CURVES ON SURFACES OF MIXED CHARACTERISTIC

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ABSTRACT. The classification of foliated surfaces, [McQ08], is applied to the study of curves on surfaces with big co-tangent bundle and varying moduli, be it purely in characteristic zero, or, more generally when the characteristic is mixed. Almost everything that one might naively imagine is true, but with one critical exception: rational curves on bi-disc quotients which aren't quotients of products of curves are Zariski dense in mixed characteristic. The logical repercussions in characteristic zero of this exception are not negligible.

I. INTRODUCTION

For $X/\mathbb{C}$ a smooth algebraic surface with $\Omega^1_{X/\mathbb{C}}$ big, a theorem, [Bog77], of Bogomolov asserts

(1)   *The set, Z, of rational or elliptic curves on X is finite*

and, better still, for H an ample divisor on X

(2)   *There is a constant $-\kappa(X, H) < 0$ such that if $f : C \to X$ is any map from*
from a smooth curve which doesn't factor through Z: $H \cdot_f C \leq -\kappa \, \chi(C)$

wherein, the notation is constructed according to the mnemonic: if ω is a metric on X (or indeed any complex manifold) with (holomorphic) sectional curvature $-K < 0$, then (Gauss-Bonnet),

$$(3) \qquad \int_C f^*\omega \leq -\frac{1}{K} \, \chi(C)$$

Nevertheless, one shouldn't, even if $Z = \emptyset$, imagine for a second that (2) is equivalent to negative sectional curvature, *e.g.* there are buckets of simply connected surfaces in Bogomolov's class with $Z = \emptyset$. Rather what (2) expresses is a weak, or (1) weaker still, algebraic vestige in complex dimension 2 ($\subset$ real dimension 4) of the implication that negative Ricci curvature implies negative sectional curvature in real dimension at most 3. Indeed, by [TY87], the canonical model of X admits (in the orbifold sense should this not coincide with the minimal model) a Kähler-Einstein metric of Ricci curvature -1, but, as we've remarked, will in all probability catastrophically fail to admit a uniformisation of the type encountered in real dimension 2 or 3. There are, of course, much better holomorphic vestiges of negative sectional curvature than (1)-(2), of which, the best to date is that Gromov's isoperimetric, [Gro07], holds, *i.e.*

(4)   *There is a constant $-K(X) < 0$ such that if $f : \Delta \to X$ is any holomorphic disc*
$$\text{area}_\omega(f) = \int_\Delta f^*\omega \leq \frac{1}{K} \, \text{length}_\omega(\partial_f) = \int_{\partial\Delta} |df|_\omega$$

where for ease of exposition, we suppose $Z = \emptyset$, which gives us the right to take ω equal to the Kähler-Einstein metric in (4). In particular, in exceptional cases where one does have uniformisation, *i.e.* ball and disc quotients, one sees that the right value of $-K = -\kappa(X, K_X)^{-1}$ whether in (2) or (4) is the holomorphic sectional curvature of the Kähler-Einstein metric, so $-2/3$ for balls, respectively $-1/2$ for the bi-disc, and, already [ACLG12], this turns out to be best possible in the restrictive algebraic context (2)-(3).

The much deeper inequality (4) while affording a revealing insight into the relation of (2) with the isoperimetric profile, is, however, only really our immediate interest in respect of the complexity, I.8, of the algebraic surface part of its proof, [McQ98], as opposed to the part, [Kle] or [Duv08], valid on all almost complex manifolds in all dimensions, and such questions of complexity are in turn subordinate to the main question of how does Bogomolov's theorem vary in moduli. This already has its own interest in characteristic zero, so we set things up accordingly: S will be an irreducible affine scheme of finite type over a Noetherian integral domain of characteristic zero, and X/S a smooth family of S-projective surfaces, with say X and its generic fibre geometrically irreducible for convenience, then we have 2 subtly different theorems

I.1. Theorem. *Suppose X/S above satisfies Bogomolov's condition $\Omega^1_{X/S}$ big, then exactly one of the following happens,*

(a) There exists a constant $-\kappa < 0$, along with a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$ and every separable map $f : C/k \to X$ from a smooth k-curve which doesn't factor through Z,

$$(5) \qquad\qquad H \cdot_f C \leq -\kappa \chi_C$$

(b) The closed points $s \hookrightarrow S$ of positive residue characteristic are dense, and the generic fibre is dominated by a modification of a bi-disc quotient (including b.t.w. the case of a product of curves).

I.2. Theorem. *Suppose X/S above satisfies Bogomolov's condition $\Omega^1_{X/S}$ big, then exactly one of the following happens,*

(a) There is a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$, every map $f : C/k \to X$ from a smooth rational or elliptic k-curve factors through Z.

(b) There exists a nowhere dense closed sub-scheme Z of X; a surjective map $S' \to U$ from an open dense affine sub-scheme $S' \subset S$, onto the spectrum of a sub-ring of $\mathbb{Q}$; finitely many bi-disc quotients Y_i/U which (over $\bar{\mathbb{Q}}$) don't admit an almost étale cover by a product of curves; and real quadratic number fields K_i (functorial in Y) such that infinitely many primes $p \in U$ are inert in each K_i; $X \times_S S'$ is dominated by an irreducible component of $Y_i \times_U S'$ for each i, and every rational or elliptic curve $C/k \to X$ which doesn't factor through Z is the image of some $f : \mathbb{P}^1_k \to Y_k$ of H-degree $O(p)$, for $\mathrm{Spec}(k) \to S$ a closed geometric point of characteristic p inert in some K_i, and f invariant by one of the natural foliations on Y_i/U arising from the bi-disc structure, yet missing the cusps should they exist.

Before proceeding to discuss the theorems in mixed characteristic, let us note that the obvious combination of I.1, Noetherian induction, the upper semi-continuity of h^0, and (to get the optimal Z as below) Riemann-Hurwitz gives

I.3. Corollary. *Let $X \to S$ be the universal smooth family of (minimal) surfaces of general type over some irreducible component of the moduli space$/\mathbb{C}$, then if the co-tangent bundle of the generic fibre is big, the set Z whose fibre over $s \in S$ is the rational and elliptic curves on X_s is a closed nowhere dense sub-scheme, and there exists a constant $-\kappa < 0$, such that for every closed geometric point $s \in S(\mathbb{C})$ and every map $f : C \to X_s$ from a smooth curve which doesn't factor through Z,*

$$(6) \qquad\qquad H \cdot_f C \leq -\kappa \chi_C$$

The case of I.3 where the bigness of Ω^1 is guaranteed by Riemann-Roch, *i.e.* $c_1^2 > c_2$ is also a theorem of Miyaoka, [Miy08], and, when the two overlap, Miyaoka's theorem is way better

since it provides explicit (and very simple) functions of the chern numbers for κ and the degree of Z, whereas I.3 says such functions exist, but they could be anything.

Now, irrespectively of whether we're in mixed characteristic or not, it's well known, and easy, V.3, that Bogomolov's theorem reduces to studying curves invariant by a foliation $\mathscr{F}$, so I.1 and I.2 are really just corollaries of

I.4. Theorem. *Let $(X, \mathscr{F}) \to S$ be a family of foliations by curves on a family X/S of smooth projective surfaces of general type with X and its generic geometric fibre irreducible for convenience, then exactly one of the following happens,*

(a) There exists a constant $-\kappa < 0$, along with a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$ and every separable $\mathscr{F} \otimes k$ invariant map $f : C/k \to X$ from a smooth k-curve which doesn't factor through Z,

$$(7) \qquad\qquad H \cdot_f C \leq -\kappa \chi_C$$

(b) The closed points $s \hookrightarrow S$ of positive residue characteristic are dense, and the generic fibre is a modification of a bi-disc quotient (including b.t.w. the case of a product of curves) with $\mathscr{F}$ one of the natural foliations induced by the bi-disc structure.

I.5. Theorem. *Hypothesis as in I.4, then exactly one of the following happens,*

(a) There is a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$, every invariant map $f : C/k \to X$ from a smooth rational or elliptic k-curve factors through Z.

(b) There exists a nowhere dense closed sub-scheme Z of X; a surjective map $S' \to U$ from an open dense affine sub-scheme $S' \subset S$, onto the spectrum of a sub-ring of $\mathbb{Q}$; a bi-disc quotient Y/U which (over $\bar{\mathbb{Q}}$) does not admit an almost étale cover by a product; and a real quadratic number field K (functorial in Y) such that infinitely many primes $p \in U$ are inert in K; $X \times_S S'$ is a bi-rational modification of an irreducible component of $Y \times_U S'$ with $\mathscr{F}/S'$ the base change of one of the natural foliations on Y/U arising from the bi-disc structure, and every $\mathscr{F}$-invariant rational or elliptic curve not factoring through Z is $\mathscr{F} \otimes k$ invariant for $\mathrm{Spec}(k) \to S$ a closed geometric point of characteristic p inert in k. Moreover for each such k, with $p \gg 0$, there is a $\mathscr{F} \otimes k$ invariant separable rational curve $f : \mathbb{P}^1_k \to Y$, with Y understood as a Deligne-Mumford champ if there are quotient singularities, missing the cusps E should these exist such that

$$(8) \qquad\qquad \frac{p}{2} \leq (K_Y + E) \cdot_f \mathbb{P}^1_k \leq (p-1)\mathrm{c}_{2,\log}(Y, E)$$

and, along which some non-classical modular form of weight $(2p, -2)$ vanishes.

At first sight I.4.(b) may appear surprising, but it obviously happens. Specifically, say $f = f_1 \times f_2 : C \to X = X_1 \times X_2$ a curve on a product of hyperbolic curves over $\mathbb{F}_p$ to avoid pointless technicalities, then f separable is only equivalent to, say, f_1 separable and there's nothing to stop us composing f_2 with a huge multiple of Frobenius. As such *separable curves of (a fixed) genus g on a product of hyperbolic curves in characteristic p are not bounded in moduli once g is (depending on X) sufficiently large*, albeit, by Riemann-Hurwitz and separable/inseparable factorisations of the f_i, the above is the only way this can happen, while, plainly, there are no rational or elliptic curves. This brings us to the case of bi-disc quotients which (up to a finite group action) aren't products of curves, and as one sees there is a clear distinction between primes p which are split, as opposed to inert in the real quadratic field K of I.2.(b). From the point of view of constructing counterexamples to boundedness in moduli of curves of genus g, the mechanism, III.9, is exactly the same as the case of products of curves when p is split, and while arguments in rigid co-homology, are more demanding than Riemann-Hurwitz, this is again, III.8, the only mechanism. Nevertheless, in the presence of cusps there is some subtlety,

i.e. bounding rational curves at split p, V.2, has more to it than just the rigid co-homology considerations of III.8.

