A GRIFFITHS' THEOREM FOR VARIETIES WITH ISOLATED SINGULARITIES

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Abstract. By the fundamental work of Griffiths one knows that, under suitable assumption, homological and algebraic equivalence do not coincide for a general hypersurface section of a smooth projective variety Y. In the present paper we prove the same result in case Y has isolated singularities.

Key words: Abel-Jacobi map, Normal function, Homological and Algebraic equivalence, Noether-Lefschetz Theory, Monodromy, Dual variety, Isolated singularity, Intersection cohomology.

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§ 0. Introduction.

By the fundamental work of Griffiths [Gri] one knows that for a smooth projective complex variety Y homological equivalence and algebraic equivalence do not coincide in higher codimension. Griffiths' proof rests on two main steps. The first one consists in showing that the algebraic part of the intermediate Jacobian of a general hyperplane section of Y vanishes, because of the irreducibility of the monodromy action on the rational cohomology: this is essentially the classical Noether-Lefschetz argument. The second one is based on a careful analysis of the normal function associated to an algebraic and primitive cycle of Y.

In the present paper, following a similar pattern, we investigate the same question for varieties with isolated singularities. As for the first step, the proof runs exactly as in the classical case, taking into account a monodromy theorem for varieties with isolated singularities recently proved in [DGF]. So this part of the proof is easy, however we reproduce it again for Reader's convenience (see Theorem 4 below). The hard points appear in adapting the second step to the singular case (cfr. Theorem 6, proof of Step 4 and Step 6). In particular, the proof of our Step 6 reduces to compare, in a certain range, the homology of Y with the homology of a *singular* hypersurface. We believe this argument may be useful also in a different context, as factoriality. More generally, our work may be useful to gain more information on Noether-Lefschetz loci.

Finally we recall that a deep improvement of Griffiths' result was made by Nori (see [N]). With a new argument based on his celebrated Connectivity Theorem, Nori proved that homological and algebraic equivalence do not coincide in a wider range. However one knows that Nori's result does not completely imply Griffiths' Theorem (cfr. [N], p. 369, Conjecture 7.4.1).

§ 1. Notation and preliminaries.

Throughout this paper $Y \subseteq \mathbb{P}^N$ denotes an irreducible, reduced, projective variety having at worst isolated singularities, with

$$\dim Y \quad = \quad n+1 \quad = \quad 2r \quad \geq \quad 4\,.$$

Furthermore $f : \widetilde{Y} \to Y$ denotes a resolution of singularities; $\Sigma_d \subseteq |\mathcal{O}_Y(d)|$ denotes the linear system on Y cut by d-degree hypersurface sections, $\Sigma_d^\circ := \left\{ b \in \Sigma_d \middle| X_b \text{ is smooth} \right\}$ its subset parameterizing smooth varieties (for $b \in \Sigma_d$, X_b denotes the corresponding hypersurface section) and $\mathcal{D} := \Sigma_d \setminus \Sigma_d^\circ$ the discriminant locus; for $b \in \Sigma_d^\circ$, $i_b : X_b \to \widetilde{Y}$ denotes the natural inclusion; for a analytical submanifold B of Σ_d° we consider the natural family over B

$$Y_B \quad := \quad \left\{ (x,\, b) \;\in\; Y \times B \;\middle|\; x \in X_b \;\right\}, \qquad \pi \,:\, Y_B \,\to\, B\,,$$

and the natural inclusion $i : Y_B \hookrightarrow \widetilde{Y} \times B$ of families over B obtained by globalizing i_b. We consider the monodromy representation associated to the universal family, i.e. over Σ_d°, $\pi_1(\Sigma_d^\circ, b) \to \mathrm{Aut}\left(H^n(X;\mathbb{Q})\right)$ where $b \in \Sigma_d^\circ$ is any point and $X = X_b$ ([V2], Ch. 3). We want to recall that as a consequence of Deligne invariant subspace Theorem the set I of invariant elements is the image of the pull-back in cohomology $i_b^* : H^n(\widetilde{Y};\mathbb{Q}) \to H^n(X;\mathbb{Q})$, in particular it is a Hodge substructure of $H^n(X;\mathbb{Q})$. Its orthogonal complement shall be denoted by V and will be called the "vanishing cohomology". So, $H^n(X;\mathbb{Q}) = I \perp V$.

We also consider the analytic map of Jacobian fibrations over B associated to the inclusion $Y_B \subseteq \widetilde{Y} \times B$, which we shall denote by i^*

$$J^r(\widetilde{Y}) \times B \quad \xrightarrow{i^*} \quad \mathcal{J} \quad := \quad \left\{J^r(X_b)\right\}_{b\in B}.$$

This map globalizes the map of Griffiths' intermediate Jacobians $i_b^* : J^r(\widetilde{Y}) \to J^r(X_b)$, where we keep the standard notation $J^r(W) := H^{2r-1}(W;\mathbb{C})\big/\left(F^r H^{2r-1}(W;\mathbb{C}) \oplus H^{2r-1}(W;\mathbb{Z})\right)$ for the Griffiths' intermediate Jacobian of any smooth projective variety W. The kernel of such i_b^* does not depend on b, in other terms the inverse image via i^* above of the trivial section $(i^*)^{-1}(\{0\}_{b\in B}) \subseteq \mathcal{J}$ is a product $K \times B$, where K is a subgroup of $J^r(\widetilde{Y})$. We furthermore denote with T the image subtori-fibration of $J^r(\widetilde{Y}) \times B$ in $\mathcal{J}$ and with $T_b = i_b^*\left(J^r(\widetilde{Y})\right)$ its fiber over b. We have an exact sequence of families over B

$$0 \longrightarrow K \times B \longrightarrow J^r(\widetilde{Y}) \times B \longrightarrow T \longrightarrow 0\,.$$

The inclusion $T \subseteq \mathcal{J}$ is closed and does not depend on the resolution of singularities $\widetilde{Y} \to Y$, likewise any inclusion $T_b \subseteq J^r(X_b)$.

