Old and New Techniques in Function Field Arithmetic

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Introduction

Unlike honest arithmetic, function field arithmetic comes equipped with a derivative, so differential algebra is a more accurate description. As such if K is the field of meromorphic functions on a proper curve S over an algebraically closed field k, the study of algebraic points on varieties $X \otimes K/K$, equivalently families $X \to S$ has some a priori relation with what in characteristic zero would be the holomorphic sectional curvature of X. Simplification is often possible since one can often profit from working at the generic point of S, but, nevertheless, the appeal to differentiation always ends up studying $\Omega_{X \otimes K/k}$. This is not, however, what is required if one wishes to relate the arithmetic of $X \otimes K/K$ to the curvature of the generic fibre, which is in fact governed by $\Omega_{X \otimes K/K}$. The new technique achieves this with moderate success, and for example, I.3, gives,
Theorem *Let $X \otimes K/K$ be a curve over a field of functions in characteristic zero, then for every $\epsilon > 0$ there is a constant $\alpha(X, \epsilon)$ such that any algebraic point f on X satisfies,*

$$h_{\omega_{X \otimes K/K}}(f) \leq (1 + \epsilon)\,\mathrm{discr}(f) + \alpha$$

Here, and throughout, heights and discriminants are normalised by the degree $(K(f) : K)$, and we ignore irrelevant dependencies of the former on a choice of models. The optimality of the theorem leads to the interpretation: $X \otimes K/K$ has arithmetic (in the function field sense) sectional curvature -1. It solves a conjecture of Vojta, [V1], and Osterlé, [O]. The method can be uniformised over the moduli of X, and/or extended to quasi-projective curves. We eschew these extensions for ease of exposition, but the former implies that α is uniform in the genus. The dependence on ϵ would appear to be ineffective beyond hope.

The trouble with the new technique is that it is very sensitive to bad reduction. This is not an unknown phenomenon when trying to relate function field arithmetic to hyperbolicity properties in characteristic zero. Indeed, by base change, we can (and will without comment) always suppose that S is as hyperbolic as we please. Hyperbolicity of the total space $X \to S$ is invariably stronger (if not as precise as a result such as I.3) than some Mordellic/boundedness property of algebraic points. By [B] this would follow if every fibre were hyperbolic. Unfortunately bad reduction makes this unlikely, but [Gr] provides a quasi-projective complement which is wholly sufficient in the case of curves, I.1. Although, in a certain sense, the above results of Brody and Green are optimal, they are in higher dimensions, even for questions of disc convergence, inadequate to treat varieties containing a large, but, say, finite to fix ideas, set of rational curves. In principle, the new technique can cope with this, but it requires work. As such, it should be possible to extend the strong hyperbolicity theorem, [M3], for surfaces with $c_1^2 > c_2$ to an 'effective Mordell type theorem' for algebraic points over function fields. Issues, however, still need to be addressed. Roughly speaking: 'Néron models for ODE's on surfaces' are required. Consequently, our second result, §II, is a little retrograde in that it emphasises the co-tangent bundle of the total space, *i.e.*

Theorem *Let $X \otimes K/K$ be a surface of general type and positive index, equivalently $c_1^2 > 2c_2$, over a field of functions in characteristic zero, then there are constants $-\kappa(X) < 0$, $\alpha(X)$ such that any algebraic point f on X satisfies,*

$$h_{\omega_{X\otimes K/K}}(f) \leq \kappa\, \mathrm{discr}(f) + \alpha$$

Here the convention $-\kappa < 0$ is chosen to emphasise the connection with curvature, since one should bear in mind that if $p : T \to S$ is the unique proper cover modeling $K(f)/K$ then, up to normalisation, the discriminant is minus the topological Euler characteristic χ_T, which also has sense for stacks, log-stacks, and in any characteristic. The second theorem is quite delicate, and uses the theory of minimal models of foliations by curves. It's very delicacy, and restrictive hypothesis (albeit that there are plenty of examples of such surfaces) suggest that the old methodology is not sustainable.

Both these theorems need characteristic zero. At the very least the first needs a co-homology theory with values in an archimedean field. The second, in an implicit way, uses local integrability of foliations, which implies, for example, some peculiarities for heights of algebraic points invariant by a foliation computed with respect to bundles which are also invariant. Already for curves, such peculiarities make α effective if we replace $1+\epsilon$ by $2+\epsilon$, but in characteristic p the optimal height vs. discriminant relation is un-clear, I.2.

These theorems constituted the last chapter of an ill-fated manuscript, 'Canonical models of foliations'. In the course of extended editorial deliberation occasioned by a proportionally long manuscript, K. Yamanoi, [Y], also proved the $1+\epsilon$ theorem. The proofs are, however, quite different, and, indeed, rather interestingly so, with [Ga] providing an excellent comparison, which we quickly summarise in regard to Yamoni, who observes that Ahlfors' isoperimetric inequality is way better than the isotrivial case, infact so much better that employed on $\mathbb{P}^1$ it would actually yield the theorem were it not for bad reduction in the moduli of divisors of degree at least 4, and to this problem the author subsequently addresses himself. The emphasis here, however, is that many big algebraic points (or for that matter discs in a suitable sense) lead to a derivative, in characteristic zero, in the projective tangent space relative to K rather than

k. As such, the key point which constitutes the new method, valid in all dimensions, is I.3.1, of which the almost trivial proof of the $1+\epsilon$ theorem is simply an illustrative example. I'm indebted to Cécile for the original typesetting, and I apologise for mucking it up by clumsy cutting and pasting.