This brings us to the amusing logical consequences of (8). More or less by definition, every theorem in algebraic geometry over $\mathbb{C}$ is an ACF_0 (first order theory of algebraically closed fields of characteristic zero) theorem. Unfortunately, and evidently, since there is quantification over a priori unbounded sets of curves in both (1) and (2) neither is a statement in ACF_0. One could, however, ask whether a theorem, $\mathscr{T}(f)$, about some type of curve, f on X, has a proof which is *essentially ACF_0*, *i.e.* does it have, for X fixed, the form

$$(9) \qquad P_i \wedge T_j(f) \Rightarrow \mathscr{T}(f)$$

where P_i are finitely many ACF_0 theorems and $T_j(f)$ is a, possibly infinite, set of tautologies valid for all curves f that we wish to study. A priori, therefore, f could be arbitrary, fixed genus, or whatever, but in practice, and mathematical precision in I.6, f is rational. For example any map $f : C/k \to X$ from a smooth curve to a geometric fibre has a factorisation $f = f_0 g$ into purely inseparable followed by separable, so, the existence of such a factorisation is such a tautology. Better still separable maps admit a derivative $f_1 := f_0' : C/k \to P_1 := \mathbb{P}(\Omega_{X/S}) \to P_0 := X$, and if L_1 is the tautological bundle then

$$(10) \qquad L_1 \cdot_{f_1} C = -\chi(C) - \mathrm{Ram}_{f_0}$$

is another tautology, which can even be iterated to an infinite sequence of tautologies, *i.e.* for $P_m \hookrightarrow \mathbb{P}(\Omega_{P_{m-1}/S})$ the mth jet bundle, L_m its tautological bundle, and f_m the mth derivative of f_0,

$$(11) \qquad L_m \cdot_{f_m} C = -\chi(C) - \mathrm{Ram}_{f_{m-1}}$$

Now Bogomolov's condition is that for some a, b the ACF_0 statement

$$(12) \qquad P_{a,b}(X) : \mathrm{H}^0(X, \mathrm{Sym}^a\Omega_X^1(-bH)) \neq \emptyset$$

is a theorem, so, choosing a, b appropriately and using just the tautology (10) the reduction of I.1-I.2 to I.4-I.5 is essentially ACF_0 in the sense of (9). On the other hand a theorem is ACF_0 iff it's ACF_p for $p \gg 0$, so

I.6. Corollary. *For every bi-disc quotient X, possibly with cusps E and quotient singularities, in which case X is to be understood as a Deligne-Mumford champ, but not a finite quotient of a product of curves, and $\mathscr{F}$ either of the natural foliations the following (true) statement has no essentially ACF_0 proof;*

$$(13) \qquad \textit{There are at most finitely many } \mathscr{F}\textit{-invariant rational curves on } X_{\mathbb{C}} \backslash E$$

Now while it's true that by I.4.(b) we could construct for high genus curves on products of curves a related statement with no *essentially* ACF_0 proof, this is only because of an evident separability rather than "bi-separability" issue which we could have excluded via a more sophisticated variation of (9). We haven't bothered to do this, since it's not only irrelevant for rational curves, but, manifestly, the weaker the statement with no essentially ACF_0 proof is, the more interesting it becomes. For example, the tautologies, 11, imply tautological inequalities,

$$(14) \qquad L_m \cdot_{f_m} C < 0$$

for rational curves, while for each m, our foliation, $\mathscr{F}$, defines a surface $X_m \hookrightarrow P_m$ in the mth jet space through which f_m must factor, so for all non-negative integers $n_1, \ldots, n_m$ with non-zero sum, n, all the ACF_0 statements

$$(15) \qquad \mathrm{H}^0(X_m, L(\underline{n}) := L_1^{\otimes n_1} \otimes \ldots \otimes L_m^{\otimes n_m}) \neq \emptyset$$

are false on bi-disc quotients which aren't quotients of products of curves. Now, if there are cusps, the $L(\underline{n})$ of (15) aren't very interesting, *e.g.* not even pseudo effective [McQ08, IV.5.7], but by (13) we have the right to do all this logarithmically, so the natural map $X_{m,\log} \to X$

is an isomorphism and the bundle in (15) (understood logarithmically if necessary) is just the nth power of the canonical of the foliation $K_{\mathscr{F}}^n$, so, we recover the well known fact, [SB95] or [McQ08, IV.5.4], that $K_{\mathscr{F}}$, which is in fact nef. in characteristic zero, has Kodaira dimension $-\infty$, which as it happens, III.3, is a lemma in the proof of I.5.(b), but, in principle I.6, or better, I.5.(b) is a much stronger fact than $\kappa(\mathscr{F}) := \kappa(K_{\mathscr{F}}) = -\infty$, c.f. I.8.(c) for another example.

As to the proof of the theorems themselves, they are largely an application of the classification theorem, [McQ08], of foliated surfaces, the theorems of which (if not the proof) are statements in ACF_0, so by model completeness they're actually ACF_0 theorems, and can be taken to be the P_i in (9). The tautologies, T_j, of op. cit. while taking (10) as their starting point are much more sophisticated. In particular they are false if the fibre over the generic point of S does not have canonical, II.3, foliation singularities, and require careful analysis, (71)-(84), of how invariant curves meet such singularities. The precise statement is the refined tautological inequality, IV.2, and it is delicate. For example in characteristic zero (14) for all m is equivalent to being rational, but not in characteristic p, III.4, and one could reasonably call such curves *pseudo-rational*. Now the refined tautological inequality gives I.4.(a)-I.5.(a) more or less gratis when the foliation is of general type, *i.e.* $\kappa(\mathscr{F}) = 2$, but it does not imply that pseudo-rational curves invariant by such foliations aren't Zariski dense in mixed characteristic. Indeed it strongly exploits the difference between rational and pseudo rational- or genus g and pseudo genus g for that matter- and it may well be the case that there is a foliation of general type in mixed characteristic with Zariski dense pseudo-rational curves, for which, by the same reasoning as above, (15) would have to be false for all $\underline{n}$ but $K_{\mathscr{F}}$ would be big. In any case since we're also supposing that X has general type in the usual sense, the only other possibility is that $\mathscr{F}$ is one of the natural foliations on a bi-disc quotient. The case of which where $\kappa(\mathscr{F}) \geq 0$ is a product of curves, and we have the pleasing fact that foliated Mori theory hits the logical obstruction of I.6 on the nose, *i.e.*

(16) $\mathscr{F}$ *has only finitely many invariant rational curves*

is always a true statement for complex surfaces of general type, but

I.7. **Corollary.** *Let $\mathscr{F}$ be a foliation on a complex surface of general type, then T.F.A.E.*

(a) *The foliated Kodaira dimension satisfies $\kappa(\mathscr{F}) \geq 0$.*
(b) *Foliated abundance holds, i.e. $\nu(\mathscr{F}) = \kappa(\mathscr{F})$.*
(c) *The (true) statement* (16) *is an essentially ACF_0 theorem.*

As such it remains to indicate the proof of I.5.(b), and, in particular the distinct behaviour of the invariant rational curves according to whether p is split or inert, which are, in fact, equivalent to the foliation being p-closed, II.5, or not. In this context, it's relevant to observe that the way [McQ98] would prove (13) is not Liouville's theorem but Baum-Bott residue theory which gives over $\mathbb{C}$ that the degree of the canonical, $K_{\mathscr{G}}$, of the other foliation along our curve is zero, and the rest is easy. Now a particular feature of the natural foliations on bi-disc quotients is that once they're p-closed, they're p-adically integrable, and with a bit of work, III.8 for rational or elliptic curves missing the cusps (whence a p-adic proof of (13) provided one chooses p correctly, *i.e.* split) and V.2 in general, one can push the Baum-Bott residue theory, respectively the proof of [McQ98], through when there is p-adic intergability. In the inert case the locus of vanishing of the p-curvature is no longer the whole surface but a divisor cut out by the non-classical modular form of I.5.(b), and if one looks at this divisor carefully enough, III.6, one finds the rational curves (8).

Before concluding this introduction it is appropriate to make

I.8. **Scholion. Complex hyperbolicity as an essentially ACF_0 theorem** We know this can't be true by I.6, but remarkably, for surfaces in the Bogomolov class, (13) with entire

instead of rational is almost the only problem. More precisely for pointed Riemann surfaces there is a tautology of the from (10) but for intersection numbers and Euler characteristics understood in the sense of Nevanlinna theory, leading to a refined tautological inequality (which was actually the motivation for IV.2) and one can ask for an *essentially* ACF_0 proof of (2), exactly as in (9) but with the implied tautologies, intersection numbers, *etc.* understood in the Nevanlinna sense. In particular, the original motivation of the classification theorem was (with a view to generalisation) to find an essentially ACF_0 proof of [McQ98]. This is discussed in the introduction to [McQ08] at length, so we won't go on about it here, but the basic irony of the proof of the classification theorem is (modulo soft theorems, Gromov's sense, of Brunella, [Bru03], and Duval, [Duv08]) is that it's essentially a corollary of what it was instead to give an essentially ACF_0 proof of, *i.e.* the main theorem on algebraic degeneracy of entire curves invariant by a foliation of [McQ98]. As such, the proof of the classification is certainly not ACF_0. Nevertheless it's statements are, so, as we've said, by model completeness they are ACF_0 theorems. Now argue as we did before, in the Nevanlinna setting the first step, *cf.* (12), is still essentially ACF_0, and we reduce to analytic curves invariant by foliations. Again, the $\mathscr{F}$ of general type case is gratis, while products of curves is easy, and again essentially ACF_0, so we're left with the class of surfaces in I.6.

