appeals, as ever, to [EGA] III, III.3.4.2 in the spirit of I.2.2 to reduce to cases that one knows, e.g. a smooth rational curve in $\mathcal{X}$ with a strictly convex neighbourhood, where one can then appeal to the Jordan decomposition of a global generator on some étale cover of the neighbourhood. More generally the same argument works generically for bundles of such, so that one can do a Noetherian induction as per the proof of I.6.7 to conclude that $\mathfrak{V}$ can be extended around all of $\mathcal{V}$. As such $\mathfrak{W}$ unlike $\mathfrak{V}$ may be a complete mess, indeed it's not even Noetherian, but of course it's a direct limit of things of 'formal finite type', which is all that matters, and which if one wants, one can add as VI.6.2(e).

Now it stands to reason that one cannot just prove something by a definition in this kind of generality, irrespective of its length. Whence we'll need that $(\mathfrak{W}, \mathcal{F})$ is *dominated by a projective* and indeed *terminally* so. More precisely there should be a stack $(\tilde{\mathcal{X}}, \mathcal{F})$ with projective moduli, canonical singularities, and $K_{\mathcal{F}}$ nef, together with a foliated pseudo irreducible formal stack $(\tilde{\mathfrak{W}}, \mathcal{F}) \subset (\tilde{\mathcal{X}}, \mathcal{F})$ precisely as per the above model example (indeed $\tilde{\mathcal{X}}$ will be an invariant modification of our original $\mathcal{X}$) together with an honest map $\rho : \tilde{\mathfrak{W}} \to \mathfrak{W}$ which is an isomorphism at every generic point of $\mathfrak{W}$ such that we have,

VI.6.3 Condition/Definition *The locus where the moduli of $\tilde{gw}$ is not isomorphic to $\mathfrak{W}$ is $K_{\mathcal{F},\mathbf{W}}$ terminal, i.e. if we find ourselves at a point where ρ isn't an isomorphism then it will be a point of $\mathfrak{V}$, so $K_{\mathcal{F},\mathbf{W}}$ should be understood as $K_{\mathcal{G}}$ terminality functorially with respect to the ideas. Infact the map ρ at the stack level will be a sequence of blow ups in smooth centres in the strict trace, and wholly transverse to $\mathcal{G}$.*

It will, of course, be the case that $K_{\mathcal{F}}$ really descends to $\mathfrak{W}$, so there's no ambiguity in the notation, and we make,

VI.6.4 Definition *We say that $K_{\mathcal{F},\mathbf{W}}$ is nef., if for every curve $\mathcal{C}$ in the pseudo trace with $K_{\mathcal{F}}.\mathcal{C} = 0$, $K_{\mathcal{F},\mathbf{W}}.\mathcal{C} \geq 0$.*

The critical thing is, therefore,

VI.6.5 Fact *Let everything be as above (in particular $(\mathfrak{W}, \mathcal{F})$ is terminally dominated by a projective, and its singularities are log-canonical) then if $K_{\mathcal{F},\mathbf{W}}$ is not nef there is an invariant rational stack $\mathcal{L}$ (generically in the smooth locus of $\mathcal{G}$) in $\mathfrak{W}$ satisfying,*

$$K_{\mathcal{F}}.\mathcal{L} = 0 \ , \ K_{\mathcal{F},\mathbf{W}}.\mathcal{L} \in [-2, 0)$$

so in particular $\mathcal{L}$ has non-empty intersection with the strict trace.

proof If $K_{\mathcal{F},\mathbf{W}}$ is not nef., there is some component of the trace or pseudo trace $\mathcal{V}$ such that on the proper transform $\tilde{\mathcal{V}}$ of $\mathcal{V}$, $\rho^* K_{\mathcal{F},\mathbf{W}}$ is not non-negative on the half space $\mathrm{NE}(\tilde{\mathcal{V}})_{K_{\mathcal{F}}=0}$. In particular there is an extremal ray R in $\mathrm{NE}(\tilde{\mathcal{V}})_{K_{\mathcal{F}}=0}$ satisfying $\rho^* K_{\mathcal{F},\mathbf{W}}.R < 0$. Quite generally R is a limit of sums of irreducible algebraic 1-cycles, $Z_i = \sum_j a_{ij} Z_{ij}$, $i \to \infty$. The Néron-Severi group is, however, finite dimensional of say dimension t, and for fixed i, $\sum_j \mathbb{R}_+ Z_{ij}$ is a polyhedral cone, so we need at most t Z_{ij}'s in our formula. After subsequencing we may suppose that for each $1 \leq j \leq t$, $a_{ij} Z_{ij}$ converges to some R_j as $j \to \infty$, so $R = R_1 + \ldots + R_t$. By hypothesis $K_{\mathcal{F}}$ is nef., so infact all the R_i must be parallel since R is extremal. Whence we can suppose that R is a limit of irreducible 1-cycles $a_i Z_i$, so in particular, $K_{\mathcal{F},\mathbf{W}}.\rho_* Z_i$is eventually strictly negative.