For any m-dimensional projective variety W, $\mathcal{Z}^q(W)$, $\mathcal{Z}^q(W)_{\mathrm{hom}}$ and $\mathcal{Z}^q(W)_{\mathrm{alg}}$ denote respectively the group of q-codimensional algebraic cycles, its subgroup of homologically trivial cycles and its subgroup of algebraically trivial cycles (compare with [F1], Ch. 19 and [F2], Appendix B). For a cycle $Z \in \mathcal{Z}^q(W)$ we shall denote the corrsponding classes in the q-Chow group and in homology respectively as $[Z] \in CH^q(W)$ and $cl(Z) \in H_{2m-2q}(W;\mathbb{C})$. The notation for the Abel-Jacobi map (in case W is smooth) shall be $\Psi_{AJ} : \mathcal{Z}^q(W)_{\mathrm{hom}} \to J^q(W)$, we also set $J^q(W)_{\mathrm{alg}} := \Psi_{AJ}\left(\mathcal{Z}^q(W)_{\mathrm{alg}}\right)$, the image of algebraically trivial cycles. Furthermore, in case W is a smooth quasi-projetive variety, $cl(Z)^\vee$ denotes the cohomology class of the cycle Z in $H^{2q}(W;\mathbb{C})$ ([V1], Ch. 11). We also recall that any local complete intersection (l.c.i.) morphism between projective varieties induces Gysin maps in homology and in cohomology ([F1], 19.2.1).

Let us go back to our situation. For a smooth d-degree hypersurface section $X = X_b$ we define $J^r(X)_0$ as the cokernel of the map of Jacobians $J^r(\widetilde{Y}) \to J^r(X)$ and eventually we set $J^r(X)_{0,\,\mathrm{alg}}$ as the image of $J^r(X)_{\mathrm{alg}}$ in $J^r(X)_0$.

Remark 1. Let $Z \in \mathcal{Z}^r(Y)$. If $[Z]|_X \in CH^r(X)_{\mathrm{hom}}$ for some smooth d-degree hypersurface section $X \hookrightarrow Y$, then $[Z]|_{X_b} \in CH^r(X_b)_{\mathrm{hom}}$ for all smooth d-degree hypersurface sections $X_b \hookrightarrow Y$.

We now observe that, in the hypothesis of the remark, the restriction $[Z]|_{X_b}$ defines an element in $J^r(X_b)$ via Ψ_{AJ}, for any X_b as above. So, there is a well-defined section (set-theoretical, for the moment) associated to our cycle Z:

$$\nu_Z \;:\; \Sigma_d^\circ \longrightarrow \mathcal{J}\,.$$

Remark 2. Denote by $Z = \sum n_i Z_i$ the decomposition of Z in its irreducible components and by L_i the subspace of Σ_d parametrizing hypersurfaces containing Z_i. For a analytic submanifold $B \subseteq \Sigma_d \setminus (\mathcal{D} \cup \bigcup L_i)$ define $Z_{i,B} := \{(x,b) \in Y_B \,:\, x \in Z_i \cap X_b\}$. Since for any $b \in B$ each Z_i meets properly X_b, then $Z_B := \sum n_i Z_{i,B}$ is a relative cycle of codimension r for the family Y_B, i.e. $Z_B \in \mathcal{Z}^r(Y_B)$ and each $Z_{i,B}$ is flat over$B$. Hence from ([V2], Theorem 7.9) we know that ν_Z is a normal function on B (i.e. it is holomorphic and horizontal on B). Actually, up to replace Z with others representatives of its class in $CH^r(Y)$, one proves that ν_Z is a normal function on *all* Σ_d°.

Remark 3. Besides ordinary cohomology we will also consider the intersection cohomology $IH^*(W;\mathbb{C})$, for a complex m-dimensional irreducible projective variety W. Here we recall some properties which we will use in the sequel, for more details see ([D2], pg. 154-161). First, $IH^*(W;\mathbb{C}) \cong H^*(W;\mathbb{C})$ when W is smooth, Poincaré Duality, Lefschetz Hyperplane Theorem and Hard Lefschetz Theorem still hold for intersection cohomology. Moreover, if W has at worst isolated singularities, then $H^q(W;\mathbb{C}) \cong IH^q(W;\mathbb{C})$ for any $q > m$. So, by Poincaré Duality, one has a natural isomorphism $H_q(W;\mathbb{C}) \cong IH^{2m-q}(W;\mathbb{C})$, for $q > m$. Finally recall that from Decomposition Theorem it follows that $IH^*(W;\mathbb{C})$ is naturally embedded in $H^*(\widetilde{W};\mathbb{C})$ as a direct summand, where $\widetilde{W}$ is a desingularization of W. So we have a natural surjection $H_*(\widetilde{W};\mathbb{C}) \to IH^*(W;\mathbb{C})^\vee \cong IH^{2m-*}(W;\mathbb{C})$. For $q > m$, assuming W has at worst only isolated singularities, this map identifies with the push-forward $H_q(\widetilde{W};\mathbb{C}) \to H_q(W;\mathbb{C})$.