At this point *op. cit* tells us that we're wasting our time, and we should look carefully at what is involved. To fix ideas, say our goal is no more than [McQ98], *i.e.* there is no Zariski dense, $\mathscr{F}$ invariant entire curve, $f : \mathbb{C} \to X'$, for X' a smooth surface in the Bogomolov class which may be no better than a modification of a bi-disc quotient X. The cusps, E, are also $\mathscr{F}$ invariant, and no invariant curve can meet them, so we certainly have $f : \mathbb{C} \to X \backslash E$. There is, however, a possible issue with quotient singularities, *i.e.* we may only have an entire curve on the moduli of the Deligne-Mumford champ of (13) and not an entire curve in the orbifold/Deligne-Mumford sense in which the curves of (13) are rational. As such one cannot a priori use Liouville, and indeed there can be rational and elliptic curves on $X \backslash E$ if this is not understood in the orbifold sense. Nor, even supposing that it can be done when there are quotient singularities, is it necessarily appropriate to try and force a Liouville style argument, since if we want to understand the logical structure then less is best. Consequently, Baum-Bott residue theory still looks to be the way to go. Again this is absolutely trivial for bi-disc quotients in a way consistent with the intuition that (13) should be trivial, and one has that the Nevanlinna degree of the canonical, $K_{\mathscr{G}}$, of the other foliation is zero for free, with the subsequent steps being trivial given the refined tautological inequality which absorbs the problem of the quotient singularities. This line of reasoning has a number of further pleasing features, *viz*:

(a) Large chunks of Baum-Bott theory, *cf.* (55), (59), and (60), are not only valid in the Zariski topology, but have perfect sense in ACF_0.

(b) Without further hypothesis, however, in characteristic p it only calculates $\mathbb{F}_p$ valued intersection numbers of invariant bundles with invariant curves, and, as it happens, this is exactly how one gets the lower bound in (8).

(c) Under the p-adic integrability condition, not only does it work, III.8, but the resulting proof, V.2, it affords of the boundedness of rational or elliptic curves at such primes is formally almost identical to what we've said above about the entire curve case over $\mathbb{C}$, and this is how it should be, *i.e.* in principle there's very little that one can say about Zariski dense entire curves that isn't also true for a Zariski dense set of rational or elliptic curves in mixed characteristic. Baum-Bott is, however, a statement which admits exactly this distinction, and it nails the logical problem posed by the rational curves of (8) on the nose, *i.e.* forget about everything one already knows, then, *in se*, the implication Baum-Bott & (8) (in fact just any lower bound going to ∞ for rational or elliptic curves) $\Rightarrow \mathscr{F}$ is not p-adically integral for $p \gg 0$ holds.

(d) It's in excellent agreement with Miyaoka's proof, [Miy08], of I.3. In general terms, this consists of fixing the curve and making finitely many ACF_0 statements about it and the surface.

Amongst these statements, the only one that doesn't have the form (9) is closure of certain global logarithmic 1-forms whose poles are allowed to depend on the curve. In the general case I.1.(a)- I.2.(a) this strategy may well fail, but for $\mathscr{F}$-invariant curves on bi-disc quotients, it should go through in the presence of p-adic integrability.

Despite these highly attractive properties, it does not follow from I.6 or even (8) that there is a logical necessity for Baum-Bott theory in proving the Green-Griffiths conjecture, [McQ98], for surfaces in the Bogomolov class. It is, however, an absolute logical necessity that some non ACF_0 statements with very similar properties to (c) must intervene. The present scholion is, however, a best possible scenario wherein model completeness in ACF_0 gets us down to the bi-disc case. The practice is, currently, much worse, *i.e.* the classification theorem, [McQ08], and whence all the results of the present article depend on the general Baum-Bott "residue estimate" of [McQ98], or some appropriate variant thereof, [Bru99]. This is much trickier than the (trivial over $\mathbb{C}$) bi-disc case, and nailing a similar "residue estimate", [McQ12], for foliated 3-folds (whence, inter alia, Green-Griffiths for surfaces with $13c_1^2 > 9c_2$) can reasonably be described as difficult.

The original motivation for investigating the variation of (1)-(2) in mixed characteristic was an ingenious (unfortunately, it's current status is we're stuck) idea of Fedor Bogomolov to reduce the study of the moduli of curves on surfaces of general type to the big co-tangent case, and I would certainly not have ventured into this area had he not prompted it. In questions of logic, I am always indebted to Ehud Hrushowski, and, in the particular, to his very informative reply to my query- with an eye to getting more out of (8)- as to whether there was a better way to formulate I.6 rather than my *ad hoc* definition (9). At the risk of exposing my ignorance in such matters, with any misunderstanding evidently being mine, his reply may usefully be noted: there is an elaboration of a language about the "generic curve" in course, but, for the moment, it couldn't be considered usable. Finally the influences of Nick Shepherd-Barron and Torsten Ekedahl, both in the classification theorem and the present article are legion. I never met Ekedahl, but Nick has frequently communicated to me many of his insights. Amongst which, I asked Nick how one proved the "folk theorem" (I had heard a similar thing from Oort) I.5.(b). He replied that it was Ekedahl who had told him, and Ekedahl had said it was just a matter of looking at where the p-curvature vanishes, and applying adjunction. As it happens, it took me a bit of thought to find the seemingly not so obvious adjunction, (35), but in terms of Ekedahl's exact sequence, (48), it's extremely natural, and to the author of the p-closed condition it would have been wholly obvious. Requiescat in pace.

II. Singularities

By a foliation (by curves) on an algebraic space, or indeed champ, X over a locally Noetherian base S we'll mean a rank 1 quotient of $\Omega_{X/S}$, *i.e.* a short exact sequence,

$$(17) \qquad\qquad 0 \to \Omega_{X/\mathscr{F}} \to \Omega_{X/S} \to K_{\mathscr{F}}.I_Z \to 0$$

This definition supposes a certain amount of regularity. If we were working with champs (which won't really be the case) then X should be Deligne-Mumford, otherwise $\Omega_{X/S}$ isn't defined, while to write the quotient as $K_{\mathscr{F}}.I_Z$, where $K_{\mathscr{F}}$ is a bundle and Z the singular locus supposed of co-dimension at least 2 amounts to supposing that the foliation is Gorenstein, *i.e.* given everywhere by a vector field non-vanishing in co-dimension 1.

If we further suppose that S is a field, k, say, of arbitrary characteristic and X is normal irreducible, then we can functorially extend the definitions of Mori theory. Details may be found in [McQ08, I.1-I.2] for surface (which is largely our present interest), and [McQ05, I.6-7], or [MP13, I.iii,III.i] in general. In particular it transpires that log-canonical and Gorenstein is equivalent to non-nilpotence (at closed points of Z of (17)) in the sense of the following

II.1. Revision. *cf.* [Mar81]. *We let A be a complete regular local ring containing a coefficient field, $\overline{k}$, supposed algebraically closed. Next let $\partial \in \mathfrak{m} \operatorname{Der}_{\overline{k}}(A)$. For every $n \in \mathbb{N}$ we have an exact sequence,*

$$(18) \qquad 0 \longrightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^n} \longrightarrow 0 \,.$$

We can consider ∂ as a $\overline{k}$-linear endomorphism, ∂_n, of $A_n = A/\mathfrak{m}^n$ for each n. Consequently ∂_n has a Jordan decomposition $\partial_{S,n} \oplus \partial_{N,n}$ into a semi-simple and nilpotent part. These are compatible with the restriction maps $A_{n+1} \to A_n$, and so on taking limits give a Jordan decomposition $\partial_S \oplus \partial_N$ of ∂.

Observe as an immediate consequence

II.2. Fact. *∂ is semi-simple iff there is a choice of generators $x_i \in \mathfrak{m}$, together with $\lambda_i \in \overline{k}$ such that,*

$$(19) \qquad \partial = \sum_i \lambda_i \, x_i \, \frac{\partial}{\partial x_i} \in \operatorname{Der}_{\overline{k}}(A)$$

Where it's important to emphasise that semi-simplicity in II.2 is to be understood in all of A, rather than just modulo $\mathfrak{m}^2$ since this is its sense in

II.3. Fact. *([MP13, III.i.3]) A foliation in characteristic zero with log-canonical Gorenstein singularities has canonical singularities iff it has no semi-simple points at which, up to scaling, the λ_i of (19) are non-negative integers.*

Describing the nilpotent part is worth the trouble. Notice: $[\partial_S, \partial_N] = 0$, so given ∂_S as above we just compute a basis for fields which commute with it. Putting $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\Lambda \cdot -$, to be the usual inner product, albeit with values in $\overline{k}$, these are easily seen to be, cf. [Mar81]

$$(20) \qquad \begin{aligned} &\text{(a) } x^Q \, x_i \, \frac{\partial}{\partial x_i}, \ \Lambda \cdot Q = 0, \ Q = (q_1, \ldots, q_n), \ q_j \in \mathbb{N} \cup \{0\} \\ &\text{(b) } x^Q \, x_i \, \frac{\partial}{\partial x_i}, \ \Lambda \cdot Q = 0, \ Q = (q_1, \ldots, q_n), \ q_j \in \mathbb{N} \cup \{0\} \text{ for } j \neq i, \ q_i = -1 \end{aligned}$$

where of course $x^Q = x_1^{q_1} \ldots x_n^{q_n}$. As such, following [McQ08, II.1.6]

II.4. Corollary. *Let ∂ be a non-singular derivation of a complete regular local ring A over an algebraically closed field $\overline{k}$ of characteristic $p > 0$ isomorphic to its residue field then there is a choice of coordinates $x, y_1, \ldots, y_n$ in the maximal ideal such that up to multiplication by a unit,*

$$(21) \qquad \partial = \frac{\partial}{\partial x} + \sum_{i=1}^n x^{p-1} \, f_i(x^p, \underline{y}) \, \frac{\partial}{\partial y_i} \,.$$

Proof. We can certainly multiply ∂ by a unit, in such a way that for some $x \in \mathfrak{m}$, $\partial x = 1$. Now consider $\widetilde{\partial} = x\partial$, and its Jordan decomposition $\partial_S \oplus \partial_N$. Trivially $\partial_S = x \frac{\partial}{\partial x}$, in some coordinate system $x, y_1, \ldots, y_n$. Observe that in our formulae for the nilpotent part we must have an exponent of x in the monomial $(x^Q x_i)$ at least 1, since $x \mid \widetilde{\partial}$, whence the claim. $\qquad \square$

This is best possible, and so could reasonably be called the characteristic p Frobenius theorem. One can only do better if,

II.5. Definition. [Eke87] *The foliation is p-closed if for some, and in fact any, local generator ∂ of the foliation the fields ∂^p and ∂ are parallel.*

The special coordinates of the divertimento can be used to 'compute' the p curvature, to wit:

II.6. Fact. *The ideal where $\partial^p \wedge \partial$ vanishes is exactly the ideal cut out by the $f_i(x^p, \underline{y})$'s whence (albeit this doesn't require special coordinates) it is invariant.*

Proof. We have $\partial^p(x) = 0$, while

$$\partial^p(y_i) = \partial^{(p-1)} x^{p-1} f_i = \sum_{a+b=p-1} \frac{\partial^b}{a!(b!)^2}(f_i) \tag{22}$$

and (f_i) is manifestly an invariant ideal. $\qquad\square$

As such, the p-curvature can only vanish if all the $f_i = 0$, so

II.7. Fact. *The following are equivalent for a smoothly foliated connected algebraic space or indeed champ,*

(I) The foliation is p closed.