I. Curves

I.1 Brody's lemma

A log-variety (X, D) which is complete hyperbolic automatically admits a constant $-\kappa(X) < 0$ such that,

$$H._f T \leq -\kappa\, \chi_{(T,B)}$$

for all maps from curves $f : (T, B) \to (X, D)$. A non-isotrivial family $X \to S$ of curves with boundary the fibres of bad reduction has, more or less, a complete hyperbolic space of complex points, whence, it's almost immediate that algebraic points $f \in X(\overline{K})$ satisfy,

$$h_{\omega_{X/S}}(f) \leq \kappa\, \mathrm{discr}(f)$$

There are plenty of proofs in the literature, [L], which make this rigorous. These proofs, however, give the impression that this requires something other than Green's complement on Brody's reparameterisation lemma, [B], [Gr]. Indeed it appears impossible to deduce from Green's lemma, the complete hyperbolicity of a semi-stable, non-isotrivial, family $X \to S$ outwith its fibres of bad reduction X_{s_i} since the lemma would require that the family is stable. Certainly, this would amount to a genuine impossibility if X were required to be smooth, but Green's lemma is wholly valid whenever the components of the boundary are $\mathbb{Q}$-Cartier, so on passing to a stable reduction we conclude,

I.1.1 Fact *Let $X \otimes K$ be a curve over a function field of characteristic zero, then there are constants $\kappa(X), \alpha(X)$ such that for all algebraic points f,*

$$h_{\omega_{X \otimes K/K}}(f) \leq \kappa\, \mathrm{discr}(f) + \alpha$$

I.2 Vojta's $2+\epsilon$ theorem

The argument here admits a positive characteristic variant, which will be similar in spirit to how we'll study surfaces. The important protagonist is the projective tangent space of the total space, and its tautological bundle, *i.e.*

I.2.1 Notation *Let $X \to B$ be a separable map of schemes, or, better, Deligne-Mumford stacks, then $\pi : P_{X/B} \to X$ will denote the projective tangent cone $\mathbb{P}(\Omega_{X/B})$ with $L_{X/B}$ it's tautological bundle, where the* EGA *convention on projectivisation is followed here, and throughout.*

In the particular case that B is a field k, of any characteristic, and $f : T \to X$ a separable map from a proper curve, there is a derivative $f' : T \to X$, satisfying,

I.2.2 Tautology $L_{X/k} \cdot_{f'} T \leq -\chi_T$

In the situation, where $X \to S$ is a family of curves, possibly iso-trivial, Riemann-Roch gives,

I.2.3 Lemma [V2] *For every rational $\epsilon > 0$, there is an effective $\mathbb{Q}$-divisor D_ϵ on $P_{X\otimes K/k}$ such that,*

$$(2+\epsilon)L_{X/k} = D_\epsilon + \pi^*\omega_{X/K}$$

Combining I.2.2 and I.2.3, immediately yields,

I.2.4 Lemma *For every $\epsilon > 0$, there is a constant $\alpha(X,\epsilon)$ such that a point f of $X \otimes K$ over the separable closure of K satisfies,*

(a) $h_{\omega_{X/K}}(f) \leq (2+\epsilon)\mathrm{discr}(f) + \alpha(X,\epsilon)$ *and/or* (b) $f' \in D_\epsilon$

Vojta's $2+\epsilon$ theorem is, up to changing α, that in characteristic 0 all points satisfy (a). Notice that changing α also allows us to assert that D_ϵ is finite at each of its generic points. On the other hand,

I.2.5 Lemma *If $\Omega_X \mid_{X\otimes K}$ is semi-stable, and D_ϵ is finite at each of it's generic points, then there is a $\mathbb{Q}$-Cartier divisor F_ϵ on the support $|D_\epsilon|$ of D_ϵ such that,*

$$(2+\epsilon)L_{X\otimes K/k}\mid_{|D_\epsilon|} = F_\epsilon + \omega_{X/K}\mid_{|D_\epsilon|}$$

This is immediate from the definitions, and whence I.2.4 (a) holds for all points unless $\Omega_{X\otimes K/\mathbb{C}}$ is unstable. In characteristic zero this is related to the Bogomolov stability of Ω_X, or more correctly the proof [B2]. Indeed taking models, allows us, any characteristic, to write the destabilising exact sequence as,

$$0 \to \omega_{X/\mathcal{F}} \to \Omega_X \to K_{\mathcal{F}} I_Z \to 0$$

for $\omega_{X/\mathcal{F}}$ the co-normal bundle to a foliation by curves, $\mathcal{F}$, with singularities Z supported in co-dimension 2. Again I.2.5 is valid on any subscheme of D_ϵ whose generic point is distinct from the section of π defined by $\mathcal{F}$. As such, we can suppose that all points not satisfying I.2.4 (a) are invariant by $\mathcal{F}$. Points f invariant by $\mathcal{F}$ have co-homology classes in,

$$\mathrm{Im}\{\mathrm{Ext}^1_X(I_Z, \omega_{X/\mathcal{F}}) \to H^1(X, \Omega_{X/k})\}$$