Now denote by $f_i : \mathcal{C}_i \to \mathfrak{W}$ the normalisation of the irreducible 1-dimensional substack of $\mathfrak{W}$ supported on $\rho(Z_i)$. Certainly $\mathcal{C}_i$ must meet the strict trace, so we can talk about whether the completion $\hat{\mathcal{C}}_i$ in this intersection (which is quite possibly all of $\mathcal{C}_i$) meets the locus where $\mathcal{G}$ on $\mathfrak{V}$ is smooth or not. If, however, it didn't it would be wholly singular, so by the log-canonicity of the singularities we'd get $K_{\mathcal{F},\mathbf{W}}.\mathcal{C}_i = K_{\mathcal{G}}.\mathcal{C}_i \geq 0$. Consequently $\mathcal{C}_i$ doesn't factor through the singular locus, and we can form the pull-back of the infinitesimal birational groupoid $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$ of [M1] II.1.3. Where one observes apart from the triviality of using # so as to preserve ˜ for proper transforms, that the bi-rational groupoid construction is well defined independently of any saturation hypothesis, whence certainly for $\mathcal{F}$, while the correction for the singularities of $\mathcal{G}$ is a local question, so infact is well defined even if $\mathcal{V}$ were only in the pseudo trace. The formal stack $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$ contains $\mathcal{C}_i$ as a section with positive (infact $-K_{\mathcal{F},\mathbf{W}}$) normal bundle, so the Chow lemma, e.g. [M1] II.2.2, to the effect that its field of meromorphic functions has transcendence degree at most 2 continues to hold. What, however, is less obvious is that the said degree actually is 2. This is where VI.6.3 enters, since it guarantees (profiting from, say, the convergent frobenius theorem to avoid some technical issues of definition) that ρ is bi-rational at the generic point of $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$. Indeed this is clear from the description of ρ in terms of blow ups, but it also follows from the terminality condition itself since any modification in a centre tangent to the foliation, yet not invariant by it leads to a non-canonical singularity. As such there is a well defined proper transform $\widetilde{\mathfrak{G}_s^{\#} \times_{f_i}} \mathcal{C}_i$ mapping finitely to $\mathcal{X} \times \mathcal{C}_i$, so indeed the space of rational functions really does have dimension 2. Consequently we find an honest 2 dimensional normal stack $s : \mathcal{S}_i \to \mathcal{C}_i$ together with a section $t(\mathcal{C}_i)$, say, such that the completion of $\mathcal{S}_i$ in $t(\mathcal{C}_i)$ is $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$ with it's natural section.

On the normal surface $\mathcal{S}_i$ we have an honest Mumford type intersection theory, NS^M or NE^M for the cone, and as per [M1] II.3.3 for every $c_i \in \mathcal{C}_i(\mathbb{C})$ we have an invariant (possibly wholly singular) rational curve $\mathcal{L}_{c_i}$ (i.e. a curve in the fibre of s through c_i) satisfying the basic estimate,

$$N.\mathcal{L}_{c_i} \leq 2 \frac{N._{f_i}\mathcal{C}_i}{-K_{\mathcal{F},\mathbf{W}}._{f_i}\mathcal{C}_i}$$

for every nef. (in the sense of Mumford) divisor N in NS^M. In particular taking N to be $K_{\mathcal{F}}$, we see that $K_{\mathcal{F}}$ is eventually $\leq$ epsilon for i sufficiently large, and since it takes discrete non-negative values $K_{\mathcal{F}}$ is actually zero on every fibre of $s : \mathcal{S}_i \to \mathcal{C}_i$. What remains, therefore, is to produce some $\mathcal{L}_{c_i}$ on which $K_{\mathcal{F},\mathbf{W}}$ is negative. The key point, irrespective of any projectivity assumptions, is that the effective Mumford cone doesn't contain lines, which allows us to assert,

VI.6.6 Claim *There is a decomposition,*

$$\mathrm{NE}^M(\mathcal{S}_i) = \mathrm{NE}^M(\mathcal{S}_i)_{K_{\mathcal{F},\mathbf{W}} \geq 0} + \sum_n \mathbb{R}_+ [\mathcal{L}_n]$$

where the $\mathcal{L}_n$ are $K_{\mathcal{F},\mathbf{W}}$ invariant (i.e. fibres of s) rational substacks of $\mathcal{S}_i$ satisfying $K_{\mathcal{F}}.\mathcal{L}_n = 0$, $-2 \leq K_{\mathcal{F},\mathbf{W}}.\mathcal{L}_n < 0$.

proof Following [K] III.1.2, we prove the closure of the right hand side is the left hand side, which is enough since $\mathcal{S}_i$ is a surface. Whence if things are false there is a nef. (Mumford sense) Cartier divisor H on $\mathcal{S}_i$ strictly positive on the right hand side, and vanishing on some extremal ray D of $\mathrm{NE}^M(\mathcal{S}_i)$. As such, arguing as before, we produce a sequence $a_m D_m$ of irreducible 1-cycles on $\mathcal{S}_i$ converging to D, but with the additional property that not only is $-K_{\mathcal{F},\mathbf{W}}.D_m > 0$ but $(-K_{\mathcal{F},\mathbf{W}}.D_m)^{-1}(H.D_m) \to 0$ as $m \to \infty$. Continuing in the same vein, one therefore finds for each point $d \in \mathcal{D}_m(\mathbb{C})$, $\mathcal{D}_m$ a stack with support D_m, a rational invariant curve $\mathcal{L}_{md}$ on $\mathcal{S}_i$ satisfying,

$$N.\mathcal{L}_{md} \leq 2\frac{N.D_m}{-K_{\mathcal{F},\mathbf{W}}.D_m}$$

for every nef. divisor N on $\mathcal{S}_i$. As such if we take N to be H we deduce that $H.\mathcal{L}_{md}$ goes to zero as $m \to \infty$ for every d. We already know, however, that $K_{\mathcal{F}}.\mathcal{L}_{md} = 0$ for all m and d, while quite generally any invariant curve satisfies $K_{\mathcal{F},\mathbf{W}}.\mathcal{L}_{md} \geq -2$, so that for m sufficiently large we contradict the strict positivity of H on the cone on the right hand side. $\square$

Back at VI.6.5 therefore, we quickly conclude that indeed there must be an invariant rational curve $\mathcal{L}$ satisfying $0 > K_{\mathcal{F},\mathbf{W}}.\mathcal{L} \geq -2$. It must, of course, meet the strict trace non-emptily, so if $\hat{\mathcal{L}}$ is its completion in the same, then $\mathcal{G}$ must be smooth at the 'generic point' of $\hat{\mathcal{L}}$, or technically better $\hat{\mathcal{L}}$ doesn't factor through $\mathrm{sing}(\mathcal{G})$, since $\mathcal{G}$ has log-canonical singularities. $\square$

We will refer to the curves guaranteed by VI.6.5 as *$K_{\mathcal{F},\mathbf{W}}$ negative curves.* Notice that if $\mathcal{L}$ is such, and the distinguished open formal substack $\mathfrak{V}$ is smooth then it's extension around $\mathcal{L}$ (whose existence has already been noted) is smooth. Indeed if $\mathcal{L}$ factors through the strict trace there is nothing to do, otherwise $\mathrm{sing}(\mathfrak{W}) \cap \mathcal{L}$ is a closed invariant substack of $\mathcal{L}$ so it certainly contains the unique point of $\mathrm{sing}(\mathcal{G}) \cap \hat{\mathcal{L}}$ (use the adjunction formula of II.3 together with the weakness of $\tilde{\mathfrak{W}}$, II..3.5) where, as ever, $\hat{\mathcal{L}}$ is the completion in the intersection. By definition, though, $\mathfrak{W}$ is smooth at $\mathrm{sing}(\mathcal{G}) \cap \hat{\mathcal{L}}$, so this is nonsense. Consequently the analysis of [M1] §III goes through verbatim, or strictly speaking a slightly more involved variant to deal with generalised weighted projective stacks, to conclude,