§ 2. Generalized Griffiths' Theorems.

Theorem 4. *Let $Y \subseteq \mathbb{P}^N$ denote an irreducible, reduced, projective variety of even dimension $n+1 = 2r \geq 4$, with isolated singularities. Let X denote the intersection of Y with a general hypersurface of degree d. Assume the vanishing cohomology V is not contained in the middle Hodge component $H^{r,r-1}(X) \oplus H^{r-1,r}(X)$. Then we have*

$$J^r(X)_{0,\,\mathrm{alg}} \quad = \quad 0\ .$$

We want to stress that in the case where Y is a projective space, this theorem reduces to the "first part" of Griffiths' Theorem as stated in ([Sh], Theorem 2.2). It has a strong analogy with Noether-Lefschetz Theorem. The present generalization is obtained by revisiting Shioda's proof ([Sh], pg. 721-722) in view of a result on the monodromy action [DGF]. As for a generalization of the "second part" of Griffiths' Theorem see Theorem 6 below and compare it with [V2], Theorem 8.25.

Remark 5. *i*) For $d \gg 0$, the hypothesis on the vanishing cohomology V is automatically satisfied; *ii*) in the hypothesis of the Theorem, also $J^r(X)_{\mathrm{alg}}$ vanishes in case the homology space $H_{n+2}(Y;\mathbb{C})$ vanishes (e.g. if $Y \subseteq \mathbb{P}^N$ is a nodal complete intersection as well as if Y is a hypersurface, and not a cone, with at most one ordinary singular point ([D1], (4.6) Corollary. (A) p. 164, and (4.17) Theorem p. 214).

The first statement holds because otherwise the orthogonal complement I of V would contain $H^{0,\,2r-1}(X)$ and the geometric genus of X would be bounded by the geometric genus of $\widetilde{Y}$, which is impossible for d sufficiently large. As for the second statement, it suffices to observe that the map of Jacobians $J^r(\widetilde{Y}) \to J^r(X)$ is induced by the restriction map $H^n(\widetilde{Y};\,\mathbb{C}) \to H^n(X;\,\mathbb{C})$ and that such map factors through $H_{n+2}(Y;\mathbb{C})$, in fact it is the Poincaré dual of the composition $H_{n+2}(\widetilde{Y};\,\mathbb{C}) \to H_{n+2}(Y;\,\mathbb{C}) \to H_n(X;\,\mathbb{C})$, where the second map is the Gysin morphism in homology.

Here, though the proof of Theorem 4 follows a well-known pattern, we shortly outline it for the sake of completeness (being Theorem 4 crucial for the sequent Theorem 6).

Proof of Theorem 4. Let $\{X_t\}_{t\in\mathbb{P}^1}$ be a general pencil of degree d hypersurface sections and fix a reference point $o \in \mathbb{P}^1 \setminus \mathcal{D}$. Let $U \subset \mathbb{P}^1 \setminus \mathcal{D}$ be a small neighborhood of o. Take a non-zero element $\gamma \in V_o$ and extend it by continuous deformation to $\gamma_t \in V_t$, $t \in U$. We have the following dichotomy: either $\gamma_t \in M_t := H^n(X_t;\mathbb{Q}) \cap \left[H^{r,r-1}(X_t) \oplus H^{r-1,r}(X_t)\right]$ for any $t \in \mathbb{P}^1 \setminus \mathcal{D}$ and any continuous deformation γ_t of γ, or the set $A_\gamma := \{t \in U \,|\, \gamma_t \in M_t\}$ is countable. In fact, by Griffiths' Transversality we know that the condition $\gamma_t \in M_t$ is an analytic condition on $t \in U$. In the first case, the submodule of V_o generated by γ under the action of $\pi_1(\mathbb{P}^1 \setminus \mathcal{D}, o)$ is contained in M_o. Since V_o is irreducible ([DGF], Corollary 3.7) then $V_o \subset M_o$, and this is in contrast with our assumption on V. So a fortiori A_γ is countable. Then also $A := \bigcup_{\gamma\in V_o,\gamma\neq 0} A_\gamma$ is countable, and for $t \in U \setminus A$ we have $M_t \cap V_t = \{0\}$. Letting $X = X_t$, $t \in U \setminus A$, on the other hand, as the orthogonal sum $H^n(X;\mathbb{Q}) = I \perp V$ respects the Hodge decomposition, we obtain

$$I \supseteq M\ , \qquad \text{equivalently} \qquad M \;=\; I \cap [H^{r,r-1}(X) \oplus H^{r-1,r}(X)]\ .$$

Therefore, taking into account that the tangent space to $J^r(X)_{\mathrm{alg}}$ at the origin is contained in $H^{r-1,r}(X)$, we obtain $J^r(X)_{0,\,\mathrm{alg}} = 0$ as required.

□

Theorem 6. *Let $Y \subseteq \mathbb{P}^N$ be an irreducible, reduced, projective variety of even dimension $n+1 = 2r \geq 4$, with isolated singularities. Let $Z \in \mathcal{Z}^r(Y)$ be a cycle with $cl(Z) \neq 0 \in H_{2r}(Y;\mathbb{C})$. For $d \gg 0$, assuming that $cl\big([Z]|_{X'}\big) = 0 \in H_{2r-2}(X';\mathbb{C})$ for some smooth d-degree hypersurface section X', a general smooth d-degree hypersurface section X of Y satisfies the following properties*

- **a)** $\Psi_{_{AJ}}\big([Z]|_X\big)$ *does not vanish in* $J^r(X)_0$;
- **b)** $[Z]|_X$ *is not algebraically trivial.*

Moreover, assuming that the singularities of Y are "mild" (see below), then

- **a′)** *property* "a)" *holds under the weaker hypothesis* $d \geq 3$;
- **b′)** *property* "b)" *holds under the hypotheses $d \geq 3$ and that the "vanishing cohomology subspace" of $H^{2r-1}(X;\mathbb{C})$ is not contained in the middle cohomology $H^{r,r-1}(X) \oplus H^{r-1,r}(X)$.*

We say that Y has *mild* singularities if for any $p \in Y$ the exceptional divisor of the blow-up of Y at p has at worst isolated singularities.