As does $c_1(\omega_{X/\mathcal{F}})$. In characteristic 0 one can blow up until only finitely many points are actually in the image of $H^1(X, \omega_{X/\mathcal{F}})$, cf. [B1], so,

$$h_{\omega_{X/\mathcal{F}}}(f) \leq 0(1), \quad \text{characteristic zero, } f \text{ invariant by } \mathcal{F}$$

In characteristic p, there is still some truth in this, but only on taking intersection numbers with values in $\mathbb{F}_p$. Whence, we use that $\deg_{X\otimes K}(K_{\mathcal{F}}) > 0$ unless $X \otimes K/K$ is iso-trivial, so that,

I.2.5 Corollary *For every $\epsilon > 0$, there is a constant $\alpha(X,\epsilon)$ such that every point f of a curve $X \otimes K$, of genus g, over the separable closure of K satisfies,*

$$h_{\omega_{X/K}}(f) \leq \begin{cases} (2+\epsilon)\,\mathrm{discr}(f) + \alpha(X,\epsilon)\,, & \text{characteristic } 0 \\ (2g-2+\epsilon)\,\mathrm{discr}(f) + \alpha(X,\epsilon)\,, & \text{characteristic } p \end{cases}$$

I.3 Vojta's $1+\epsilon$ conjecture

Needless to say this involves replacing 2 by 1 in I.2.5. We will need characteristic zero, and in many ways the full force of $\mathbb{C}$, which from now on is our base. Singularities will also be important, so say $X \to S$ a semi-stable family over a smooth proper curve S. The key point is I.3.1 below, if it held for logarithmic derivatives then singularities would be trivial to handle. Unfortunately, the proof fails because $x \mapsto x^{-1}$ is not an integrable function of a real variable at zero. However $x^{\frac{1}{n}-1}$ is integrable, so for each $n \in \mathbb{N}$ consider algebraic stacks $X_n \to X$, with moduli X, obtained by extracting nth roots about every irreducible component of every singular fibre. The order in which this is done is un-important, and one obtains a stack whose local monodromy at a non-scheme like point is $(\mathbb{Z}/n)^r$, where r is the number of components of the fibre through the point. Indeed the (étale) fundamental group of a strictly local regular affine scheme punctured in a strict simple normal crossing divisor with r components is $\hat{\mathbb{Z}}^r$, so this is manifestly well defined, cf. [V], proof of proposition 2.8. Now let f_i be a sequence of algebraic points on X ordered by increasing height for which the $1+\epsilon$ conjecture fails. We view these as S-maps from proper covers $p_i : T_i \to S$, with T_{in} a normalised component of $T_i \times_X X_n$. As such, we have liftings f_{in}, and one should note that,

$$\deg(\omega_{T_{in}/\mathbb{C}}) \leq \deg(\omega_{T_n/\mathbb{C}}) + 0((T_n : S))$$

for the simple reason that $X_n \to X$ is ramified only in fibres. Furthermore X_n is smooth in its étale topus, so smooth function, distribution *etc.* have perfect sense. In particular on an étale neighbourhood a basis of differential forms permits a non-canonical identification of forms with a finite dimensional vector space of functions, and, in an a priori basis dependent way, we can talk about L_p, $1 \leq p \leq \infty$ forms, or measured valued forms, or whatever ones favourite function space is. Plainly, however, the induced topology on the space of (canonically defined) smooth forms is independent of the basis, so that after completion in the relevant topology we get sheaves of topological vector spaces $L_p^{i,j}$ of forms of type (i,j) with L_p coefficients, or even just $M^{i,j}$ with measure coefficients. Furthermore for any stack with a moduli space, a partition of unity on the moduli space allows one to conclude that the $L_p^{i,j}$ are actually sheaves of Banach spaces. Similarly on measured valued forms, M^{top}, of top weight the dual of pulling back bounded Borel functions from the moduli to the stack defines a push-forward to top weight measured valued forms on the moduli, which one can compose with integration on the moduli, to define integration on any stack with a moduli space. As it happens, this is a somewhat stupid way to proceed, and integration is best defined wholly moduli space free, [M5]. Nevertheless such a degree of sophistication is irrelevant to the current simple context which wholly suffices to define currents of integration,

$$F_{in} : A^{1,1}(X_n)|_{L_\infty^{1,1}} \to \mathbb{C} : \tau \mapsto \frac{1}{H._{f_{in}} T_{in}} \int_{T_{in}} (f_{in})^* \tau$$

where A is for smooth, albeit understood here in the topology induced by its inclusion in L_∞, and H is an ample divisor on X. A smooth metric on X_n may be taken as the pull-back of a metric on X plus dd^c of a suitable Hölder continuous function, *c.f.* [M6] III.2.1, so by Stokes, the integral of F_{in} against a smooth metric on X_n is bounded independently of i. Consequently, we can subsequence in i, so that the F_{in} converge to some Φ_n in the weak dual of L_∞ forms, *i.e.* in $M^{1,1}$. Similarly, we obtain on differentiating,

$$F'_{in} : A^{1,1}(P_{X_n/\mathbb{C}})|_{L^{1,1}_\infty} \to \mathbb{C} : \tau \mapsto \frac{1}{H._{f_{in}} T_{in}} \int_{T_{in}} (f'_{in})^* \tau$$

Supposing the f_i violate the $1+\epsilon$ conjecture, these will also have bounded mass, so, without loss of generality, there's no trouble subsequencing to a weak limit Φ'_n such that, $(\pi)_*\Phi'_n = \Phi_n$. Or, better, do this latter subsequencing first, and simply define Φ_n as the push-forward.