VI.6.7 Fact *The deformations of $K_{\mathcal{F},\mathbf{W}}$ negative curves in $\mathfrak{W}$ are unobstructed. Better still the local germ of deformation is (up to étale covering) that of an invariant curve for a bundle of radial foliations in generalised weighted projective stacks over a locally smooth component of sing($\mathcal{G}$), and the formal neighbourhood structure in $\mathfrak{W}$ splits,* cf. [M1] V.4.

At which point,

VI.6.8 Difficulties *We face the following problems,*

(a) *We need to guarantee that the local germ of deformation converges.*

(b) *Even after we do (a) we have to avoid a foliated variant of a Hironaka type example [H] - which certainly exist by the way.*

Both problems are resolved by VI.6.3. In case (a), as in the proof of VI.6.5, it guarantees that if ρ is an isomorphism at the generic point of the germ of deformation - infact even on the original curve- then the said germ has a well defined proper transform. By the definition of $\tilde{\mathfrak{W}}$ all we have to do is check that the dimension of the deformation space is correct, which itself reduces to a surface question, i.e. does a rational curve in the broad sense of nodal and every sub-curve of negative arithmetic genus & trivial normal bundle move, which indeed it does, so this is okay. As to (b), suppose we're presented with a $K_{\mathcal{F},\mathbf{W}}$ negative curve $\mathcal{L}^0$. If it doesn't move, we don't care. It may, however, move and break up into $\mathcal{L}_1^0, \ldots, \mathcal{L}_1^n$. Certainly all the $\mathcal{L}_1^i$ are invariant, $K_{\mathcal{F}}.\mathcal{L}_i^0$ for each i, so we can find one of them $\mathcal{L}^1$, say, which is a $K_{\mathcal{F},\mathbf{W}}$ negative curve. There is, though, no guarantee that a Hironaka example won't form, i.e. this will continue ad nauseam.

Again, VI.6.3 comes to the rescue, since all of the individual specialisations $\mathcal{L}^0 \rightsquigarrow \mathcal{L}^1 \rightsquigarrow \mathcal{L}^2 \rightsquigarrow$, etc. can be lifted to $\tilde{\mathcal{X}}$ without any loss (indeed there may be a gain) in the number of extra components introduced when a given $\mathcal{L}^i$ breaks. Consequently,

VI.6.9 Definition/Summary *Given a $K_{\mathcal{F},\mathbf{W}}$ negative irreducible curve $\mathcal{L}^0$ there exists another (possibly the same) $K_{\mathcal{F},\mathbf{W}}$ negative irreducible curve $\mathcal{L}$ such that every germ of deformation of $\mathcal{L}$ that may exist by VI.6.7 not only converges, but every curve obtained by the deformation is also $K_{\mathcal{F},\mathbf{W}}$ negative and irreducible. Such a curve without breaking will be termed pseudo extremal.*

Now pseudo extremality, as opposed to extremality, is all that one needs for the analysis of [M1] III.3-6 to go through, i.e.

VI.6.10 Fact *The deformations of a pseudo extremal $K_{\mathcal{F},\mathbf{W}}$ negative curve $\mathcal{L}$ in $\mathfrak{W}$ sweep out a bundle of radially foliated generalised weighted projective stacks $\mathcal{P}$, with the base of the bundle $\mathcal{Z}$ a smooth substack of the strict trace. Indeed $\mathcal{Z}$ is a connected component of* $\mathrm{sing}(\mathcal{G})$, *and the normal bundle of $\mathcal{P}$ in $\mathfrak{W}$ is, up to étale covering, locally on $\mathcal{Z}$ a sum of relatively negative line bundles whenever $\mathcal{P} \neq \mathfrak{W}$.*

The formal form of the flip theorem, [M1] IV.2 & IV.5, therefore goes through without change. Indeed the only quibble that one might have is at the pseudo trace, but since the flip is actually a flap this is guaranteed by Theorem 3.1 of [A], and the unicity of contraction, which extends op. cit. to stacks. Whence,

VI.6.11 Fact *Suppose the $\mathcal{P}$ of* VI.6.10 *is not all of $\mathfrak{W}$ then putting $\mathfrak{W}_- = \mathfrak{W}$ there is a flip/flap diagram of pseudo irreducible formal stacks,*

$$\begin{tikzcd}
 & \mathfrak{W}^{\#} \arrow[dl, "\text{weighted blow up}"'] \arrow[dr, "\text{weighted blow down}"] & \\
\mathfrak{W}_- \arrow[rr, dashed] \arrow[dr, "K_{\mathcal{F},\mathbf{W}_-}\ \text{negative}"'] & & \mathfrak{W}_+ \arrow[dl, "K_{\mathcal{F},\mathbf{W}_+}\ \text{positive}"] \\
 & \mathfrak{W}_0 &
\end{tikzcd}$$

In addition having started from the situation that the distinguished irreducible open $(\mathfrak{V}_-, \mathcal{G}_-)$ is smooth with log-canonical singularities, the same is true of $(\mathfrak{V}_+, \mathcal{G}_+)$, while the centre of $\mathfrak{W}^{\#} \to \mathfrak{W}_+$ is contained in the strict trace of $\mathfrak{W}_+$ and everywhere transverse to $\mathcal{G}_+$.