Proof of Theorem 6.
Step 1. We first explain how to deduce the Theorem for $d \gg 0$ from the following property ($\star$).

($\star$) For $d \gg 0$, there exists a d-degree hypersurface section X_o such that a Zariski general line $\ell \in \big\{$ lines in Σ_d through $X_o\big\}$ does not contain a ball $U \subseteq \ell \setminus \mathcal{D}$ with

$$\Psi_{_{AJ}}\ ([Z]|_{X_b}) \quad \in \quad T_b \ = \ \mathrm{image}\big(J^r(\widetilde{Y})\big) \quad , \qquad \forall\ b \in U$$

(note that this is the vanishing in $J^r(X_b)_0$ condition).

Proof of Step 1. Consider the set $G2 \subseteq \Sigma_d^\circ$ of d-degree smooth hypersurface sections X_b satisfying a). Since the normal function is analytic and T is closed in $\mathcal{J}$, the complement of $G2$ in Σ_d° is analytic. As a consequence, we have the dichotomy: either $G2$ is empty, or it is the complement of a proper analytic subset of Σ_d°. As the first case can be excluded for otherwise ($\star$) would be contradicted, then $G2$ is dense in Σ_d°. By Theorem 4, which can be applied in view of remark 5, the set $G1$ of sections satisfying $J^r(X_b)_{0,\,\mathrm{alg}} = 0$ contains the complement of a countable union of proper analytic subvarieties of Σ_d. So, the same holds for the set $G1 \cap G2$ as well. Finally, for X_b with $b \in G1 \cap G2$ property b) holds. In fact, if $[Z]|_{X_b}$ were algebraically trivial, then $\Psi_{_{AJ}}\ ([Z]|_{X_b})$ would belong to $J^r(X_b)_{0,\,\mathrm{alg}} = 0$ (this is because $b \in G1$), and this would contradict the fact that $b \in G2$.

□

Now the strategy is the following: we introduce a particular hypersurface section X_o (step 2), we state properties of a Zariski general pencil through X_o (step 3), then we prove property ($\star$).

Step 2. For $d \gg 0$, there exists a d-degree hypersurface section X_o of Y intersecting properly each component of Z, containing Y_{sing}, and such that $\widetilde{X}_o$ is irreducible, smooth, and very ample, where $\widetilde{X}_o$ denotes the strict transform of X_o in the desingularization $\widetilde{Y}$ of Y. Note that, in particular, $X_o \cap Y_{\mathrm{smooth}}$ is smooth.

Proof of Step 2. One can construct $f : \widetilde{Y} \to Y$ via a sequence of blowings-up along smooth centers supported in Y_{sing} ([L], Theorem 4.1.3 pg. 241). Denote by $g : \widetilde{\mathbb{P}} \to \mathbb{P}^N$ the projective variety obtained with the same sequence of blowings-up of $\mathbb{P}^N$ and observe that $\widetilde{Y} \subset \widetilde{\mathbb{P}}$. Since the centers are smooth, then $\widetilde{\mathbb{P}}$ is nonsingular and its Picard group is generated by the pull-back of the hyperplane $g^*(H)$ and the components $E_i \subset \widetilde{\mathbb{P}}$ of the exceptional divisor (see [H], Proposition 7.16 and Theorem 8.24, and [F1], Proposition 6.7.e). Therefore, for suitable integers $d \gg 0$ and m_i, the line bundle $\mathcal{O}_{\widetilde{\mathbb{P}}}\big(dg^*(H) - \sum m_i E_i\big)$ must be very ample. Let $\widetilde{S}_o \in |\mathcal{O}_{\widetilde{\mathbb{P}}}(dg^*(H) - \sum m_i E_i)|$ be a general divisor. Then $X_o := Y \cap g(\widetilde{S}_o)$ verifies all requests. In fact $\widetilde{X}_o$ is equal to $\widetilde{Y} \cap \widetilde{S}_o$, which is irreducible, smooth and very ample. Moreover X_o meets properly each component Z_i of Z for $\widetilde{X}_o$ meets properly $Z_i \setminus g^{-1}(Y_{\mathrm{sing}})$. Finally, X_o contains Y_{sing} because $\widetilde{S}_o$ is very ample.

□

Step 3. A Zariski general pencil $\ell \cong \mathbb{P}^1$ through X_o satisfies the following properties:

- for any $b \in \ell$ the fiber X_b of ℓ meets all components of our cycle Z properly, so $Z_{\ell \setminus \mathcal{D}}$ is a relative cycle for the family $Y_{\ell \setminus \mathcal{D}}$ (cfr remark 2);

• $\ell\backslash\{X_o\}$ has finitely many singular fibers, not meeting $Y_{\rm sing}$ and having only one ordinary double point.