Notation done, the key point is as follows: $P_{X_n/S}$ embeds in $P_{X_n/\mathbb{C}}$, with a distinguished component D which is the closure of the unique component over K, and we assert,

I.3.1 Claim *If the normalised heights of the f_i go to infinity, then,*

$$(\pi)_* \mathbb{1}_{P_{X_n/\mathbb{C}} \setminus D} \Phi'_n = 0$$

proof Suppose otherwise, then we require to understand what it means to be a distance $\delta > 0$ off D. The situation is most complicated at the singularities, so we'll exclusively work there. By construction we have analytic local coordinates x, y, s such that X_n is given by $s = (xy)^n$, with s a coordinate on S. As such an equation for D may be identified with the differential form,

$$xdy + ydx = \frac{1}{n}(xy)^{-(n-1)} ds$$

Whence if $Z_n \to S$ is the stack with moduli S obtained by taking nth roots at points where the fibre is singular, then for $|.|$, and $\|.\|$ metricisations of the tangent bundles of Z_n, and X_n respectively, to be a distance at least δ from D amounts to,

$$\|dx\| + \|dy\| \ll \frac{1}{\delta}|dz|$$

where, of course, z is a local coordinate on Z_n. Whence if w is a smooth metric on X_n, then,

$$\int_{f_{in}^{-1}(\mathrm{dist_D} > \delta)} (f'_{in})^* w \ll_n \frac{1}{\delta}(T_i : S)$$

where the implied constant may depend on n. $\square$

Now, by I.2.2, the $1+\epsilon$ conjecture will follow if for sequences f_i whose height goes to infinity, we can show,

I.3.2 Further Claim *For n sufficiently large,*

$$\omega_{X/S}.\Phi_n \le L_{X_n/\mathbb{C}}.\Phi'_n + \epsilon$$

proof This follows, trivially, from I.3.1 & [M6] III.3.3 if $X \to S$ is smooth, and the stack structure is there so as to make better, and better approximations to this situation. Specifically $D \to X_n$ is a blow up in some bunch of geometric points $\{z\}$ lying over the singularities, so denote by E_{nz} an exceptional divisor over any such, then it will suffice to show that given ϵ we can find a large n such that $E_{nz}.\Phi'_n \leq \epsilon$. To this end let B be an irreducible curve in the fibre of $X \to S$ through z with B_n its nth root in X_n, and $\widetilde{B}_n$ its proper transform in D. By hypothesis, and I.3.1,

$$E_{nz}.\mathrm{II}_D \Phi'_n = -\widetilde{B}_n.\mathrm{II}_D \Phi'_n$$

On the other hand,

$$\widetilde{B}_n.\mathrm{II}_D \Phi'_n \geq \widetilde{B}_n.\mathrm{II}_{\widetilde{B}_n} \Phi'_n = \frac{1}{n}(B-C)\mathrm{II}_B \Phi_0$$

where C is the other curve in the fibre through z. Since Φ_0 can be fixed as a current on X independent of n we conclude. $\square$

II. Surfaces

II.1 Riemann-Roch calculations

Unlike curves, the sectional curvature of surfaces does not enjoy any a priori relation to the ampleness of any line bundle. As such, we'll need some Riemann-Roch calculations at the generic point of a proper family $X \to S$ of surfaces, which is generically smooth, and, for convenience, say X non-singular, since from now on the base will be $\mathbb{C}$. Over the generic fibre, we have a short exact sequence,

$$0 \to \mathcal{O}_{X\otimes K} \to \Omega_{X\otimes K/\mathbb{C}} \to \Omega_{X\otimes K} \to 0$$

So that in particular, $s_t\,(\Omega^\vee_{X\otimes K/\mathbb{C}}) = s_t(\Omega^\vee_{X\otimes K/K})$, where s_t is the Segre polynomial, and whence: $s_2\,(\Omega^\vee_{X\otimes K/\mathbb{C}}) > 0$, if $s_2\,(\Omega^\vee_{X\otimes K/K}) > 0$. On the other hand from the point of view of the projective bundle, $\pi : P_{X\otimes K/\mathbb{C}} \to X \otimes K$ this says $L^4_{X\otimes K/\mathbb{C}} > 0$, and so, as ever by Riemann-Roch,

$$\chi\,(P_{X\otimes K/\mathbb{C}}, L^{\otimes n}_{X\otimes K/\mathbb{C}}) \sim \frac{n^4}{4!}\,L_{X\otimes K/\mathbb{C}},$$