On a slightly technical point: one should note that 'weighted blow up' on the left includes the possibility of extracting a root of a Cartier divisor if $\mathcal{P}$ is such or the identity if this is unnecessary. This allows us to assert that the distinguished open $(\mathfrak{V}^{\#}, \mathcal{G}_-)$ again is smooth with log-canonical singularities. Plainly if $\mathfrak{W}_-$ is terminally dominated by a projective in the sense of VI.6.3 & prequel then after sufficient modification of the ambient space (including extraction of invariant roots if necessary) there is, thanks to the transversality of the $\mathfrak{W}^{\#} \to \mathfrak{W}_+$ centre, no problem to conclude the same for $\mathfrak{W}_+$ since everything is actually $\mathcal{F}$ invariant, while $K_{\mathcal{F}}$ itself obviously descends to $\mathfrak{W}_+$ by a standard theorem of formal functions type argument. Taken in combination with VI.6.5, together with the termination of flipping thanks to the destruction of a singular component of $\mathcal{G}$ at each stage, we obtain,

VI.6.12 *Let $(\mathfrak{W}, \mathcal{F})$ be a weak foliated pseudo irreducible formal stack inside a stack $(\mathcal{X}, \mathcal{F})$ with projective moduli and $K_{\mathcal{F}}$ nef. such that the distinguished open $(\mathfrak{V}, \mathcal{G})$ is smooth with log-canonical singularities, then there are diagrams,*

(a) *The minimal model algorithm,*

$$(\mathfrak{W}, \mathcal{F}) = (\mathfrak{W}_0, \mathcal{F}) \dashrightarrow (\mathfrak{W}_1, \mathcal{F}) \dashrightarrow \dots \dashrightarrow (\mathfrak{W}_n, \mathcal{F})$$

The arrows here are the $K_{\mathcal{F},\mathbf{W}}$ minimal model algorithm as described above, so either $K_{\mathcal{F},\mathbf{W}}$ is nef. in the sense of VI.6.4 or $(\mathfrak{W}_n, \mathcal{F})$ is a radially foliated bundle of generalised weighted projective stacks, *with the base of the bundle the unique smooth component of* $\mathrm{sing}(\mathcal{G})$. *In any case all the* $(\mathfrak{V}_i, \mathcal{G})$*'s are smooth with log-canonical singularities.*

(b) *Domination by a projective for all* i,

$$(\mathcal{X}_i, \mathcal{F}) \underset{j_i}{\hookleftarrow} (\tilde{\mathfrak{W}}_i, \mathcal{F}) \underset{\rho_i}{\twoheadrightarrow} (\mathfrak{W}_i, \mathcal{F})$$

Indeed each of the $(\tilde{\mathfrak{W}}_i, \mathcal{F})$*'s are as in the exemplary discussion of a foliated pseudo irreducible stack post* VI.6.2, *with the* j_i*'s embeddings in stacks with projective moduli, while the* ρ_i*'s are as per* VI.6.3.

(c) *Composition by invariant modification,*

$$\begin{tikzcd} (\mathcal{X}_i, \mathcal{F}_i) \arrow[d] & (\tilde{\mathfrak{W}}_i, \mathcal{F}) \arrow[l, "j_i"'] \arrow[d] \\ (\mathcal{X}_{i-1}, \mathcal{F}_{i-1}) & (\tilde{\mathfrak{W}}_{i-1}, \mathcal{F}) \arrow[l, "j_{i-1}"'] \end{tikzcd}$$

All of the vertical maps $i \in \mathbb{N}$ *are modifications in* $K_{\mathcal{F}}$ *nil invariant centres. Infact they are compositions of weighted blow ups in the broad sense post* VI.6.11.

It's worth closing by way of,

VI.6.13 Remark *As one can imagine from the emphasis on* $K_{\mathcal{F}}$ *centres, it would certainly have been preferable to preserve the immersion* $(\mathfrak{W}, \mathcal{F}) \hookrightarrow (\mathcal{X}, \mathcal{F})$, *i.e. flop. Already in dimension 3 if one wants to preserve projectivity of the moduli then it's necessary to create nodes. Of course, one can think to finesse the situation in the spirit of* II.9 *combined with* VI.6.3 *but this is in general impossible when the weak branch has co-dimension at least 2. The technical difficulty in working with nodes is rather small, but would still require several more pages of justification than those required to define* $\mathfrak{W}$. *Irrespectively, the key would remain* VI.6.5.

VI.7 Inductive Remarks

To begin with, for $\mathfrak{V}$ a node of the weak branching graph $\mathbb{W}$ as defined in VI.4 let's see how VI.6.12 applies to the study of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, where $\mathbb{I}_{\mathbf{W}}$ is now understood functorially with respect to the ideas for $\mathfrak{W}$ the associated pseudo irreducible formal stack with $\mathfrak{V}$ the distinguished open, and this even agrees with the previous rabbit out of a hat notation VI.4.3. Anyway, as post VI.4.2 we can modify $\mathcal{X}$ as we please, and so assume that $\mathcal{X}$ is the $\mathcal{X}_n$ of VI.6.12. Plainly if $(\mathfrak{W}, \mathcal{F})$ is a modification of a radially foliated weighted projective stack then we're rather happy, and the discussion finishes, otherwise there is a map $\rho : (\mathfrak{W}, \mathcal{F}) \to (\mathfrak{W}_+, \mathcal{F})$ such that $\rho^* K_{\mathcal{F}, \mathbf{W}_+}$ is non-negative on all $K_{\mathcal{F}}$ nil curves in the pseudo trace of $\mathfrak{W}$. Now let $d\nu$ be a component of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a closed substack $\mathcal{V}$ factoring through $\mathfrak{W}$. Everything has projective moduli, and we just project $d\nu$ into $\mathrm{NS}_1(\mathcal{V})$, i.e. view it as an operator on bundles, whence this projection $[d\nu]$ is in $\mathrm{NE}_1(\mathcal{V})_{K_{\mathcal{F}}=0}$, so by construction $\rho^* K_{\mathcal{F}, \mathbf{W}_+}.d\nu \geq 0$. We further assert,

VI.7.1 Claim *For every* $d\nu$ *as above,* $\rho^* K_{\mathcal{F}, \mathbf{W}_+}.d\nu = K_{\mathcal{F}, \mathbf{W}_+}.\rho_* d\nu = 0$.