(II) There exists a closed point ξ such that in the complete local ring $\hat{\mathscr{O}}_{X,\xi}$ there are coordinates $(x, y_1, \ldots, y_n)$, and the foliation has the form,

$$(x, y_1, \ldots, y_n) \mapsto (x^p, y_1, \ldots, y_n) \tag{23}$$

(III) For every point ξ there are coordinates in the complete local ring $\hat{\mathscr{O}}_{X,\xi}$ such that the foliation has the form,

$$(x, y_1, \ldots, y_n) \mapsto (x^p, y_1, \ldots, y_n) \tag{24}$$

Consequently, even if Z is empty, it's far from true that curves invariant by a foliation by curves have to be smooth, and this is the case even if the foliation isn't p closed, For example,

$$\partial = \frac{\partial}{\partial x} + x^{p-1}(y^2 + x^p)\frac{\partial}{\partial y} \tag{25}$$

then the invariant curve $y^2 + x^p$ certainly isn't smooth, even though the foliation isn't p closed, and, indeed, by II.4 the singular locus of the set where the p-curvature vanishes is essentially arbitrary. It's therefore perhaps a little surprising that there are natural criteria whereby the locus of vanishing of the p-curvature is extremely well behaved at the singular points, *e.g.*

II.8. Fact. *Suppose A of II.4 has dimension 2, and that ∂ has 2-invariant smooth transverse branches while being semi-simple modulo $\mathfrak{m}^2$ with the ratio, λ_0, of the eigenvalues not belonging to $\mathbb{F}_p$, then there is a choice of formal coordinates such that, up to rescaling by a constant,*

$$\partial = x\frac{\partial}{\partial x} + \lambda(x^p, y^p)y\frac{\partial}{\partial y}, \quad \lambda(0,0) = \lambda_0 \tag{26}$$

so, in particular, the ideal defined by $\partial^p \wedge \partial$ is just (xy).

Proof. Just apply (20)- by hypothesis we don't have to worry about (b). $\qquad\square$

III. Rational curves on bi-disc quotients

Now let us apply these considerations to find rational curves on bi-disc quotients, X, which are not (a precision that will subsequently be eschewed) finite quotients at a set of positive characteristics of positive (in fact about half) density. We begin with the smooth case, so in characteristic 0, Ω_X splits as

$$\Omega_X = K_{\mathscr{F}} \coprod K_{\mathscr{G}} \tag{27}$$

for 2-integrable foliations $\mathscr{F}$, and $\mathscr{G}$. By [ESBT, 6.2.(iii)] there is a set of primes of positive density where $\mathscr{F}$ isn't p-closed (and, as it happens, at the same primes $\mathscr{G}$ isn't closed either, III.7, but we don't need this for the moment). We work over the reduction modulo p at such a

prime, with say base change to the algebraic closure k for convenience, and, of course $p \gg 0$ to guarantee not just good reduction, but also the splitting (27). The essential fact that we will use in analysing the locus where the p-curvature vanishes is

III.1. Revision. *([SB95] or [McQ08, IV.5.4]) In characteristic zero $K_{\mathscr{F}}$ and $K_{\mathscr{G}}$ are nef. line bundles of Kodaira dimension $-\infty$, so in particular*

$$(28) \qquad\qquad K_{\mathscr{F}}^2 = K_{\mathscr{G}}^2 = 0, \quad 2K_{\mathscr{F}} \cdot K_{\mathscr{G}} = \mathrm{c}_2 > 0, \quad K_X, \text{ is ample.}$$

So that before getting underway let's make

III.2. Warning. Over our current choice of k, $K_{\mathscr{F}} \otimes k$ and $K_{\mathscr{G}} \otimes k$ will be big, whence very far from nef., although (manifestly) (28) will continue to hold.

The first step is to write the locus of vanishing of the p-curvature (of $\mathscr{F}$) as a sum of irreducible divisors

$$(29) \qquad\qquad P := pK_{\mathscr{F}} - K_{\mathscr{G}} = \sum_i n_i C_i$$

so that by II.6 we know the Cartier divisors $\mathscr{O}_X(-n_i C_i)$ are $\mathscr{F}$-invariant. We wish, however, to know that the C_i themselves are $\mathscr{F}$-invariant. This will follow if we know that $p \nmid n_i$, which follows a fortiori from

III.3. Fact. *Let N_p be the maximum of the n_i's in (29) then $\limsup_p p^{-1} N_p = 0$.*

Proof. Suppose to the contrary that $\limsup_p p^{-1} N_p = \varepsilon > 0$, then for some infinite set of primes p, there is a curve D_p in characteristic p such that

$$(30) \qquad\qquad K_{\mathscr{F}} - \varepsilon D_p \geq 0 \in \mathrm{NE}_1(X \otimes k) \otimes \mathbb{Q}$$

Consequently, the degrees of the D_p are bounded independently of p, so they must belong to finitely many components of the Hilbert-scheme. Sub-sequencing as necessary, we can suppose that this component is the same for all p, and whence it's non-empty in characteristic zero. As such (30) may be supposed to hold for $D_p = D_0$ independent of p from which $K_{\mathscr{F}} - \varepsilon D_0$ is pseudo-effective in characteristic zero. However, cf. [McQ08, IV.5.7], by Zariski decomposition and Hodge-index in characteristic zero this forces $K_{\mathscr{F}}$ to be numerically effective (whence effective since $q(X) = 0$) in characteristic zero contradicting III.1. $\square$

As such for $\nu_i : \tilde{C}_i \to C_i$ the normalisations of the curves in (29); $-\chi_i = 2g_i - 2$ their geometric Euler characteristics; and r_i the ramification of the ν_i we obtain

$$(31) \qquad\qquad K_{\mathscr{F}} \cdot C_i = -\chi_i - r_i$$

Again, before proceeding, let us make another

III.4. Warning. It is immediate, from (28-29) that there are curves with $K_{\mathscr{F}} \cdot C_i < 0$ and a moments thought even shows that the tangent sheaf of C_i is even a bundle isomorphic to $K_{\mathscr{F}}^{\vee} |_{C_i}$. This does not, however, imply that the C_i are rational, since tangent vectors do not necessarily lift to the normalisation in positive characteristic. A perfectly good example is provided by (25), which has the further curious property that the derivative $\nu_i' : \tilde{C}_i \to P_1 := \mathbb{P}(\Omega_X^1)$, and indeed all subsequent derivatives, $\nu_i^{(m)}$, to all higher jet spaces, P_m, has exactly the same ramification as ν_i. In particular if L_m were the tautological bundle on such spaces, then

$$(32) \qquad\qquad L_m \cdot_{\nu_i^{(m)}} \tilde{C}_i \text{ (here } = K_{\mathscr{F}} \cdot C_i) < 0, \quad \text{for all } m$$

which for many proof theory purposes is just as interesting as C_i rational. Nevertheless, in characteristic p, the *pseudo rationality* condition (32) is not equivalent to rationality.

Complementary to (31) we have the tangency divisor of C_i with $\mathscr{G}$, *i.e.* the composition of

$$(33) \qquad \mathscr{O}_{C_i}(-C_i) \to \Omega_X|_{C_i} \to K_{\mathscr{G}}|_{C_i}$$

which is necessarily non-zero since k is algebraically closed- albeit perfect, which is natural here, since everything is arithmetic would do. Consequently if t_i is its degree then

$$(34) \qquad t_i = (K_{\mathscr{G}} + C_i) \cdot C_i \geq 0$$

Putting (31) and (34) together, we therefore obtain

$$(35) \qquad (pK_{\mathscr{F}} + K_{\mathscr{G}}) \cdot C_i = -p\chi_i - C_i^2 + (t_i - pr_i)$$

Now observe that by (28) $(pK_{\mathscr{F}} + K_{\mathscr{G}}) \cdot P = 0$, so there is at least one curve such that (34) is not positive, and we assert

III.5. Claim. *If, for C_i in the support of P, $(pK_{\mathscr{F}} + K_{\mathscr{G}}) \cdot C_i \leq 0$ then C_i is rational.*

Proof. In the first place, we show $C_i^2 < 0$, since, otherwise $P \cdot C_i \geq 0$, and whence

$$(36) \qquad K_{\mathscr{F}} \cdot C_i \geq (p+1)^{-1} K_X \cdot C_i > 0$$

so that the left hand side of (35) is at least $2p(p+1)^{-1} K_X \cdot C_i > 0$, which is nonsense. Now suppose that C_i isn't rational, *i.e.* $\chi_i \leq 0$ then we must have

$$(37) \qquad pr_i > t_i$$

To exclude this follows a fortiori if it doesn't happen locally at any branch through a singular point of C_i, so to this end we take a generator, ∂, of $\mathscr{F}$ in the special coordinates, and notation, of II.4 with A the completion of $\mathscr{O}_X$ in the said singularity, so that our branch is defined by a prime factor $g(x^p, y)$ of $f(x^p, y)$ being equal to zero, while the local contribution to the tangency divisor of the branch is the order of of $g_y(x^p, y)$ along it. Now consider the normalisations, $\nu : \Delta \to \Delta^2$, $\nu' : \Delta \to \Delta^2$ of the irreducible plane curves $g(x^p, y) = 0$, and $g(z, y) = 0$ where $\Delta := \mathrm{Spf} k[[t]]$. These fit into a diagram

$$(38) \qquad \begin{array}{ccc} \Delta & \xrightarrow{t \mapsto (\bullet t^m, \bullet t^n)} & \Delta^2 \\ {\scriptstyle t \mapsto \bullet t^d} \downarrow & & \downarrow {\scriptstyle (x,y) \mapsto (x^p, y)} \\ \Delta & \xrightarrow{s \mapsto (\bullet s^{m'}, \bullet s^{n'})} & \Delta^2 \end{array}$$

where (m, n), respectively (m', n') are relatively prime positive integers, and $\bullet$ denotes some unit. As such, we obtain

$$(39) \qquad pm = dm', \, n = dn'$$

and there are two cases to consider. In the first place suppose that $p|d$, then $m = m'$, $n = pn'$, and the ramification is at most $m - 1$. If, however, we write $g(x^p, y)$ as a power series in monomials $x^a y^b$, then we must have $am + bn \geq mn$, so the tangency divisor is at least

$$(40) \qquad (m-1)n \geq p(m-1)$$

which contradicts (36). In the other case, $m' = pm$, $n = n'$, so the ramification is at most $n - 1$, while if we write $g(z, y)$ as a power series in monomials $z^a y^b$, then we must have $am' + bn' \geq m'n'$, and whence the tangency is at least

$$(41) \qquad n(mp - 1)$$

From which (36) implies $n - 1 > n(m - 1/p)$, so $m = 1$, and the absurdity that the branch is unramified. $\qquad \square$