grows positively in n, for $c_1(X\otimes K)^2 > c_2(X\otimes K)$. We calculate the cohomology,

$$H^i(P_{X\otimes K/\mathbb{C}}, L^{\otimes n}_{X\otimes K/\mathbb{C}}) \xrightarrow{\sim} H^i(X\otimes K, \mathrm{Sym}^n\,\Omega_{X\otimes K/\mathbb{C}}),$$

where we profit from $R^j\pi_* L^{\otimes n}_{X\otimes K/\mathbb{C}} = 0$ for $j > 0$. So if we're interested in conditions for large h^0, we need only concern ourselves with the H^2, which is isomorphic to $H^0(X\otimes K, \mathrm{Sym}^n T_{X\otimes K/\mathbb{C}} \otimes \omega_{X\otimes K/\mathbb{C}})$. Whence we may conclude that L is big provided that $T_{X\otimes K/\mathbb{C}}$ is not big. In fact,

II.1.1 Lemma *Suppose that the generic fibre $X \otimes K$ is minimal of general type, and the family is not iso-trivial, then,* $h^0\left(X \otimes K, \mathrm{Sym}^n T_{X\otimes K/\mathbb{C}}\right) = 0$, $\forall\, n \in \mathbb{N}$.
Proof. Suppose otherwise, and let C be a generic member of a sufficiently high multiple of $\omega_{X\otimes K/K}$. Then $\Omega_{X\otimes K/K}|_C$ is semi-stable, and $\omega_{X\otimes K/K}.C > 0$ so in fact, $\Omega_{X\otimes K/K}|_C$ is ample. Now choose the smallest $n \geq 1$ such that $h^0\left(X \otimes K, \mathrm{Sym}^n T_{X\otimes K/\mathbb{C}}\right) \neq 0$, and consider the exact sequence,

$$0 \to \mathrm{Sym}^{n-1}\,\Omega_{X\otimes K/\mathbb{C}} \to \mathrm{Sym}^n\,\Omega_{X\otimes K/\mathbb{C}} \to \mathrm{Sym}^n\,\Omega_{X\otimes K/K} \to 0$$

then if $n \geq 2$ we obtain an element of $H^0\left(X \otimes K, \mathrm{Sym}^n\, T_{X\otimes K/K}\right)$, while if $n = 1$, given that $H^0\left(X \otimes K, T_{X\otimes K/K}\right) = 0$, $\Omega_{X\otimes K/\mathbb{C}}$ must be a split extension of $\Omega_{X\otimes K/K}$ by $\mathcal{O}_{X\otimes K}$, and so $X \otimes K/K$ is isotrivial. $\square$

Consequently, $h^0(L^{\otimes n}_{X\otimes K/\mathbb{C}})$ grows like n^4 for $c_1^2 > c_2$ on the generic fibre, so applying I.2.2 we obtain,
II.1.2 Fact *Let $X \otimes K/K$ be a surface of general type with $s_2(X \otimes K) > 0$, then for H ample, there are constants $-\kappa(X) < 0$, $\alpha(X) > 0$, and a divisor $D \subset P_{X\otimes K/\mathbb{C}}$ such that any algebraic point f satisfies,*

(a) $h_H(f) \leq \kappa\,\mathrm{discr}(f) + \alpha$ *and/or* (b) $f' \in D$

The extra dimensions make (b) more difficult to deal with. To be sure I.3.1 is valid in all dimensions on replacing X_n by X, so we can almost reduce to the main lemma of [M3]. Unfortunately, there are several issues of bad reduction not just for the family $X \to S$, but also of foliations, and currents to deal with, so we'll proceed in a spirit more akin to I.2. Regrettably, this involves a serious weakening of the surfaces we may study, since we appeal to,
II.1.3 Fact (cf. [Mi], [Lu]) *Again let $X \otimes K$ be a non-isotrivial minimal surface of general type, but with positive topological index, τ, equivalently i.e. $c_1(X \otimes K)^2 > 2c_2(X \otimes K)$. Furthermore let $D \subset P_{X\otimes K/\mathbb{C}}$ be a divisor each generic point of which dominates $X \otimes K$, then $L_{X\otimes K/\mathbb{C}}|_D$ is big.*
proof We may suppose D is irreducible, so there is an integer m and a line bundle M on $X \otimes K$ such that, $\mathcal{O}_P\,(D) \xrightarrow{\sim} L^{\otimes m}_{X\otimes K/\mathbb{C}} \otimes \pi^*\, M^\vee$. We wish to calculate the intersection number, $L^3_{X\otimes K/\mathbb{C}}.D$ on the generic fibre. To this end, observe:

$$L^3_{X\otimes K/\mathbb{C}}.D = L^3_{X\otimes K/\mathbb{C}}.(m\, L_{X\otimes K/\mathbb{C}} - M) = m\, s_2\,(T_{X\otimes K/\mathbb{C}}) - \omega_{X\otimes K/\mathbb{C}}.M\,.$$