proof By VI.5.3, we know that $K_{\mathcal{F}, \mathbf{W}}.\mathbb{I}_{\mathbf{W}} d\nu \leq 0$, while $K_{\mathcal{F}, \mathbf{W}}$ is related to $K_{\mathcal{F}, \mathbf{W}_+}$ by a formula of the form,

$$K_{\mathcal{F}, \mathbf{W}} = \rho^* K_{\mathcal{F}, \mathbf{W}_+} + \mathcal{E}$$

where without loss of generality $\mathcal{E}$ is an effective Cartier divisor in the ambient space, and $\mathcal{E} \cap \mathfrak{W}$ is an effective Cartier divisor supported on $K_{\mathcal{F}}$ nil invariant rational curves. It may, however, happen that $\mathcal{E} \cap \mathfrak{W}$

isn't contained in the strict trace, so VI.5.3 doesn't apply, and to conclude what is obviously sufficient, i.e. $\mathcal{E}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, we may have to try to use VI.5.4. Whence let $\mathcal{Y}$ be a component of the trace in $\mathcal{E} \cap \mathfrak{W}$ which isn't strict, with $\mathcal{E}_{\mathcal{Y}}$ the component of $\mathcal{E}$ affording $\mathcal{Y}$ which by blowing up we may not only assume is irreducible, as we may indeed suppose $\mathcal{Y}$ smooth, but that,

$$N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} N_{\mathcal{Y}/\mathbf{W}} \oplus N_{\mathcal{Y}/\mathcal{E}_{\mathcal{Y}}}$$

Now to prove that $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, it will suffice, thanks to VI.5.3., to do it on any modification which only involves blowing up in things inside the strict trace. Necessarily $\mathcal{Y}$ is covered by $\mathcal{F}$ invariant rational curves $\mathcal{C}_t$ all meeting sing$(\mathcal{G})$ without factoring through it, so by blowing up enough we can assume that $N^{\vee}_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{C}_t}$ is ample for generic $\mathcal{C}_t$, where, without loss of generality, generic means not factoring through the strict trace. Furthermore if I_p is the sheaf of ideals used to define the weighted blow ups $\mathcal{X}^p \to \mathcal{X}$ employed in VI.5.3, adapted to $\mathcal{Y}$ as in VI.5.4, then we have an exact sequence,

$$0 \longrightarrow N^{-p}_{\mathcal{Y}/\mathbf{W}} \longrightarrow I_p/I_p^2 \longrightarrow N^{\vee}_{\mathcal{Y}/\mathcal{E}_{\mathcal{Y}}} \longrightarrow 0$$

and an extension of ample by ample is ample, so without loss of generality $I_p/I_p^2|_{\mathcal{C}_t}$ is ample for $\mathcal{C}_t$ not factoring through the strict trace, which we'll denote $\mathcal{Z}$. Consequently if $\pi : \mathcal{E}^p_{\#} \to \mathcal{Y}$ is the projection of the exceptional divisor of $\mathcal{X}^p$ then we organise our computation of $\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}}$ according to,

$$\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}} = \mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}\backslash\pi^{-1}(\mathcal{Z})} d\mu^p_{\mathcal{X}/\mathcal{F}} + \mathcal{E}^p_{\#}.\mathbb{I}_{\pi^{-1}(\mathcal{Z})}\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}}$$

The part off $\mathcal{Z}$ pushed forward to $\mathcal{Y}$ comes from a measure $d\lambda$ on $\mathcal{Y}/\mathcal{F}$, i.e. the moduli (even coarsely speaking) of our rational curves, so that,

$$\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}\backslash\pi^{-1}(\mathcal{Z})} d\mu^p_{\mathcal{X}/\mathcal{F}} = \int_{\mathcal{Y}/\mathcal{F}} d\lambda(t) \int_{\pi^{-1}(\mathcal{C}_t)} \mathcal{E}^p_{\#}.d\mu_t$$

for $d\mu_t$ some invariant measure on $\pi^{-1}(\mathcal{C}_t)$. By construction, however, $\mathcal{O}_{\pi^{-1}(\mathcal{C}_t)}(-\mathcal{E}^p_{\#})$ is ample, so this is at most zero. The part over $\mathcal{Z}$ is as before, i.e. here the induced foliation on $\mathcal{E}^p_{\#}$ has π as a first integral, hence we may argue exactly as in the proof of VI.5.3 by way of vigorous application of VI.4.1 to deduce that the conditions of VI.5.4 hold. $\square$

This completes the discussion if our original singular component of interest meets the locus where the induced foliation on a single weak branching stack in $\mathcal{X}$ is smooth. Unfortunately, as we've noted post VI.4.2 the general situation is that we need to go down a chain $\mathfrak{V}^1 \supset \ldots \supset \mathfrak{V}^p$ with $\mathfrak{V}^{i+1}$ a weak branch in $\mathfrak{V}^i$ before concluding to generic smoothness. This is more notationally fastidious than anything else. For example given that $\mathfrak{V}^1$ is defined as the smallest weak branching stack in $\mathcal{X}$ containing our component the notation $\mathfrak{V}^1 \supset \mathfrak{V}^2$ is unambiguous, but $\mathfrak{V}^1 \supset \mathfrak{V}^2 \supset \mathfrak{V}^3$ may not be, depending on whether $\mathfrak{V}^3$ extends to a weak branch in $\mathfrak{V}^1$ or not. Consequently we do some relabeling/throwing out so that $\mathfrak{V}^2$ is actually the smallest weak branch in $\mathfrak{V}^1$ containing our component, idem for $\mathfrak{V}^3$ in $\mathfrak{V}^2$ etc., and write this as $\mathfrak{V}^1 > \ldots > \mathfrak{V}^p$. As such $\mathfrak{V}^{i+1}$ in $\mathfrak{V}^i$ has the additional property that if it contains a component of the singular locus of the induced foliation $\mathcal{G}^{i-1}$, say, on $\mathfrak{V}^{i-1}$ then it actually equals it. Necessarily we can suppose that each $(\mathfrak{V}^i, \mathcal{G}^i)$ is smooth with log-canonical singularities, indeed even EmbLCR, and we have associated pseudo irreducible formal stacks $(\mathfrak{W}^i, \mathcal{F})$ defined in the obvious way as post VI.6.2, i.e. throw in the pseudo trace components of $\mathfrak{V}^i$, and extend $\mathfrak{V}^i$ were possible. A moments reflection shows that $(\mathfrak{W}^i, \mathcal{F})$ is actually contained in $(\mathfrak{W}^{i-1}, \mathcal{F})$, and plainly what we'll have to do is prove that $K_{\mathcal{F},\mathbf{W}^p}$ is nef. in some suitable sense. Certainly $K_{\mathcal{F},\mathbf{W}^p}$ is wholly well defined as a bundle on $\mathfrak{W}^p$, but just as the definition of nefness for $K_{\mathcal{F},\mathbf{W}^1}$ was only for $K_{\mathcal{F}}$ nil curves we'll have to do something similar here - this results from the relations governing $K_{\mathcal{F},\mathbf{W}^{i+1}}$ in terms of $K_{\mathcal{F},\mathbf{W}^i}$. In this respect observe, apart from the ubiquitous we're done if $\mathfrak{W}^i$ is foliated in conics (which we'll systematically ignore from now on to simplify the discussion), that the minimal model algorithm has nothing to do with weak branching, i.e. no flipping variety ever meets a weak branch, so infact $K_{\mathcal{F},\mathbf{W}^1}$ is $K_{\mathcal{F},\mathbf{W}^1_+}$ around $\mathfrak{W}^2$, in an obvious extension of the notations pre VI.7.1.