It therefore remains to consider the general bi-disc quotient. In characteristic zero, after re-solving the cusps, these have at worst isolated quotient singularities. In particular they are smooth champs in a particularly simple way, *i.e.* they admit a finite étale covering by a space, and since our interest is to find rational curves on such champs, there is absolutely no loss of generality in supposing that there are no quotient singularities. As such, we can work on a smooth surface X with simple normal crossing boundary E, each connected component of which is a polygon of rational curves, and whence (27) generalises to a splitting

$$\Omega_X(\log E) = K_{\mathscr{F}} \coprod K_{\mathscr{G}} \tag{42}$$

Remaining in characteristic zero for the moment, the foliation singularities, whether of $\mathscr{F}$ or $\mathscr{G}$, are identical with the singularities of E (which, b.t.w., is invariant by both foliations) and can be described [McQ08, IV.2.2] in suitable formal coordinates by a generator

$$\partial = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} \tag{43}$$

where λ is a real quadratic irrationality, and $xy = 0$ is the local equation for E. Consequently if we work modulo sufficiently large primes such that X, E have good reduction; (42) continues to hold; and $(\mathbb{F}_p(\lambda) : \mathbb{F}_p) = 2$, then II.8 applies. In particular, therefore, if we understand the locus of vanishing of the p-curvature logarithmically, then it does not contain any component of E, *i.e.* (29) holds with no C_i in the support of the cusps. On the other hand, [McQ08, IV.2.2], $K_{\mathscr{F}} \cdot E' = K_{\mathscr{G}} \cdot E' = 0$ for every E' in the support of E, so P is disjoint from E, while $K_X + E$ descends to an ample divisor on the contraction of the cusps, whence everything is exactly as in the smooth case, and we deduce

III.6. **Proposition.** *Let (X, E) be a model over the integers of a smooth algebraic champ obtained by resolving the cusps (if any) of a bi-disc quotient no finite étale cover of which is a product of curves, then for $\mathscr{F}$ either of the natural foliations arising from the bi-disc structure there is a set, $\mathscr{P}$, of primes of positive density such that for k an algebraically closed field of characteristic $p \in \mathscr{P}$ there is a $\mathscr{F}$-invariant rational curve $f_p : \mathbb{P}^1_k \to X\backslash E$, and constants c, C, depending on X, H, where the latter is an ample divisor, such that*

$$pc \leq H \cdot_{f_p} \mathbb{P}^1_k \leq pC \tag{44}$$

Proof. Suppose first that $\liminf_{p \in \mathscr{P}} H \cdot_{f_p} \mathbb{P}^1_k$ is finite, then there is an irreducible component of the Hilbert scheme (say of a finite projective étale cover of X to avoid some technicalities) which contains a rational 1-cycle missing E for infinitely many primes, whence a non-trivial rational curve $f_{\mathbb{C}} : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}\backslash E$, which is nonsense. On the other hand, *cf.* (55) & (60), the residue theory of Baum-Bott, [BB72], is valid in the De-Rham co-homology over k, and f misses E, so $K_{\mathscr{G}} \cdot_{f_p} \mathbb{P}^1_k$ vanishes modulo p, while $K_{\mathscr{F}} \cdot_{f_p} \mathbb{P}^1_k$ is at most -2. Consequently, since $K_{\mathscr{F}} + K_{\mathscr{G}}$ is ample on contracting the cusps, we must have $K_{\mathscr{G}} \cdot_{f_p} \mathbb{P}^1_k = pd$, $d \in \mathbb{N}$. Supposing, therefore, $p \gg 0$ to avoid bad reduction, we can take c in (44) anything such that $H - cK_{\mathscr{G}}$ is ample, while by (29), C is at worst $K_{\mathscr{F}} \cdot H$. $\square$

Before proceeding to analyse the complimentary primes let us make

III.7. **Remark.** If some étale cover of our bi-disc quotient were a product of curves, then, of course, the foliation is p-closed for all p and the divisor (29) never exists, nor, are there any rational curves. Similarly, in the hypothesis of III.6, we can- [ESBT, remark, pg.23] when there are no cusps, and [McQ08, II.2.3] otherwise- take, for $p \gg 0$, the complement, $\mathscr{P}'$, to $\mathscr{P}$ to be primes where both $\mathscr{F}$ and $\mathscr{G}$ are p-closed. The decomposition $\mathscr{P} \coprod \mathscr{P}'$ of sufficiently large primes can be naturally identified with those, $\mathscr{P}$, which are inert, respectively, $\mathscr{P}'$, split in some real quadratic extension (depending naturally on X) $K/\mathbb{Q}$. It's both curious, and typical of this sort of problem, that the relation of K to the geometry is more obvious when there are singularities, *i.e.* in the notation of (43), it's just $\mathbb{Q}(\lambda)$.

As to what happens at $\mathscr{P}'$ the following is surprisingly, *cf.* III.9, precise

III.8. **Proposition.** *Let (X, E) be as in III.6 with (up to excluding a finite set of primes of bad reduction) $\mathscr{P}'$ the set of primes complimentary to $\mathscr{P}$ then for $f : C/k \to X$ any separable map from a smooth curve over k of characteristic $p \in \mathscr{P}'$ which doesn't factor through E there is a $m = m(f) \in \mathbb{N} \cup \{0\}$ such that*

$$(45) \qquad (K_X + E) \cdot_f C \le -(1 + p^m)\chi(C \backslash f^{-1}(E))$$

where for D a divisor on C, the Euler characteristic $\chi(C \backslash D)$ is $\chi(C)$ minus the number, D_{red} of points in D counted without multiplicity. In particular, therefore, for any such curve, $\chi(C \backslash f^{-1}(E)) < 0$, and over k, $X \backslash E$ is "algebraically hyperbolic".

Proof. By hypothesis we have a non-trivial map

$$(46) \qquad f^* \Omega_X(\log E) \to \omega_C(\log f^{-1}(E))$$

and wild ramification is a help rather than a hindrance, *i.e.* the degree of the right hand side in (46) is at most $-\chi(C \backslash f^{-1}(E))$. Consequently if both the induced maps from $f^* K_{\mathscr{G}}$ and $f^* K_{\mathscr{F}}$ arising from combining (42) with (46) are non-trivial, then we get (45) with $m = 0$, so without loss of generality f is $\mathscr{F}$, but not $\mathscr{G}$, invariant.

This much is characteristic independent, but by the definition of $\mathscr{P}'$, and [ESBT, remark.(ii), pg.23] in the smooth case, respectively II.8 should there be cusps, both our natural foliations are p-integrable. Even better, [ESBT, remark.(iii), pg.23], they are actually p-adically integrable, so in particular $\mathscr{F}$ is a height m-foliation in the sense of [Eke87] for all $m \in \mathbb{N}$, *i.e.* one can replace p by p^m in II.7. This means that there are compatible factorisations,

$$(47) \qquad X \xrightarrow{\rho_m} Y_m \xrightarrow{\sigma_m} X^{(m)}$$

of the mth powers, F_X^m, of the geometric Frobenius together with natural exact sequences

$$(48) \qquad 0 \to K_{\mathscr{F}}^{p^m} = \rho_m^* \sigma_m^* K_{\mathscr{F}}^{(m)} \to \rho_m^* \Omega_{Y_m}(\log E_m) \to \Omega_X(\log E) \to K_{\mathscr{F}} \to 0$$

where E_m is the divisor induced by the cusps on Y_m which although it will be singular if the cusps are non-empty has the structure of a (wild) Deligne-Mumford champ which we suppose in defining $\rho_m^* \Omega_{Y_m}(\log E_m)$. Furthermore, the height $(m+1)$-structure defines a section of the left hand side of (48) so, in the sense of smooth champs should there be cusps, we again have a splitting

$$(49) \qquad \Omega_{Y_m}(\log E_m) = \sigma_m^* K_{\mathscr{F}}^{(m)} \coprod K_{\mathscr{G}_m}, \quad \rho_m^* K_{\mathscr{G}_m} = K_{\mathscr{G}}$$

Now suppose inductively (we already have the case $m = 1$) that $\rho_m f$ admits a factorisation

$$(50) \qquad \begin{array}{ccc} C & \xrightarrow{\quad f \quad} & X \\ {\scriptstyle F_C^m} \downarrow & & \downarrow {\scriptstyle \rho_m} \\ C^{(m)} & \xrightarrow{\quad f_m \quad} & Y_m \end{array}$$

where F_C is Frobenius, and f_m is separable. If, however, this were to fail for $\rho_{m+1} f$, then the composition

$$(51) \qquad f_m^* K_{\mathscr{G}_m} \to f_m^* \Omega_{Y_m}(\log E_m) \to \omega_{C^{(m)}}(\log f_m^{-1}(E_m))$$

implied by (49) and (46) is non-trivial, so by (49)-(50),

$$(52) \qquad K_{\mathscr{G}} \cdot_f C \le -p^m \chi(C \backslash f^{-1}(E))$$

and, by the definition of $\mathscr{F}$-invariance, this already holds with $m = 0$ for $K_{\mathscr{F}}$ whence (45).