On the other hand, D defines a map, $0 \to \pi^*\, M \to L^{\otimes m}_{X\otimes K/\mathbb{C}}$, which we may push forward to obtain maps,

$$M \to \mathrm{Sym}^n \Omega_{X\otimes K/\mathbb{C}} \to \mathrm{Sym}^n \Omega_{X\otimes K/K},$$

while D irreducible implies the composite map $M \to \mathrm{Sym}^m\, \Omega_{X_K}$ is non-zero, so we may apply $\omega_{X\otimes K/K}$ semi-stability of the generic cotangent bundle to obtain,

$$M \cdot \omega_{X\otimes K/K} \leq \frac{m}{2}\, c_1\,(X \otimes K)^2 \quad \text{and whence,} \quad L^3_{X\otimes K/\mathbb{C}} \cdot D \geq \frac{3m}{2}\, \tau\,(X \otimes K) > 0\,.$$

To determine the cohomology $H^i\,(D, L^{\otimes n}_{X\otimes K/\mathbb{C}})$, we use the exact sequence,

$$0 \to L^{\otimes n}_{X\otimes K/\mathbb{C}}\,(-D) \to L^{\otimes n}_{X\otimes K/\mathbb{C}} \to L^{\otimes n}_{X\otimes K/\mathbb{C}}\,|_D \to 0$$

so, that on taking n sufficiently large, we are reduced to excluding the possibility that $H^2\,(P_{X\otimes K/\mathbb{C}}, L^{\otimes n}_{X\otimes K/\mathbb{C}})$ grows in dimension like n^3. Arguing as before we must simply exclude that $H^0\,(C, \mathrm{Sym}^n\, T_{X\otimes K/\mathbb{C}})$ admits a similar estimate, for C a generic member of a suitable multiple of $\omega_{X\otimes K/\mathbb{C}}$. Proceeding exactly as in II.1.1 we see that we are done unless $H^0\,(C, T_{X\otimes K/\mathbb{C}}) \neq 0$. This of course forces $T_{X\otimes K/\mathbb{C}}\mid_C$ to split, and better still the semi-stability of $\Omega_{X\otimes K/K}\mid_C$ obliges,

$$0 \to \Omega_{X\otimes K/K}\mid_C \to \Omega_{X\otimes K/\mathbb{C}}\mid_C \to \mathcal{O}_C \to 0$$

to be the Harder-Narismhan filtration for the bundle $\Omega_{X\otimes K/\mathbb{C}}$, and whence the tangent sheaf of the canonical model of $X\otimes K$ splits, which again implies that $X\otimes K$ is isotrivial. $\square$

Now components of D appearing in II.1.2 (a) which don't dominate $X\otimes K$, are the pull-backs of curves. If these curves aren't rational or elliptic, then any of the height bounds of §I can be used to deduce a height bound of the form II.1.2 (b), there are, however, by [B1] only finitely many of these under the chern class inequalities that we've given, so without loss of generality D satisfies II.1.3. Again, we may appeal to II.2.2, change our constants to account for non-dominant components of the appropriate linear system, and deduce,

II.1.4 Fact *Let $X\otimes K/K$ be a surface of general type with $\tau(X\otimes K) > 0$, then for H ample, there are constants $-\kappa(X) < 0$, $\alpha(X) > 0$, and a sub-scheme $Y \subset P_{X\otimes K/\mathbb{C}}$, finite over $X\otimes K$ at each of its generic points such that any algebraic point f satisfies,*

(a) $h_H(f) \leq \kappa\,\mathrm{discr}(f) + \alpha$ *and/or* (b) $f' \in D$

II.2 Foliations by curves

Needless to say the goal is to prove that on surfaces of general type and positive index all algebraic points satisfy II.1.4(a). The obstruction is II.1.4(b), so (b) implies (a) will do. This has nothing to do with chern numbers, and we assert,

II.2.1 Claim *Let $X\otimes K/K$ be a surface of general type and $Y \subset P_{X\otimes K/\mathbb{C}}$ a sub-scheme finite over $X\otimes K$ at each of its generic points, then for H ample there are constants $-\kappa(X,Y) < 0$, $\alpha(X,Y) > 0$ such that algebraic points f with $f' \in Y$ satisfy,*

$$h_H(f) \leq \kappa\,\mathrm{discr}(f) + \alpha$$

The proof will occupy the rest of the manuscript. Plainly we may suppose Y geometrically irreducible, and the points f are Zariski dense. The condition $f' \in Y$ is a first order O.D.E.. Sine the points f also lift to Y by differentiation, we can replace X by Y without changing their discriminants, so without loss of generality the O.D.E. is linear, *i.e.* it's a foliation by curves $\mathcal{F}$. The condition $f' \in Y$ may thus be replaced by f is invariant by $\mathcal{F}$, which in turn is given by a short exact sequence,

$$0 \to \Omega_{X/\mathcal{F}} \to \Omega_{X\otimes K/\mathbb{C}} \to K_{\mathcal{F}} I_Z \to o$$

where the kernel is reflexive rank 2, $K_{\mathcal{F}}$ is the bundle of forms along the leaves, and Z the (generic) singular sub-scheme of $\mathcal{F}$. As such Y is in fact,

$$\mathrm{Proj}\left(\sum K_{\mathcal{F}.}^{\otimes n}\, I_Z^n\right)$$

The singularities are important, and we blow up to make them as good as possible. This means, functorially with respect to the ideas, canonical, [M4] I.6.2. With current hypothesis such a modification exists, [S]. Consequently,
II.2.2 Fact *Let everything be as in* II.2.1, *then for every $\epsilon > 0$, there is a proper sub-variety $V_\epsilon \subset X \otimes K$ such that,*

$$h_{K_{\mathcal{F}}}(f) \leq \mathrm{discr}(f) + \epsilon h_H(f),\;\; f \notin V_\epsilon$$