This suggests (for notational reasons if nothing else) that we proceed back to front, since VI.6.3 is already a given for the image $\mathfrak{W}^2_1$ of $\mathfrak{W}^2$ in $\mathfrak{W}^1_+$. Now define *nefness* for $\mathfrak{W}^2$ to mean that $K_{\mathcal{F},\mathbf{W}^2}$ is non-negative on curves which are $K_{\mathcal{F},\mathbf{W}^1_+}$, and $K_{\mathcal{F}}$ nil, or better: just define a $K_{\mathcal{F},\mathbf{W}^2}$ *negative curve* to be an invariant rational curve in $\mathfrak{W}^2$ on which $K_{\mathcal{F}}$, $K_{\mathcal{F},\mathbf{W}^1_+}$ are nil, and $K_{\mathcal{F},\mathbf{W}^2}$ is negative. Rather helpfully all of VI.6.7-11 is valid, independently of VI.6.5 so without having verified that it satisfies the nefness criterion, we can nevertheless define $\mathfrak{W}^2_+$ by the condition that we've exhaustively flipped all pseudo extremal $K_{\mathcal{F},\mathbf{W}^2}$ negative curves in $\mathfrak{W}^2_1$, and whence there are no more such. In addition we can use all the other arrows in VI.6.12 so that $\mathfrak{W}^1_+$ gets modified just as $\mathcal{X}$ in op. cit. by weighted blow up in $K_{\mathcal{F},\mathbf{W}^1_+}$ invariant centres, and needless to say $\mathcal{X}$ gets modified too. Subsequently we define $K_{\mathcal{F},\mathbf{W}^3}$ negative curves in the obvious way, flip all of these, and continue till we eventually get,

VI.7.2 Diagram

$$\begin{array}{ccccccc}
\mathfrak{W}^p & \subset \ldots\ldots \subset & \mathfrak{W}^2 & \subset & \mathfrak{W}^1 & \subset & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{W}^p_1 & \subset \ldots\ldots \subset & \mathfrak{W}^2_1 & \subset & \mathfrak{W}^1_+ & & \\
\downarrow & & \downarrow & & & & \\
\mathfrak{W}^p_2 & \subset \ldots\ldots \subset & \mathfrak{W}^2_+ & & & & \\
\downarrow & & & & & & \\
\vdots & & & & & & \\
\downarrow & & & & & & \\
\mathfrak{W}^p_+ & & & & & &
\end{array}$$

where all the arrows $\rho^!_{*\bullet} : \mathfrak{W}^!_* \to \mathfrak{W}^!_\bullet$ are a sequence of weighted blow ups in centres contained in the strict trace, and wholly transverse to the induced foliation around the distinguished open $\mathfrak{V}^!_\bullet$, and any extensions thereof. Now for each $1 \leq i \leq p$, $j < i$, and each component $\mathcal{V}$ of the trace or pseudo trace of $\mathfrak{W}^i$, let $\mathrm{NE}(\mathcal{V})_j$ be the curves in $\mathrm{NE}(\mathcal{V})$ on which $(\rho_+)^*K_{\mathcal{F},\mathbf{W}^k_+}$ is nil for all $k \leq j$, including $K_{\mathcal{F}}$ nil, $k = 0$. Having thus defined everything, it remains to check that it works. Obviously it would be very disappointing if the following proved insufficient,

VI.7.3 Fact *$(\rho^p_+)^*K_{\mathcal{F},\mathbf{W}^p_+}$ is non-negative on $\mathrm{NE}(\mathcal{V})_{p-1}$ for all $\mathcal{V}$ in the trace or pseudo trace of $\mathfrak{W}^p$.*

proof VI.6.5 has been constructed to generalise easily to these circumstances. Since, however, we're determined to sail close to the border where this sort of thing fails, we give some extra details. We go by induction on q, $1 \leq q \leq p$, with $q = 1$ being precisely VI.6.5. To do the case q (≥ 2), supposing $q-1$ we use VI.7.2, but with q instead of p.

On starts therefore, for an appropriate $\mathcal{V}$, with an extremal ray R in $\mathrm{NE}(\mathcal{V})_{q-1}$, supposed to have $(\rho^q_+)^*K_{\mathcal{F},\mathbf{W}^q_+}.R < 0$, while the inductive hypothesis allows us to suppose R is a limit of irreducible 1-cycles a_iZ_i such that,

$$\frac{(\rho^r_+)^*K_{\mathcal{F},\mathbf{W}^r_+}.Z_i}{-(\rho^q_+)^*K_{\mathcal{F},\mathbf{W}^q_+}.Z_i} \to 0\ ,\ i \to \infty$$

for each $r < q$, including $K_{\mathcal{F}}$ for $r = 0$. The log-canonical nature of the singularities of $(\mathfrak{V}^q_+, \mathcal{G}^q)$ together with the structure of the left most column of VI.7.2, i.e. $K_{\mathcal{F},\mathbf{W}^q_+}$ terminality, allows us to play the bi-rational groupoid trick. Whence for $f_i : \mathcal{C}_i \to \mathfrak{W}^q_+$ the normalisation of the substack supported on $(\rho^q_+)(Z_i)$ we produce a normal surface $s : \mathcal{S}_i \to \mathcal{C}_i$ together with a section whose normal bundle is $\mathcal{O}_{\mathcal{C}_i}(-K_{\mathcal{F},\mathbf{W}^q_+})$, so that for every $c_i \in \mathcal{C}_i(\mathbb{C})$ there is an invariant rational curve $\mathcal{L}_{c_i}$ satisfying the basic estimate,