Proof of Step 3. Let Z_i be an irreducible component of Z. Since $\dim(Z_i) = r > 0$ then Z_i imposes at least two conditions to Σ_d. Therefore the subspace $L_i \subset \Sigma_d$ parametrizing hypersurfaces containing Z_i has codimension at least 2. Since $X_o \notin L_i$, then a general line in Σ_d passing through X_o does not meet L_i. This proves the first property.
As for the second property, embed Y via the d-Veronese and interpret Σ_d as the linear series cut by hyperplane sections. We have $\mathcal{D} = Y^* \cup p_1^* \cup \cdots \cup p_h^*$, where Y^* denotes the dual variety of Y, and the p_i^* denote the dual hyperplanes of the singular points p_i of Y. Let ℓ be a general pencil through X_o and let $X_o, X_{b_2}, \cdots, X_{b_k}$ be its singular fibers, namely the fibers corresponding to the points of the intersection $\ell \cap \mathcal{D}$. Since $X_o \in \bigcap p_i^*$ then ℓ meets $\bigcup p_i^*$ only at X_o. Moreover, since $\dim(Y^*)_{\rm sing} \leq \dim\Sigma_d - 2$, then $\ell \cap (Y^*)_{\rm sing} \subseteq \{X_o\}$. Therefore $\{b_2, \cdots, b_k\} \subset Y^* \setminus [(Y^*)_{\rm sing} \cup \bigcup p_i^*]$. Assuming $\{b_2, \cdots, b_k\}$ is not empty (for otherwise we would be done), Y^* has no dual defect and it is not a cone with vertex X_o. We have to prove that the pencil ℓ meets Y^* transversally at b_j for any $j = 2, \cdots, k$ (this means that each X_{b_j} has only one ordinary double point). Proceeding by contradiction, if the general line through X_o is tangent to $Y^* \setminus (Y^*)_{\rm sing}$ at some point, then the projection (possibly internal) of Y^* from X_o has the image of $R \setminus \{X_o\}$ equal to the image of $Y^* \setminus \{X_o\}$, where $R := \{b \in Y^* : X_o \in T_{b,Y^*}\}$ denotes the ramification locus of the projection. Since the image of $Y^* \setminus \{X_o\}$ has the same dimension of Y^*, then the ramification locus is all Y^*. Therefore for a general $b \in Y^*$ we have $X_o \in T_{b,Y^*}$. Now let $J \subset \Sigma_d$ be the cone with vertex X_o and basis Y^* (i.e. the embedded join of X_o and Y^*). By Terracini's Lemma ([FOV], Proposition 4.3.2.) we know that for a general $b \in Y^*$ and a general $c \in \overline{bX_o}$ the tangent space to J at c is spanned by X_o and the tangent space to Y^* at b. Since $X_o \in T_{b,Y^*}$ then $\dim J = \dim Y^*$, i.e. $J = Y^*$. Hence Y^* is a cone with vertex X_o, and this is in contrast with previous assumption.

□

Now, for a pencil $\ell \cong \mathbb{P}^1$ as in step 3, let $B = \mathbb{P}^1 \setminus \mathcal{D}$ denote the set of smooth sections and consider the natural completion of the family $\pi : Y_B \to B$ (cfr §1)

$$\begin{array}{ccc} Y_B & \hookrightarrow & Y_\ell \\ \pi \downarrow & & \downarrow \pi \\ B & \hookrightarrow & \mathbb{P}^1 \end{array}$$

where Y_ℓ is the blow-up of Y along the base locus of the pencil. Finally, consider $Z_B \in \mathcal{Z}^r(Y_B)$ as introduced in §1, remark 2, and its class $cl(Z_B)^\vee \in H^{2r}(Y_B;\mathbb{Z})$.

We are now ready to prove property ($\bigstar$). For this, we proceed by contradiction.

Step 4. Let ℓ be a general pencil as in step 3. If there exists a ball $U \subseteq B$ such that $\nu_z(b) \in T_b$ for all $b \in U$, then there exists an element $\xi \in J^r(\widetilde{Y})$ such that $i_b^*(\xi) = \nu_z(b)$, $\forall\, b \in B$.

Proof of Step 4. First, we introduce a natural commutative diagram

$$\begin{array}{ccccccc} & & & & J^r(\widetilde{Y}) & & \\ & & & & \downarrow i_b^* & & \\ [Z]|_{X_b} & \in & CH^r(X_b)_{\rm hom} & \xrightarrow{\Psi_{AJ}} & J^r(X_b) & \ni & \nu_z(b) \\ & & i_{b*}\downarrow & & \downarrow i_{b\star} & & \\ i_{b*}([Z]|_{X_b}) & \in & CH^{r+1}(\widetilde{Y})_{\rm hom} & \xrightarrow{\Psi_{AJ}} & J^{r+1}(\widetilde{Y}) & & \end{array}$$

where the left vertical map i_{b*} denotes the push-forward and the map of Jacobians $i_{b\star}$ at the right-side is the map induced by the map $i_{b\star} : H^{2r-1}(X_b;\mathbb{Z}) \to H^{2r+1}(\widetilde{Y};\mathbb{Z})$ (which is the Gysin morphism in cohomology associated to $i_b : X_b \to \widetilde{Y}$). We now observe the following properties:
i) the vertical composition $i_{b\star} \circ i_b^*$ can be interpreted as the map induced by the map $H^{2r-1}(\widetilde{Y};\mathbb{Z}) \to H^{2r+1}(\widetilde{Y};\mathbb{Z})$ given by the cap product in homology (modulo Poincaré duality) with the class of the