This is immediate from the substantially more general [M4] V.6.1. Since, we essentially only have 2-dimensions to worry about, one can also do this by modifying the original proof of the refined tautological inequality of [M1], as found in [M2] VI.2. The upshot is that we can try to get down to a curve problem by studying the linear systems $K_{\mathcal{F}}^{\otimes n}$. This requires some of the theory of minimal models of foliations. In the current situation this enjoys some substantial simplifications because the composite of the natural maps,

$$\omega_{K/\mathbb{C}} = \mathcal{O}_{X\otimes K} \to \Omega_{X\otimes K/\mathbb{C}} \to K_{\mathcal{F}}$$

is non-zero at every f, which are dense, so indeed it's non-zero, and defines a section Γ, which we confuse with the curve that it defines, of $K_{\mathcal{F}}$. Consequently, though we could appeal to the minimal model theory of [M4], we can also blow up so that Γ is simple normal crossing, and just do things by hand, *i.e.*
II.2.3 Facts *Let $C \subset X \otimes K$ be a curve then either,*
(a) C is not invariant by $\mathcal{F}$, and $(K_{\mathcal{F}} + C).C \geq 0$, or,
(b) C is invariant, and, say smooth, and contained in the support of Γ, then,

$$K_{\mathcal{F}}.C = -\chi_C + s_Z(C)$$

proof (a) is immediate from the definitions, so let's concentrate on (b), including defining the segre class s_Z at the singularities. Since C is invariant, it admits a lifting $\widetilde{C}$ to Y, and we have a surjective map,

$$\Omega_{C/\mathbb{C}}\mid_{\widetilde{C}} \to K_{\mathcal{F}}(-E)\mid_{\widetilde{C}} \to 0$$

where E is the, not necessarily reduced, exceptional divisor. By definition, $s_Z(C) = E.\widetilde{C}$, so $C \subset \Gamma$ implies (b), and even the nullity of the natural map,

$$\Omega_{X/\mathcal{F}} \to \Omega_{C/K} \qquad \Box$$

The first of these tells us that the conditions $K_{\mathcal{F}}.C < 0$, and C contractible are incompatible, the second that C must be a rational curve. The self intersection may well not be -1, so contracting it may lead to a quotient singularity. As such there is a minimal smooth stack whose moduli is that of the contraction,

[V]. On the said stack, the non-scheme like points are terminal for the induced foliation, so, in fact, smooth, [M4] I.6.11. Since there is a unique smooth invariant hypersurface through a co-dimension 2 terminal singularity, the push forward of Γ to the contracted surface followed by it's pull-back to the stack has an invariant part which is still simple normal crossing, while II.2.3 (a) is evidently still valid for smooth stacks. Consequently, there is a contraction $\rho : X \otimes K \to X_0 \otimes K$ to a variety with quotient singularities, with canonical bundles related by,

$$K_{\mathcal{F}} = \rho^* K_{\mathcal{F}_0} + \sum_z E_z$$

where z are the centres of the contractions, E_z are $\mathbb{Q}$-divisors supported on chains of invariant rational curves and $K_{\mathcal{F}_0}$ is nef. As such, $K^2_{\mathcal{F}_0} \neq 0$ implies II.2.1 by II.2.2, while if the Kodaira dimension is zero, then a cyclic cover of $X_0 \otimes K$ is isotrivial. This leaves the possibility that the push-forward Γ_0 of Γ is nef., non-zero, and $\Gamma_0^2 = 0$. To cope with this we have to know how curves in Γ intersect, *i.e.*

II.2.4 Lemma *Two integrable hypersurfaces C, C' in the support of Γ can only meet generically in a foliation singularity.*

proof Suppose there is no singular point in the intersection then there is a neighbourhood U of C containing $C \cap C'$ which does not contain any foliation singularities, so,

$$c_1\left(C' \mid_U\right) \in \operatorname{Im}\left\{H^1\left(U, \Omega_{X/\mathcal{F}}\right) \to H^1\left(U, \Omega_{X\otimes K/\mathbb{C}}\right)\right\}.$$

Consequently, $c_1\left(C' \mid_C\right)$ is in the image of the natural map from $H^1\left(U, \Omega_{X/\mathcal{F}}\right)$ to $H^1\left(C, \Omega_{C/K}\right)$ which we've noted is zero, which is absurd. Otherwise there are n singular points $z_1, \ldots, z_n$, say, in the intersection, which by base change we may suppose is simple so if we blow up in these points to obtain $\widetilde{C}$ and $\widetilde{C'}$, say, then by the above $\widetilde{C} \cdot \widetilde{C'} = 0$, and so $C \cdot C' = n$ as required. $\square$