$$N.\mathcal{L}_{c_i} \leq 2\frac{N._{f_i}\mathcal{C}_i}{-K_{\mathcal{F},\mathbf{W}^q_+}\cdot_{f_i}\mathcal{C}_i} \quad (\to 0,\ \text{as } i \to \infty)$$

for every nef N (as ever Mumford sense) divisor on $\mathcal{S}_i$. In particular, as per VI.6.5, $K_{\mathcal{F}}$ is nil on every $\mathcal{L}_{c_i}$, and more generally on every fibre of s.

To get from here to a contradiction we require to prove that all the other $K_{\mathcal{F},\mathbf{W}^r_+}$'s, for $r < q$ are nef., stricta dictum, on $\mathcal{S}_i$. To do this one does an induction within an induction à la VI.6.6. Indeed with VI.6.6 as is, and the nullity of $K_{\mathcal{F}}$ on the fibres of s, one sees that $K_{\mathcal{F},\mathbf{W}^1_+}$ is already nef. on $\mathcal{S}_i$, so for $i >> 0$ the basic estimate applies to deduce that $K_{\mathcal{F},\mathbf{W}^1_+}$ is nil on all the fibres of s too. Now reprove VI.6.6, but for $K_{\mathcal{F},\mathbf{W}^2_+}$ with the polyhedral part generated by $K_{\mathcal{F},\mathbf{W}^2_+}$ negative curves (bearing in mind this includes $K_{\mathcal{F}}$, $K_{\mathcal{F},\mathbf{W}^1_+}$ nullity). Since there are no such curves, $K_{\mathcal{F},\mathbf{W}^2_+}$ is nef., and so forth till we get up to $K_{\mathcal{F},\mathbf{W}^q_+}$ nef, so as to contradict $K_{\mathcal{F},\mathbf{W}^q_+}.\mathcal{C}_i < 0$. $\square$

By the usual expedient of projecting into Néron-Severi, it goes, without saying, that we have,

VI.7.4 Corollary *For every component $d\nu$ of a transverse parabolic measure $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a substack factoring through the trace or pseudo trace of some $\mathfrak{W}^q$, $(\rho^q_+)^* K_{\mathcal{F},\mathbf{W}^q_+}.d\nu \geq 0$, provided $(\rho^r_+)^* K_{\mathcal{F},\mathbf{W}^r_+}.d\nu \geq 0$, for each $0 \leq r < q$.*

The next thing to observe is that the proof of VI.5.3 doesn't depend on the fact that $\mathcal{X}$ is a stack, but that $d\mu_{\mathcal{X}/\mathcal{F}}$ satisfies V.1.2 (b) & (c) - (a) & (d) are irrelevant- provided that Cartier divisor is understood in the more expedient sense of directly relevant to the proof of VI.5.3 (4), so, infact one can even tone down V.1.2 (c) to (notations as per op. cit.) ρ an invariant (including root extraction) modification of pseudo irreducible formal stacks. Consequently since VI.5.3 is as valid on any modification which in current notation is $\mathcal{G}^1$ invariant seen from $\mathfrak{V}^1$ as the $\mathcal{X}$ we first thought of, we can induct through the $\mathfrak{W}^i$ to conclude,

VI.7.5 Fact *For any $1 \leq q \leq p$, and any component of the strict trace of $\mathfrak{W}^q$, $\mathcal{Y}.\mathbb{I}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, where, as ever $\mathbb{I}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}}$ is the sum of components of the trace or pseudo trace of $\mathfrak{W}^q$, and, of course, these are the only numerically relevant components as per* VI.5.1/2.

Now we have to carefully use VI.7.2 to induct up to p, by way of the formulae,

$$(\rho^q_+)^* K_{\mathcal{F},\mathbf{W}^q_+} + (\rho^q_{q-1})^*(\mathcal{E}_{q-1} + \mathcal{E}') = (\rho^q_{q-1})^* K_{\mathcal{F},\mathbf{W}^{q-1}_+}$$

where the effective Cartier divisors $\mathcal{E}_{q-1}, \mathcal{E}'$ are the differences between $K_{\mathcal{F},\mathbf{W}^q_{q-1}}$ and $K_{\mathcal{F},\mathbf{W}^q_+}$, respectively $K_{\mathcal{F},\mathbf{W}^{q-1}_+}$. As such $\mathcal{E}'$ is in the strict trace of $\mathfrak{W}^q_{q-1}$, and ρ^q_{q-1} is even an isomorphism here, so VI.7.5 is well adapted to dealing with this, as it is for any components of $\mathcal{E}_{q-1}$ in the strict trace. Where we have to be cautious is at a component $\mathcal{E}$, say, which lies outside the strict trace of $\mathfrak{W}^q_{q-1}$. In this respect observe that the role of the projectivity of $\mathcal{X}$ is limited. The place (in this context) where we need this is the implication that numerically relevant things must be of pseudo trace type. This, however, we can do a priori in $\mathcal{X}$ with a possibly different notion of pseudo trace which pushes forward under $\rho^!_{*\bullet}$ to what we want. Consequently the difficulty rests in knowing that if we start associating weighted blow ups, $\sigma^{n,q-1}_+ : \mathfrak{W}^{n,q-1}_+ \to \mathfrak{W}^{q-1}_+$ to $\mathcal{E}$ inside $\mathfrak{W}^{q-1}_+$ with associated exceptional divisors $\mathcal{E}^n_{\#}$ and modify the whole of VI.7.2 appropriately then $\mathcal{E}^n_{\#}.\mathbb{I}_{\mathbf{W}^{n,q-1}_+} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, since otherwise VI.7.1 goes through without change. To this end, therefore, consider a generic curve $\mathcal{C}$ in $\mathcal{E}$, evidently rational & invariant, which doesn't lie in the strict trace of $\mathfrak{W}^q_{q-1}$. Viewed in $\mathfrak{W}^q_+$ it's either in the strict trace or it isn't. If it is then we're happy since all the $\rho^!_{*\bullet}$'s pull back strict trace to strict trace, so we've got what we need by VI.7.5, which is valid for any modifications which seen from $\mathfrak{V}^q$ are $\mathcal{G}^q$ invariant. Otherwise we look at the proper transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ in $\mathfrak{W}^{q-1}_{q-2}$. If this is in the strict trace of $\mathfrak{W}^{q-2}_+$ then we stop. If it isn't then we appeal to the nullity of $K_{\mathcal{F},\mathbf{W}^{q-2}_+}$ along $\mathcal{C}$ and II.3. Indeed the index set J of op. cit. can be viewed as the number of curves contracted by ρ^{q-1}_{q-1+} meeting $\mathcal{C}$, which has to become an inded set I viewed from $\mathfrak{W}^{q-2}_+$, so infact J is empty, i.e. ρ^{q-1}_{q-1+} is an isomorphism around $\mathcal{C}$. Thus,