As such, there remains the possibility that (50) holds for all m. Here, if it were true, that f could be lifted to characteristic zero, then it would follow from the residue theory of Baum-Bott, [BB72], that the degree of $K_{\mathscr{G}}$ along f would be zero. Now, while f may well fail to be liftable,

it's always "liftable in co-homology", and a p-adic variant of [BB72] works. More precisely let W be the Witt vectors over k, with K its quotient field, and $\mathrm{H}_{\mathrm{rig}}$ the rigid co-homology of Monsky-Washnitzer, [LS07], which is, canonically, isomorphic to the algebraic De-Rham co-homology H_{DR} with values in K for any proper k-variety admitting a lifting over W. By hypothesis we already have a lifting over W of X, while every curve is liftable by [Gro63, Exposé III, 7.4], so working over W, and denoting the special fibre by the subscript k, we have maps

$$(53) \qquad \mathrm{H}^2_{\mathrm{DR}}(X_W) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{rig}}(X_k) \xrightarrow{f^*_{k,\mathrm{rig}}} \mathrm{H}^2_{\mathrm{rig}}(C_k) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{DR}}(C_W)$$

Now the rigid co-homology of an affine is the De-Rham cohomology of the weak completion (*i.e.* some p-adic convergence condition which here is always satisfied) so we can calculate $f^*_{k,\mathrm{rig}}$ via the local global spectral sequence of any affine covering $U = \coprod_a U_\alpha \to X$, and a choice of local liftings of f_α of f_k. On the other hand, we have the Hodge spectral sequence for $\mathrm{H}_{\mathrm{DR}}(X)$, so, by the aforesaid local considerations, for the usual degree reasons, $f^*_{k,\mathrm{rig}}$ is zero on $\mathrm{E}^{2,0}_{\mathrm{Hodge}}\mathrm{H}^2_{\mathrm{DR}}(X)$, and (quite generally for any curve on any surface) we get a map

$$(54) \qquad \mathrm{H}^1(X_W, \Omega^1_{X/W}) \otimes K \xrightarrow{f^*_{k,\mathrm{rig}}} \mathrm{H}^1(C_W, \Omega^1_{C/W})$$

Now consider first the case where there are no cusps, then $d : \Omega^1_{X/W} \to \Omega^2_{X/W}$ restricted to $K_{\mathscr{G}}$ yields a "leafwise" connection,

$$(55) \qquad \nabla : K_{\mathscr{G}} \to K_{\mathscr{G}} \otimes K_{\mathscr{F}}$$

so the image in $\mathrm{H}^1(X_W, K_{\mathscr{F}})$ of the chern class, $\mathrm{c}_1(K_{\mathscr{G}}) \in \mathrm{H}^1(X_W, \Omega^1_{X/W})$ is zero, or equivalently, it belongs to $\mathrm{H}^1(X_W, K_{\mathscr{G}})$. Again, this much is quite general, *i.e.* it is independent of whether p belongs to $\mathscr{P}$ or $\mathscr{P}'$, and the co-cycle in $\mathrm{H}^1(X_W, K_{\mathscr{G}})$ affording $\mathrm{c}_1(K_{\mathscr{G}})$ can be taken in the Zariski topology, while (still independent of p) in terms of the covering spectral sequence $\mathrm{E}^{i,j}_{\mathrm{cov}}$, if $g_{\alpha\beta}$ are the transition functions for local generators ω_α of $K_{\mathscr{G}}$ then

$$(56) \quad \mathrm{c}_1(K_{\mathscr{G}}) = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} = \nu_{\alpha\beta} + \tau_\alpha - \tau_\beta \in \check{\mathrm{H}}^1(U, \mathrm{H}^1_{\mathrm{DR}}), \quad \nu_{\alpha\beta} \in \mathrm{H}^0(U_{\alpha\beta}, K_{\mathscr{G}}), \tau_\alpha \in \mathrm{H}^0(U_\alpha, \Omega^1_X)$$

where $d\omega_\alpha = \omega_\alpha \wedge \tau_\alpha$, and which (by the functoriality, a.k.a. main theorem, of rigid co-homology) on choosing local liftings f_α of f_k, restricts to

$$(57) \qquad f^*_{k,\mathrm{rig}}\mathrm{c}_1(K_{\mathscr{G}}) = f^*_\alpha \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} = f^*_\beta \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} \in \mathrm{E}^{(1,1)}_{\mathrm{cov}} = \mathrm{H}^1(C_W, \Omega^1_{C/W}) \otimes K$$

for the covering $f^{-1}(U) \to C/k$. Now, by II.7, albeit with p^m instead of p, we can (depending on m) find an étale covering $U \to X$ such that the ω_α are closed modulo p^m, or, equivalently the τ_α vanish modulo p^m. Similarly, since (50) holds for all m, we can choose the local liftings f_α such that the restriction of any local section of $K_{\mathscr{G}}$ vanishes modulo p^m, so by (56), the restriction (57) must be zero modulo p^m. Consequently, since this must hold for all m,

$$(58) \qquad K_X \cdot_f C = K_{\mathscr{F}} \cdot_f C \leq -\chi(C)$$

while K_X is ample, so $\chi(C) < 0$, and, (45) holds a fortiori with $m = 0$. To understand the case with cusps, E, in the notation of (17), and again quite generally, exterior derivation always yields a leafwise connection

$$(59) \qquad \nabla : \Omega_{X/\mathscr{F}} \to \Omega_{X/\mathscr{F}} \otimes K_{\mathscr{F}}$$

which is bad since we'd have to take the τ_α of (56) to be $K_{\mathscr{F}}$ valued rather than honest 1-forms. If, however, one has canonical singularities, without nilpotent structure, and an invariant simple normal crossing divisor E with 2 branches through each singularity, then it's always true that one has a connection

$$(60) \qquad \nabla : \Omega_{X/\mathscr{F}}(E) \to \Omega_{X/\mathscr{F}} \otimes K_{\mathscr{F}} I_Z$$

while, in our immediate context, $K_{\mathcal{G}} \xrightarrow{\sim} \Omega_{X/\mathcal{F}}(E)$, so (56) holds with the τ_α as stated, and better $\nu_{\alpha\beta} \in \mathrm{H}^0(U_{\alpha\beta}, \Omega_{X/\mathcal{F}})$. Plainly, everything works as before, except that there could be a residue about Z, which in the context of the usual Baum-Bott theory over $\mathbb{C}$ is already zero in the presence of (60). Here, however, the pull back, (57), in rigid co-homology is more fastidious, and there could be a residue modulo p^m for some m, so we still need to know that $K_{\mathcal{G}}$ is locally generated by forms ω_α which are closed modulo p^m for every m. Should $m = 1$, then, [McQ08, II.1.3], the Jordan decomposition of II.1 and p-closure combine to give what we need, $i.e.$ in the notation of II.8, albeit with $\lambda_0 \in \mathbb{F}_p$, the requisite local generator of $K_{\mathcal{G}}$ is

$$(61) \qquad \frac{dy}{y} - \lambda \frac{dx}{x} \quad \mathrm{mod}\,(p);\ \lambda \in \mathbb{Z}_p$$

and what we need in general is that there is a semi-simple local generator of $\mathcal{F}$ modulo p^m for every m. Proceeding by induction on m, with $m = 1$ just being a re-statement of (61), and taking a generator normalised by $\partial(x) = 0$ for $x = 0$ a local branch of E, linear algebra reveals that if ∂ is semi-simple modulo p^m, then we still have a Jordan decomposition $\partial_S + \partial_N$ modulo p^{m+1}, with the nilpotent part exactly as in (20).(a), $i.e.$ for $xy = 0$ the local equation of E,

$$(62) \qquad \partial_N = \sum_{i+j\lambda = 0\,(p^{m+1})} (\varepsilon_{ij} x^i y^j) y \frac{\partial}{\partial y}, \quad \varepsilon_{ij} = 0\,(p^m)$$

Now to obtain a local generator, ∂_m, of the foliation at the corresponding singular point of the resulting foliation on Y_m of (47) from ∂_{m-1} one applies the difference operator,

$$(63) \qquad \delta(\partial_{m-1}) := p^{-1}(\partial_{m-1}^p - \partial_{m-1})$$

by first working modulo p^2, then reducing modulo p. Better still the complete local ring of Y_m is easily expressed in the Jordan coordinates, $i.e.$

$$(64) \qquad k[[x^i y^j]], \quad i + j\lambda = 0\,(p^m)$$

and we deduce that the Jordan decomposition of ∂_m is

$$(65) \qquad \delta^m(\partial_S) + p^{-m}\partial_N$$

Consequently, [McQ08, II.1.3] applies again to deduce that $\partial_N = 0\,(p^{m+1})$, and whence

$$(66) \qquad (K_X + E) \cdot_f C = K_{\mathcal{F}} \cdot_f C \le -\chi(C \backslash f^{-1}(E))$$

which completes the proof since $K_X + E$ is ample on contracting E, so, $\chi(C \backslash f^{-1}(E)) < 0$. $\quad\square$

Having achieved our goal let us observe that III.8 is optimal

III.9. **Remark.** Consider first the case where $X = X_1 \times X_2$ is a product of smooth geometrically irreducible hyperbolic curves, and everything is defined over $\mathbb{F}_p$ to avoid some technicalities. Then for any map $f : C/\mathbb{F}_p \to X$ from another smooth geometrically irreducible curve, if both projections, f_i, are separable, then by Riemann-Hurwitz we get (45) with $m = 0$. The condition of III.8, however, is only that f is separable, so there's nothing to stop us replacing $f = f_1 \times f_2$ by $g_m := f_1 \times F^m f_2$ for as large a multiple, F^m, of Frobenius as we like. As such $K_X \cdot_{g_m} C$ grows linearly in p^m, while $\chi(C)$ is constant, so the appearance of p^m in (45) is unavoidable and *separable curves of bounded genus on X are not bounded in moduli*. Similarly, if one toys with the examples in [ACLG12], exactly the same mechanism should yield examples of products of curves where

$$(67) \qquad K_X \cdot_{f_i} C_i \ge -(1-\varepsilon)(1 + p^m)\chi(C_i)$$

for infinitely many separable curves $f_i : C_i \to X$. As such, if, for any bi-disc quotient, X, be it a product of curves or not, were we to define $\mathscr{P}_X$, $\mathscr{P}_X'$ as primes where the natural foliations aren't, respectively are, p-closed then III.6, (which, III.7, is void for products of curves) and III.8 (just apply Riemann-Hurwitz and separable/inseparable factorisation) hold

exactly as stated. In particular, (45) is in no way the result of non-classical behaviour of bi-disc quotients at $p \in \mathscr{P}'_X$- indeed the overwhelming evidence suggests that one ought to have p-adic uniformisation at such primes, while from the proof we see that the mechanism giving rise to the factor p^m in *op. cit.* is exactly the same as that for products of curves. Similarly, to see that the factor p^m is necessary in (45) one can exploit, at least when there are no cusps, that the canonical model of Y_m occurring in (47) behaves, [Eke88], exactly as in characteristic zero. Specifically, the 5-canonical map is very ample, while Y_m has exactly the same chern numbers as X, so there is a very ample bundle $H = 5K_{Y_m}$ whose generic member is a smooth curve $C_m \hookrightarrow Y_m$ with $\chi(C_m)$ independent of m. The pull-back, $\rho_m^* C_m$, is, generically, an irreducible curve C' whose normalisation, $f : C \to C'$, satisfies (50). As such the degree of f grows like p^m, while $\chi(C)$ remains bounded. Again, however, in all likelihood, the examples in [ACLG12] can be tweaked to give to give the lower bound (67) on the nose, *i.e.* $\varepsilon = 0$ up to the addition of a constant $o(p^m)$.