"divisor X_b" (in particular it does not depend on b);
ii) as the diagram above commutes and $i_{b*}([Z]|_{X_b}) \in CH^{r+1}(\widetilde{Y})_{\hom}$ does not depend on b, the image $i_{b\star} \circ \nu_Z(b)$ does not depend on b as well.
We now conclude the proof of step 4 under the assumption that the restriction $i_{b\star}|_{\mathrm{image}(i_b^*)}$ is an isogeny on its image. Let us now go back to our exact sequence $0 \to K \times B \to J^r(\widetilde{Y}) \times B \to T \to 0$ of families over B. Working modulo the identification $\frac{J^r(\widetilde{Y})}{K} \times B \cong T$ of fibrations over B, we replace

$$J^r(\widetilde{Y}) \times B \xrightarrow{i^*} T \xrightarrow{i_\star} J^{r+1}(\widetilde{Y}) \times B$$

with

$$J^r(\widetilde{Y}) \times B \longrightarrow \frac{J^r(\widetilde{Y})}{K} \times B \xrightarrow{\hat{\imath}_\star} J^{r+1}(\widetilde{Y}) \times B$$

where $i_\star$ denotes the map of Jacobian fibrations that globalizes $i_{b\star}$. Now, our normal function ν_Z takes values to T and therefore induces a analytic section $\hat{\nu} : B \to \frac{J^r(\widetilde{Y})}{K} \times B$. As the image $i_{b\star} \circ \nu_Z(b)$ does not depend on b, the composition $\hat{\imath}_\star \circ \hat{\nu}$ is constant (as a matter of language, a section of a trivial fibration is said to be *constant* if its image is the graph of a constant function). By our assumption stating that the restriction $i_{b\star}|_{\mathrm{image}(i_b^*)}$ is an isogeny on its image, the kernel of $\hat{\imath}_\star$ is discrete (by definition, the kernel of a map of fibrations is the inverse image of the zero section, and it is said to be *discrete* if its restriction to any fiber is a discrete subset). As a consequence, the inverse image of $\hat{\nu}$ itself must be constant.
We are left to prove that the restriction $i_{b\star}|_{\mathrm{image}(i_b^*)}$ is an isogeny on its image. First we may note that in the smooth case X_b is ample on $Y = \widetilde{Y}$ and $i_{b\star} \circ i_b^*$ is an isogeny by the Hard-Lefschetz theorem (and the interpretation we have given above).
Going back to our situation, we set $i_b,\ j_b,\ f$ as in the diagram

$$\begin{array}{ccc} & & \widetilde{Y} \\ & {\scriptstyle i_b} \nearrow & \downarrow {\scriptstyle f} \\ X_b & \xrightarrow{j_b} & Y \end{array}$$

and we investigate more closely the map $i_{b\star} \circ i_b^*$. As we already said, the map $i_{b\star} \circ i_b^*$ descends from the map $H^{2r-1}(\widetilde{Y}; \mathbb{Z}) \longrightarrow H^{2r+1}(\widetilde{Y}; \mathbb{Z})$ which, passing to complex coefficients, turns out to be equal to the composition of all maps at the first row in the diagram below (cfr remark 3 on intersection cohomology):

$$\begin{array}{ccccc} H^{2r-1}(\widetilde{Y}; \mathbb{C}) \cong H_{2r+1}(\widetilde{Y}; \mathbb{C}) & \xrightarrow{f_*} & H_{2r+1}(Y; \mathbb{C}) \cong IH^{2r-1}(Y; \mathbb{C}) \cong^{(1)} IH^{2r+1}(Y; \mathbb{C}) \subseteq & & H^{2r+1}(\widetilde{Y}; \mathbb{C}) \\ \downarrow {\scriptstyle i_b^*} & & \downarrow {\scriptstyle j_b^\star} & & \updownarrow \cong \\ H^{2r-1}(X_b; \mathbb{C}) \quad \cong & & H_{2r-1}(X_b; \mathbb{C}) & \xrightarrow{i_{b*}} & H_{2r-1}(\widetilde{Y}; \mathbb{C}). \end{array}$$

Here i_b^* is the natural pull-back and $j_b^\star$ is the Gysin morphism in homology and the isomorphism (1) is the Hard-Lefschetz isomorphism. Note that the map of Jacobians $i_b^* : J^r(\widetilde{Y}) \to J^r(X_b)$ descends from the first one and $i_{b\star}$ descends from the composition of the bottom row with the right "vertical" isomorphism (Poincaré duality). Finally, an element in $H^{2r-1}(\widetilde{Y}; \mathbb{C})$ that maps to zero in $H^{2r+1}(\widetilde{Y}; \mathbb{C})$ must vanish in $H_{2r+1}(Y; \mathbb{C})$ and therefore must vanish in $H^{2r-1}(X_b; \mathbb{C})$.
This proves that the restriction of the bottom row to the image of i_b^* is injective, and therefore that the restriction $i_{b\star}|_{\mathrm{image}(i_b^*)}$ is an isogeny on its image as required.

□

Step 5. The cohomology class of Z_B in $H^{2r}\big(Y_B; \mathbb{C}\big)$ is zero.