Now, suppose every curve in Γ_0 were invariant, and write $\rho^*\Gamma_0 = \sum a_j\, C_j$. By II.2.4 this is a normal crossing divisor with the crossings occurring only in the foliation singularities. Moreover it is nef. of square 0 so we have the formulae,

$$-\,a_j\, C_j^2 = \sum_{k\neq j} a_k\, C_k \cdot C_j \quad \text{and} \quad (\omega_{X\otimes K/K} - K_{\mathcal{F}}).\, C_j \leq -\,C_j^2 - \sum_{k\neq j} C_k \cdot C_j$$

for all j. So that multiplying the latter by a_j and combining we obtain,

$$\omega_{X\otimes K/K} \cdot p^*\,\Gamma_0 \leq K_{\mathcal{F}} \cdot p^*\,\Gamma_0 = 0,$$

which is absurd since $\omega_{X\otimes K/K}$ is big.

Consequently Γ_0 contains a non-invariant irreducible curve C, so by II.2.3(a) (stack version) and the index theorem C is parallel to Γ_0 in Néron-Severi. We use the foliation to move C. This is most conveniently done on the minimal smooth stack over $X_0 \otimes K$. Since $X \otimes K$ has general type, C cannot be rational, so it's pre-image in the said stack admits an étale neighbourhood U which is everywhere scheme like, and we, legitimately, confuse C with an irreducible pre-image in U. If either C were not smooth in U or there were induced foliation singularities on C then II.2.3(a) would be a strict inequality, which is absurd. Thus the foliation is given on a cover $\coprod U_\alpha \to U$ by non-vanishing vector fields ∂_α, C by coordinate functions $x_\alpha = 0$, and $\Gamma_0 \mid_U = nC$, for some $n \in \mathbb{N}$. We further assert,

II.2.5 sub-claim *Let U_m, $m \in \mathbb{N}$ be the mth thickening of C in U, then, $\mathcal{O}_{U_m}(C)$ is at worst $n+1$ torsion.*

proof $m = 1$ follows from II.2.3(a), so without loss of generality we have a torsion bundle L on U with transition functions $\zeta_{\alpha\beta}$ such that $L^\vee \mid_C \xrightarrow{\sim} \mathcal{O}_C(C)$, and we go by induction. Quite generally there is an exponential sequence,

$$H^0(\mathcal{O}^\times_{U_m}) \xrightarrow{\delta} H^1(C, I_C^m/I_C^{m+1}) \longrightarrow \mathrm{Pic}(U_{m+1}) \longrightarrow \mathrm{Pic}(U_m) \longrightarrow 0\,.$$

Consequently, we may choose our local equations for C, so that the transition functions $f_{\alpha\beta}$ satisfy $f_{\alpha\beta} = \zeta_{\alpha\beta}(1 + h_{\alpha\beta})$ for $h_{\alpha\beta}$ a 1 co-cycle in I_C^m/I_C^{m+1}. As such if $g_{\alpha\beta}$ are the transition functions for $K^\vee_{\mathcal{F}}$ then,

$$\partial_\alpha x_\alpha = \zeta_{\alpha\beta}\exp((m+1)h_{\alpha\beta})g_{\alpha\beta}\partial_\beta x_\beta \pmod{I_C^{m+1}}$$

leads to a trivialisation of $L + (m+1)C + K_{\mathcal{F}}$ mod I_C^{m+1}, and we're done since $\mathrm{Im}\delta$ is a K-vector space. $\square$

Dismissing cases where $X \otimes K$ is an irregular surface is straightforward, so we get an actual Kodaira fibration, $q : X_0 \otimes K \to B := |K_{\mathcal{F}}^{\otimes n}|$, for some appropriately large n. Furthermore, the natural map $q^*\Omega_{B/\mathbb{C}} \to K_{\mathcal{F}}$, allows us to conclude that we may descend the foliation, and obtain generically a map of foliated varieties $(X, \mathcal{F}) \to (B, \mathcal{G})$, *i.e.* leaves go to leaves. Without loss of generality, however, the images of our algebraic points in B are Zariski dense, so by [J], $\mathcal{G}$ has a first integral. This first integral won't be generically flat over S, so we take models, and denote by $p : X_0 \to C$ the Stein factorisation of the composition of the Kodaira fibration with the first integral, where C is a curve over $\mathbb{C}$. The fibres are a family of foliated surfaces of foliated Kodaira dimension 1, with the Kodaira fibration transverse to the foliation. Since $X \otimes K$ has general type, this implies that the foliation on the generic fibre over p is the suspension of a representation in the automorphism group of a curve of genus at least 2, so the fibre is an isotrivial family, over S, of curves of genus at least 2. Consequently, there are certainly constants $-\kappa < 0$, and α such that any map $f : T \to X$ from a smooth curve factoring through a fibre of p satisfies,

$$\omega_{p^{-1}(c)} \cdot_f T \leq -\kappa\,\chi_T + \alpha$$

while by adjunction $\omega_{p^{-1}(c)} = \omega_{X_0} \mid_{p^{-1}(c)}$, and we conclude to II.2.1 $\square$

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