IV. Refined tautology

Again, let X be an algebraic space or Deligne-Mumford champ over a locally Noetherian base S. For $\mathrm{Spec}(k) \hookrightarrow S$ a closed (of any characteristic) point, and C/k a proper smooth curve (or indeed proper 1 dimensional smooth Deligne-Mumford champ), we consider a separable map $f : C \to X$. As such if $P := \mathbb{P}(\Omega^1_{X/S})$, then by base change $\mathbb{P}(\Omega_{X/k}) = P \otimes_S k$, to which, by the definition of separability, there is a derived curve, $f' : C \to \mathbb{P}(\Omega_{X/k})$, admitting the following

IV.1. **Tautology.** *Let L be the tautological bundle on P, and $\chi_C = 2 - 2g(C)$ the geometric Euler-characteristic of the curve, then*

$$(68) \qquad\qquad L \cdot_{f'} C = -\chi_C - \mathrm{Ram}_f \leq -\chi_C$$

In the case that $f : C \to X$ is invariant by a foliation by curves $\mathscr{F}$, with X/S smooth to fix ideas, this can, in the notation of (17), be re-written as follows: outside of Z, the foliation defines a section of $\mathbb{P}(\Omega_{X/S}) \to X$, whose closure over Z is the blow up $\pi : \widetilde{X} \to X$ in Z, including any implied nilpotent structure. As such there is an exceptional divisor E, $L \mid_{\widetilde{X}} = \pi^* K_{\mathscr{F}}(-E)$, and, provided that f doesn't factor through Z, the intersection $E \cdot_{f'} C$ can be identified with the Segre class $\mathrm{s}_Z(f)$, so the first part of the tautology (68) becomes,

$$(69) \qquad\qquad K_{\mathscr{F}} \cdot_f C = -\chi_C - \mathrm{Ram}_f + \mathrm{s}_Z(f)$$

We'd like to refine this by "removing" the Segre class term. This can be done for S of finite type over $\mathbb{Z}$, and $(X, \mathscr{F}) \to S$ a family of foliations by curves with canonical singularities over each generic point of S- otherwise it's false, but we don't need that. The precise statement is,

IV.2. **Fact.** *Let $S/\mathbb{Z}$ be of finite type, with $(X, \mathscr{F}) \to S$ a family of foliations by curves on a proper (over S) Deligne-Mumford champ with canonical $\mathbb{Q}$-Gorenstein singularities over each generic point of S, then for every $\epsilon > 0$ there is a closed, nowhere dense, sub-champ Z_ϵ of X such that every curve $f : C/k \to X$ over every closed point $\mathrm{Spec}(k) \hookrightarrow S$ invariant by $\mathscr{F} \otimes_S k$ which doesn't factor through Z_ϵ satisfies,*

$$(70) \qquad\qquad K_{\mathscr{F}} \cdot_f C \leq -\chi_C + \epsilon H \cdot_f C$$

where H is a line bundle on the moduli of X which is big, relative to S, over every generic point of the same.

This is stated in [McQ05] V.6.1 for S a field of characteristic zero, wherein the situation even involves invariant discs, which don't have so much sense in mixed characteristic. The proof though, which is all of op. cit. §V, continues to have perfect sense, and works in the generality stated. With respect to the immediate interests of this article, we only need the case of X/S a smooth geometrically connected family of projective surfaces, and since the theorem is stable

under base change and throwing away nowhere dense closed sets (precisions which may well be subsequently eschewed) we can further suppose that each singularity is a section. Plainly the difficulty is to estimate $s_Z(f)$. This is, however, subordinate to a local question: how does a local invariant curve meet Z, so we can suppose everything complete in some connected component of Z. Now, as it happens, the pure characteristic 0 case isn't much easier than that of mixed characteristic. Nevertheless, the latter is a little bit more subtle, so we confine ourselves to it, while supposing, for ease of exposition, that S is an open subset of the spectrum of the ring of integers of a number field, K. In particular, after appropriate localisation and without loss of generality, we can suppose that the completion in Z is an affinoid $\mathrm{Spf}(B)$ with trace S on which the foliation is defined by an everywhere singular, but non-vanishing in co-dimension 1, vector field ∂.

To fix ideas, let's begin with the easy case, *i.e.* a so called saddle node over K, equivalently the semi-simple part (over K) has only 1 non-zero eigenvalue. As such, over K, one can refine (20) to find formal coordinates x, y in $B_{\{K\}}$ (where $\{\}$ is formal localisation, which, [McQ02], is not to be confused with localisation) such that

$$(71) \qquad \partial = x\frac{\partial}{\partial x} + \frac{y^{r+1}}{1+\nu y^r}\frac{\partial}{\partial y}, \quad \nu \in K,\, r \in \mathbb{N}$$

Now, while there is no comparably simple formula in all of B, observe that IV.2 is stable under blowing up in the singularities, which we can suppose done a priori so we have an exceptional divisor $E \supset Z$ which, without loss of generality, in (71) is the plane $y = 0$. Similarly, straightforward linear algebra shows that the plane $x = 0$ is a well defined $\mathscr{F}$-invariant formal sub-scheme in all of B, so over B we can suppose that our generator has the form

$$(72) \qquad \partial = x\frac{\partial}{\partial x} + y^{r+1}b(y,x)\frac{\partial}{\partial y}, \quad b(0,0) = 1$$

With this is mind, we next consider what happens at $s \in S$ of characteristic p, *i.e.* reduce modulo $k(s)$, and complete $Z \otimes k(s)$ to get a new local ring A_s in which II.1 and (20) are valid. Consequently, for a change of coordinates of the form $(x,y) \mapsto (x, z = yu_s(x,y))$ where u_s is a unit in A_s, we can improve (72) to a Jordan decomposition

$$(73) \qquad \partial = x\frac{\partial}{\partial x} + z^{r+1}b_s(z,x^p)\frac{\partial}{\partial z}, \quad b_s(0,0) = 1,\, b_s \in A_s$$

Plainly, this depends on s, but it has the manifest advantage that if $f = (x(t), z(t)) : \Delta_s \to \mathrm{Spf}(A_s)$ is a $\mathscr{F}$-invariant map from the formal disc over $\bar{k}(s)$ then it must be invariant by both the semi-simple and nilpotent parts. As such if $x(t) \neq 0$ then, by invariance under the semi-simple part, $\dot{x} \neq 0$ and $z = z(t^p)$. Consequently for ord the order of vanishing under f^* at the closed point of Δ,

$$(74) \qquad s_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}} \leq \mathrm{ord}(x) - \mathrm{ord}(\dot{x}) \leq 1 \leq \frac{1}{p}(E \cdot_f \Delta)_{\mathrm{loc}}$$

where the Segre class and intersection number are understood locally in the obvious way. In particular, for ε given as in IV.2 and $p \gg \varepsilon^{-1}$ we certainly have the kind of bound that we require to deduce (70) from (69). To deal with the alternative possibility that $x(t)$ is identically zero: observe that for any $d \in \mathbb{N}$ this forces

$$(75) \qquad s_Z(f)_{\mathrm{loc}} \leq \frac{n+1}{d}s_{Z_d}(f)_{\mathrm{loc}}$$

for Z_d the subscheme defined in all of $\mathrm{Spf}(B)$ by the ideal (x, y^d) of (72). Now over the generic point Z_d has length d, while the number of sections of a multiple mH of an ample bundle grows like m^2, so throwing away a nowhere dense (dependent on d) Zariski closed subset of X, we

have the global inequality

$$(76) \qquad \frac{n+1}{d} \mathrm{s}_{Z_d}(f) \ll \frac{1}{\sqrt{d}} H \cdot_f C$$

which is again an appropriate bound for $d \gg \varepsilon^{-2}$.

This reduces us to singularities, Z, whose semi-simple part over the generic point has 2 non-zero eigenvalues. To avoid some pointless technicalities we can (since it doesn't change $K_{\mathscr{F}}$) suppose that the singularities over the generic point are reduced in the sense of Seidenberg, *i.e.* either the previous case of nodes, or the ratio of the eigenvalues is not in $\mathbb{Q}_{>0}$. The Seidenberg property is stable under blowing up, so again, for convenience, we can suppose that there is an exceptional divisor $E \supset Z$, and whence the semi-simple part over K has the form

$$(77) \qquad y\frac{\partial}{\partial y} + \lambda x \frac{\partial}{\partial x}$$

for $y = 0$ a local equation for E, and, after base change if necessary, $\lambda \in K \backslash \mathbb{Q}_{>0}$. The first of a couple of additional subtleties is that unlike (72) the invariant branch $x = 0$ of $\mathscr{F}$ which exists (just apply (20) and the Seidenberg condition) in the completion at the generic point, may fail to exist in the completion $\mathrm{Spf}(B)$ over all of S. More precisely, for $d \in \mathbb{N}$ some large integer to be chosen, linear algebra, *cf.* (20), reveals that the obstruction to finding a coordinate hypersurface $x = 0$ other than the exceptional divisor such that

$$(78) \qquad \partial(x) \in (x) + I_Z^{d+1}$$

occurs at points $s \in S$ such that

$$(79) \qquad \lambda \in \{1, \dots, d\} \bmod \mathfrak{m}(s)$$

Thanks to our Seidenberg hypothesis this only occurs at finitely many closed points- indeed their residue characteristics, p, must even satisfy $p \ll d$. Such quantification isn't so important however, since $d \gg \varepsilon^{-2}$ will be fixed once we have global sections of $H^{\otimes m}(x, y^d)$ over the generic point for m of size commensurable to $\sqrt{d}$, so we just discard fibres where (79) could happen.