<u>*Proof of Step 5*</u>. First recall the exact sequence defining the Jacobian fibration: $0 \to R^{2r-1}\pi_*\mathbb{Z}/\text{torsion} \to \mathcal{H}^{2r-1}/F^r\mathcal{H}^{2r-1} \to \mathcal{J} \to 0$, where $\mathcal{H}^{2r-1} := R^{2r-1}\pi_*\mathbb{Z} \otimes \mathcal{O}_B$. From previous step we

know there exists $\xi \in J^r(\widetilde{Y})$ such that $i_b^*(\xi) = \nu_z(b)$ for any $b \in B$. Let ξ' be a lifting of ξ in $H^{2r-1}(\widetilde{Y};\mathbb{C})$. Then $\{i_b^*(\xi')\otimes 1\}_{b\in B} \in H^0(B,\mathcal{H}^{2r-1}/F^r\mathcal{H}^{2r-1})$ is a global section whose image in $H^0(B,\mathcal{J})$ is equal to ν_z (here, i_b^* is the pull-back $H^{2r-1}(\widetilde{Y},\mathbb{C}) \to H^{2r-1}(X_b,\mathbb{C})$). Therefore the image of ν_z in $H^1(B,R^{2r-1}\pi_*\mathbb{C})$ vanishes. On the other hand the Leray filtration of $Y_B \to B$ induces a natural map $\ker(H^{2r}(Y_B;\mathbb{C}) \to H^{2r}(X_b;\mathbb{C})) \to H^1(B,R^{2r-1}\pi_*\mathbb{C})$, which is an isomorphism because $B \subset \mathbb{P}^1$. On the other hand, under this identification, one knows that the image of ν_z corresponds to Z_B ([V2], Lemma 8.20).

□

Remembering our strategy, to find a contradiction, hence to conclude the proof of the Theorem for $d \gg 0$, it suffices to prove step 6 below. Here we shall make a strong use of the fact that ℓ is as in step 3, a assumption that was not necessary for steps 4 and 5.

Step 6. Keep our previous notation, in particular ℓ is as in step 3 and $B = \mathbb{P}^1 \setminus \mathcal{D}$. Then,

$$cl(Z_B)^\vee = 0 \in H^{2r}\big(Y_B;\mathbb{C}\big) \qquad \Longrightarrow \qquad cl(Z) = 0 \in H_{2r}(Y;\mathbb{C}).$$

Proof of Step 6. Let Y_ℓ be the blow-up of Y along the base locus of ℓ, let $\pi : Y_\ell \to \ell \cong \mathbb{P}^1$ be the natural map, consider the exact sequence of the pair $(Y_\ell,\, \pi^{-1}(\mathcal{D}))$ and maps and elements as indicated in the diagram below:

$$\begin{array}{ccccccc}
 & & H_{2r}(Y;\mathbb{C}) & & H^{2r}(Y_B;\mathbb{C}) & \ni\ cl(Z_B)^\vee & (=0) \\
 \nearrow{\scriptstyle \oplus j_{b_i *}} & & \downarrow{\scriptstyle \lambda^\star} & & \downarrow{\scriptstyle \cong} & \updownarrow & \\
 \oplus H_{2r}(X_{b_i};\mathbb{C}) & \xrightarrow{\varpi} & H_{2r}(Y_\ell;\mathbb{C}) & \xrightarrow{\rho} & H_{2r}(Y_\ell,\, \pi^{-1}(\mathcal{D});\mathbb{C}) & & \\
 \sum \xi_i & \mapsto & \lambda^\star(cl(Z)) & \mapsto & & \rho(\lambda^\star(cl(Z))) & (=0)
\end{array}$$

where, $i)$ $\{b_0 = o, b_1, ..., b_k\} = \mathcal{D}\cap\ell$ denotes the discriminant locus of the pencil; $ii)$ $j_{b_i} : X_{b_i} \hookrightarrow Y$ denotes the natural inclusion; $iii)$ $\lambda : Y_\ell \to Y$ denotes the natural projection, and so $\lambda^\star : H_{2r}(Y;\mathbb{C}) \to H_{2r}(Y_\ell;\mathbb{C})$ is the Gysin morphism in homology (note that λ is a l.c.i. morphism because the base locus of ℓ is contained in Y_{smooth}); $iv)$ we work under the natural identification $H_{2r}(\pi^{-1}(\mathcal{D});\mathbb{C}) \cong \oplus H_{2r}(X_{b_i};\mathbb{C})$ induced by the disjoint union decomposition $\pi^{-1}(\mathcal{D}) = \dot\cup X_{b_i}$, note that $\varpi = \oplus\iota_{b_i *}$ where $\iota_{b_i} : X_{b_i} \hookrightarrow Y_\ell$ is the natural inclusion.

We claim that, as $cl(Z_B)^\vee = 0$ by hypothesis,
(6.1) there exists an element $\sum\xi_i$ as indicated in the diagram and satisfying $\sum j_{b_i *}(\xi_i) = cl(Z)$.
The proof of (6.1) involves two statements, the first one is that $cl(Z_B)^\vee$ and $\rho(\lambda^\star(cl(Z)))$ correspond to each other under Lefschetz Duality [Sp] (the vertical isomorphism at the right of the diagram), and this is clear. So, as a consequence of the exactness of the pair sequence there exists an element $\sum\xi_i$ mapping to $\lambda^\star(cl(Z))$ via ϖ above. The second one is the chain of equalities

$$[\oplus j_{b_i *}]\left(\sum\xi_i\right) = \lambda_*\varpi\left(\sum\xi_i\right) = \lambda_*(\lambda^\star(cl(Z))) = cl(Z)$$

where, the first equality is trivial, the second one follows by the definition of $\sum\xi_i$, the third one is the non-trivial one and follows by ([F1], Proposition 6.7,.b). This concludes the proof of claim (6.1).