VI.7.6 Possibilities *Continuing we get exactly one of,*

(a) *We find some $\mathfrak{W}^r_+$, $1 \leq r \leq q$ containing the proper transform $\mathcal{E}^r$ of $\mathcal{E}$ in its strict trace. Better still for each $r < s \leq q$ an invariant rational curve $\mathcal{C}$ in the proper transform $\mathcal{E}^s_+$ of $\mathcal{E}$ in $\mathfrak{W}^s_+$ is either* *wholly contained in the strict trace, or meets it, infact in a unique point, and all the singularities of the induced foliation on $\mathfrak{W}^s_+$ around $\mathcal{E}^s_+$ are contained in the strict trace.*

(b) *$\mathcal{E}$ even meets the smooth locus of $\mathcal{X}$, with everything as in (a), except that $r = 0$.*

Now to conclude we argue as follows. To begin with, albeit that it's convenient rather than important, everything in VI.7.2 is smooth around $\mathcal{E}$ and its proper transforms for the habitual reason, i.e. singularity is an invariant condition, yet everything is smooth around the distinguished opens $(\mathfrak{V}^!, \mathcal{G}^!)$. Next we a priori blow up a lot in $\mathrm{sing}(\mathcal{G}^q)$ to guarantee that around an invariant rational curve $\mathcal{C}$ in $\mathcal{E}$, not in the strict trace, the ampleness of various bundles such as , $N^\vee_{\mathcal{C}/\mathbf{W}^\bullet_+}|_{\mathcal{C}}$, $N_{\mathbf{W}^\bullet_+/\mathbf{W}^!_+}|_{\mathcal{C}}$, etc. The entire discussion is $\mathcal{G}^q$ invariant, so this changes nothing. Our starting point therefore (with the obvious notation) is $\mathcal{E}^{r+1}_r$ inside $\mathfrak{W}^r_+$ mapping to $\mathcal{E}^r_+$, so that necessarily (II.3.5) the points where this isn't an isomorphism are in the strict trace of $\mathfrak{W}^{r+1}_r$. Next, in a notation that's rapidly getting out of hand, form for each n weighted blow ups $\mathfrak{W}^{r,n}_+ \to \mathfrak{W}^r_+$ associated to the nth power of the pull back of the ideal of $\mathcal{E}^{r+1}_+$ viewed in $\mathfrak{W}^r_+$, with $\mathcal{E}^{r,n}_\#$ the total exceptional divisor, and $\mathcal{Z}$ the locus where $\mathcal{E}^{r+1}_r$ isn't isomorphic to $\mathcal{E}^{r+1}_+$. At which point we have everything we need: $\mathcal{E}^{r,n}_\#.\mathbb{I}_{\mathbf{W}^{r,n}_+} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, by VI.7.6 (a) or (b), $\mathcal{E}^{r,n}_\#.\mathbb{I}_{\mathcal{E}^{r,n}_\# \setminus \mathbf{W}^{r,n}_r} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$ as in VI.7.1, i.e. off $\mathcal{Z}$ we have our ampleness argument, while over $\mathcal{Z}$ the induced foliation on $\mathcal{E}^{r,n}_\#|_{\mathcal{Z}} \to \mathcal{Z}$ has $\mathcal{Z}$ as a first integral, and the numerically relevant components all lie in $\mathbb{I}_{\mathbf{W}^{r+1}_r} d\mu_{\mathcal{X}/\mathcal{F}}$ by VI.5.4, or more correctly its proof. More generally the same argument works for any scheme structure, or better sequence of scheme structures, supported on $\mathcal{E}^{r+1}_+$ which when pulled back to $\mathfrak{W}^{r+1}_r$ differ from that on $\mathfrak{W}^{r+1}_+$ only around the strict trace of $\mathfrak{W}^{r+1}_r$. All of which is equally valid for any of steps,

$$\begin{array}{ccc} \mathfrak{W}^{s+1}_s & \longrightarrow & \mathfrak{W}^s_+ \\ \downarrow & & \\ \mathfrak{W}^{s+1}_+ & & \end{array}$$

for any $q - 1 \geq s \geq r$, with the exception of the positivity of the ambient weighted blow ups on $\mathfrak{W}^s_+$, which we obviously do by induction down from r. Whence,

VI.7.7 Fact *For every $1 \leq q \leq p$, $(\rho^q_{q-1})^* \mathcal{E}_{q-1}.\mathbb{I}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.*

Thus by VI.7.4 we obtain,

VI.7.8 Final Fact *For any singular component $d\nu$ of a parabolic invariant measure in the strict trace of $\mathfrak{W}^p$, $(\rho^p_+)^* K_{\mathcal{F},\mathbf{W}^p_+}.d\nu = 0$.*

In the particular case that our foliated stack $(\mathcal{X}, \mathcal{F})$ comes from an ODE on a 2-dimensional algebraic stack $\mathcal{S}$ with projective moduli by VI.4.2, and as noted pre VI.6.1, this provides the missing piece of the jigsaw required to extend VI.3.5 to singular components, so that indeed in the presence of LCR (I.6.1), VI.2.3 holds as asserted. To conclude from here to the statements of the introduction is an immediate application of [M4].

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