Now the easy sub-case occurs at $s \in S$ where λ isn't in the prime sub-field of $k(s)$. By II.8, $\mathscr{F}_s$ cannot be p-closed, while, conversely, the locus where $\partial^p \wedge \partial$ is zero contains all invariant curves, so our invariant curve must be one of the coordinate hypersurfaces in (26). One of these is the exceptional divisor, so we can ignore this possibility, while the other agrees with $x = 0$ of (78) to order d, and in either case the maximum possible value of the local contribution

$$(80) \qquad \mathrm{s}_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}}$$

is 1. As such, if we again put Z_d to be sub-scheme cut out by (x, y^d), but here with x as in (78), then

$$(81) \qquad \mathrm{s}_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}} \ll \frac{1}{d} \mathrm{s}_{Z_d}(f)_{\mathrm{loc}}$$

and, as in (76), we have what we need.

This leaves us with the possibility that λ belongs to the prime field $\mathbb{F}_p \hookrightarrow k(s)$, and we effect a change of coordinates $(x, y) \mapsto (x_s, y)$ to identify (on completion in $Z \otimes k(s)$) the semi-simple part

$$(82) \qquad \partial_S(s) = y\frac{\partial}{\partial y} + \lambda_s x_s \frac{\partial}{\partial x_s},$$

of a re-scaling of ∂ over $k(s)$, wherein the hypersurface $x_s = 0$ may be supposed in agreement with $x = 0$ to order $d + 1$. Writing our invariant curve as $f : t \mapsto (x_s(t), y(t))$, observe that the maximum possibility for (80) is still 1, and it would actually be negative if f were widely ramified. Plainly, this latter possibility can be ignored, but equally if $t \mapsto y(t)$ were wild then (74) would hold, while if $t \mapsto x_s(t)$ were wild then (81) holds as soon as $p \geq d$. Consequently,

we can suppose that both the coordinate projections of f are tame. As such if m, n are the smallest positive integers whose ratio mod p is λ_s, then for ord again the order of vanishing on restriction to Δ, invariance by the semi-simple part (82) implies

$$\text{(83)} \qquad \qquad \text{ord}(x_s) \geq m, \ \text{ord}(y) \geq n$$

or one of f^*x_s, f^*y vanishes identically. This latter possibility is exactly what we encountered in the previous discussion of $\lambda_s \notin \mathbb{F}_p$, whence, on treating it in the same way, we can suppose that (83) holds. On the other hand since $\lambda \in K\backslash\mathbb{Q}_{>0}$, any Arakelov intersection of $\lambda \in \mathbb{P}^1_S$ with m/n is, up to the irrelevance of a choice of metric on $\mathscr{O}(1)_{\mathbb{P}^1_\mathbb{C}}$, non-negative, so

$$\text{(84)} \qquad \qquad \max\{m,n\} \gg p$$

Finally, therefore, if $p \geq d$, then, either $n \gg p$ and we have the estimate (74) or $m \gg p$ and we appeal to (81). Having thus estimated all the possibilities for the local contribution to (80) we deduce the refinement (70) of the tautology (69).

V. SURFACES OF GENERAL TYPE

Throughout this section S is an irreducible affine scheme of finite type over a Noetherian integral domain with generic point of characteristic zero, *e.g.* $\mathbb{Z}$, and X/S is a smooth family of S-projective surfaces of general type, with H a S-ample line bundle. The principle conclusions, III.6 and IV.2 of III, respectively IV, combine with the minimal model theory of [McQ08] to yield

V.1. *Proof of I.4.* By III.6 and III.9, (a) is hopelessly false should (b) occur, so it remains to prove (a) otherwise. By the classification theorem, [McQ08], this amounts to the distinct possibilities

(1) The (foliation) canonical $K_{\mathscr{F}}$ is big on the generic fibre.
(2) The generic fibre is as (b) but points of positive residue characteristic aren't dense.

The first case is immediate by IV.2 and a suitable choice of ε. In the second case, notation as in III.8 albeit in characteristic zero and including the case of products of curves, we have by classical Baum-Bott residue theory, *i.e.* a much simpler variant of the final steps of the proof of *op. cit.*, or by *op. cit.* itself if one wants to be cute about it, that (66) holds, while in characteristic zero an invariant curve can't meet E, so the left hand side of (66) is $-\chi(C)$. $\square$

V.2. *Proof of the subtler theorem, I.5.* Plainly (a) holds for products of curves, and their quotients by finite group actions, while any other bi-disc quotient is defined over $\bar{\mathbb{Q}}$, and by III.6 we have invariant rational curves satisfying (8) whenever p is inert. As such, by I.4, it remains to deal with bi-disc quotients at split primes, and we can, of course, suppose that we're on a model with generically canonical singularities. Putting ourselves again in the notation of III.8, this follows from (45) if there are no cusps, E. Otherwise, there is some subtlety, since invariant curves can meet E in positive characteristic. In the proof of III.8, if (50) holds for all m, then in fact we have

$$\text{(85)} \qquad \qquad (K_X + E) \cdot_f C \leq K_{\mathscr{F}} \cdot_f C$$

and there's nothing to do by IV.2. If, however, we're in the infinitely more likely possibility that (50) holds for some m, but fails for some $m + 1$, then (since C is rational or elliptic) we only get

$$\text{(86)} \qquad \qquad (K_X + E) \cdot_f C \leq (1 + p^m)\text{cardinality}(f^{-1}(E)_{\text{red}})$$

In the particular case that $m = 1$, we've already seen in (82)-(84) that

$$\text{(87)} \qquad \qquad \text{cardinality}(f^{-1}(E)_{\text{red}}) \ll \frac{1}{p}E \cdot_f C$$

where the implied constant may be no better than the Arakelov height of λ, *i.e.* it's arbitrary. Nevertheless, this bounds the degree along $K_X + E$ by a constant multiple of the intersection with E, and this works in general- just do (82)-(84) modulo p^m instead of modulo p, which actually has some simplification since $xy = 0$ of *op. cit.* can be taken to be the equation of E, while by (61)-(65) we have full p-adic Jordan decomposition. As such we deduce ($p \gg 0$ to avoid any bad reduction issues) that any rational or elliptic curve $f : C/k \to X$ where the characteristic of k is a split prime, satisfies

$$(88) \qquad (K_X + E) \cdot_f \ll E \cdot_f C$$

for an implied constant depending only on X.

Now suppose such rational or elliptic, over $\bar{k}(s)$ are Zariski dense with $s \in \Sigma$ of residue characteristic, p, a split prime. By hypothesis, and the fact that we know, I.4, that the theorem is true is in characteristic zero we can, *cf.* proof of III.6, find for an infinite set of split p, a rational or elliptic curve, $f_p : C_p/\bar{k}(s) \to X$ such that for H ample

$$(89) \qquad \liminf_{p \in \Sigma} H \cdot_{f_p} C_p = \infty$$

On the other hand, for $s \in \Sigma$ we have a specialisation map, [Ful98, 20.3] on Néron-Severi

$$(90) \qquad i_s : \mathrm{NS}^1(X \otimes k(S))_{\mathbb{R}} \to \mathrm{NS}^1(X \otimes k(s))_{\mathbb{R}}$$

whence, by duality, classes in $\mathrm{NS}_1(X \otimes k(S))_{\mathbb{R}}$ defined by

$$(91) \qquad [\phi_p] : \mathrm{NS}^1(X \otimes k(S))_{\mathbb{R}} \to \mathbb{R} : D \mapsto \frac{1}{H \cdot_{f_p} C_p} i_s(D) \cdot_{f_p} C_p$$

which for $p \gg 0$ certainly intersects any ample divisor, A, positively. On the other hand $nH - A$ is ample for $n \gg 0$, and vice-versa, so for A fixed, $A \cdot [\phi_p]$ is bounded, and weak limits of the $[\phi_p]$ exist. Choose one, ϕ, say, and throw the rest of Σ, *i.e.* $[\phi_p]$ not converging to ϕ, away, then the underlying f_p are still Zariski dense by (89), so ϕ may be identified with a (non-zero) nef. class in $\mathrm{NS}^1(X \otimes k(S))_{\mathbb{R}}$. Now, up to introducing some harmless quotient singularities, there's no loss of generality in identifying $X \otimes k(S)$ with a resolution, π, of the cusps on the canonical model, [McQ08, III.3], $(X_0, \mathscr{F}_0)$ of $\mathscr{F}$, and since ϕ is nef., then, sub-sequencing as necessary, by IV.2

$$(92) \qquad \pi^* K_{\mathscr{F}_0} \cdot \phi = 0$$

while by (88), $E \cdot \phi > 0$. We have, however,

$$(93) \qquad \phi = \pi^*(\pi_* \phi) + \frac{E \cdot \phi}{E^2} E$$

and $\phi^2 \geq 0$, so $\pi_* \phi$ is big and nef. in the Mumford intersection theory of X_0, which by Hodge affords the absurdity, [McQ08, IV.5.5], that $K_{\mathscr{F}_0}$ is numerically trivial. $\qquad \square$

Now we can apply this, following, [Bog77], to curves on surfaces of general type in the usual way

V.3. *Proof of I.1,* $\Omega^1_{X/S}$ *big.* Let $\pi : P = \mathbb{P}(\Omega_{X/S}) \to X$ be the projective tangent space, and L its tautological bundle, then, by hypothesis, there is a $\delta > 0$ such that $L - \delta\pi^* H$ is effective. Again, by hypothesis, the derivative, $f' : C \to P$ of our curve exists, and satisfies IV.1. As such, we're done unless f' factors through a divisor $D \hookrightarrow P$ dominating X. Such a divisor defines a foliation, $\mathscr{F}$, by curves (on itself) given over the generic point by

$$(94) \qquad \Omega^1_{D/S} \xrightarrow{\sim} \pi^* \Omega^1_{X/S} \to L|_D$$

so, without loss of generality, curves not satisfying (a) are invariant by a foliation on a surface dominating X, and we conclude by I.4. $\qquad \square$

V.4. *Proof of the subtler theorem, I.2.* Exactly as V.3, but use I.5 rather than I.4. $\qquad \square$

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