Now, for any $b \in \mathbb{P}^1$, we consider the Gysin map $j_b^\star$ associated to the inclusion $j_b : X_b \hookrightarrow Y$ and the diagram

$$(6.2) \qquad \begin{array}{ccc}
H_{2r+2}(Y;\mathbb{C}) & \xrightarrow{\gamma} & H_{2r}(Y;\mathbb{C}) \\
{\scriptstyle j_b^\star}\downarrow & & \nearrow{\scriptstyle j_{b *}} \\
H_{2r}(X_b;\mathbb{C}) & &
\end{array}$$

where γ denotes the cap-product with $cl(X)^\vee \in H^2(Y;\mathbb{C})$ (namely $\gamma = \cap\, cl(X)^\vee$, where X is any d-degree hypersurface section of Y). This diagram is commutative and therefore, in particular, the composition $j_{b *}\circ j_b^\star$ does not depend on b. We now show the following

(6.3) $\qquad$ for any $i \in \{0, ..., k\}$, the Gysin map $j^{\star}_{b_i}$ is surjective.

To prove (6.3), we first examine the case $i = 0$. Consider the following natural commutative diagram:

$$\begin{array}{ccc} H_{2r+2}(\widetilde{Y};\mathbb{C}) & \longrightarrow & H_{2r}(\widetilde{X}_o;\mathbb{C}) \\ \downarrow & & \downarrow \\ H_{2r+2}(Y;\mathbb{C}) & \xrightarrow{j^{\star}_o} & H_{2r}(X_o;\mathbb{C}), \end{array}$$

where the horizontal maps denote Gysin maps, and the vertical ones denote push-forwards. Since $\widetilde{X}_o$ is a smooth and very ample divisor on $\widetilde{Y}$ (see step 2), then the upper horizontal map is an isomorphism in view of Poincaré Duality and Hyperplane Lefschetz Theorem. Moreover the vertical map at the right is surjective: this follows by the Decomposition Theorem because the restriction $f|_{\widetilde{X}_o} : \widetilde{X}_o \to X_o$ is a desingularization of X_o (see remark 3). Thus, since the diagram commutes, then $j^{\star}_o$ is surjective.
Next assume $i \in \{1, \dots, k\}$, i.e. assume that $X := X_{b_i}$ has only one ordinary double point at a smooth point p of Y. Denote by $B_p(Y)$ and $B_p(\mathbb{P}^N)$ the blowing-up of Y and $\mathbb{P}^N$ at p, by E_Y and $E_{\mathbb{P}}$ the exceptional divisors, and by $\phi : B_p(Y) \to Y$ and $\psi : B_p(\mathbb{P}^N) \to \mathbb{P}^N$ the natural projections. Let $\widetilde{X}$ be the strict transform of X in $B_p(Y)$. Then $\widetilde{X}$ is a smooth Cartier divisor on $B_p(Y)$ and it is also very ample on $B_p(Y)$: in fact $\mathcal{O}_{B_p(Y)}(\widetilde{X}) \cong \mathcal{O}_{B_p(Y)}(\phi^*(dH_Y) - 2E_Y)$ is the restriction to $B_p(Y)$ of $\mathcal{O}_{B_p(\mathbb{P}^N)}(\psi^*(dH_{\mathbb{P}}) - 2E_{\mathbb{P}})$, which is very ample on $B_p(\mathbb{P}^N)$ (here we denote by $H_{\mathbb{P}}$ the hyperplane in $\mathbb{P}^N$, and by H_Y its restriction to $Y \subset \mathbb{P}^N$). Now, as before, consider the natural commutative diagram:

$$\begin{array}{ccc} H_{2r+2}(B_p(Y);\mathbb{C}) & \longrightarrow & H_{2r}(\widetilde{X};\mathbb{C}) \\ \downarrow & & \downarrow \\ H_{2r+2}(Y;\mathbb{C}) & \longrightarrow & H_{2r}(X;\mathbb{C}). \end{array}$$

As before the right vertical map is surjective because $\widetilde{X}$ is a desingularization of X. Moreover, since $\widetilde{X}$ is smooth and very ample on $B_p(Y)$, then, by Lefschetz Hyperplane Theorem for "intersection cohomology", the restriction map $IH^{2r-2}(B_p(Y);\mathbb{C}) \to IH^{2r-2}(\widetilde{X};\mathbb{C})$ is an isomorphism (remark 3). On the other hand, since $B_p(Y)$ has only isolated singularities, then $IH^{2r-2}(B_p(Y);\mathbb{C}) \cong H_{2r+2}(B_p(Y);\mathbb{C})$. Furthermore, as $\widetilde{X}$ is smooth, then $IH^{2r-2}(\widetilde{X};\mathbb{C}) \cong H^{2r-2}(\widetilde{X};\mathbb{C}) \cong H_{2r}(\widetilde{X};\mathbb{C})$. As a consequence, the map $H_{2r+2}(B_p(Y);\mathbb{C}) \to H_{2r}(\widetilde{X};\mathbb{C})$ is surjective. Finally, the commutativity of the diagram shows that also the map $H_{2r+2}(Y;\mathbb{C}) \to H_{2r}(X;\mathbb{C})$ is surjective. This concludes the proof of (6.3).
We now conclude the proof of step 6. By (6.3), for any $i \in \{0, ..., k\}$ there exists an element η_i satisfying $j^{\star}_{b_i}(\eta_i) = \xi_i$. Thus, we have

$$cl(Z) \quad = \quad \sum j_{b_i *}(\xi_i) \quad = \quad \sum j_{b_i *}\left(j^{\star}_{b_i}(\eta_i)\right) \quad = \quad \sum \gamma(\eta_i) \quad = \quad \gamma\left(\sum \eta_i\right)$$

where the first equality is (6.1), the second one is obtained substituting ξ_i with its expression $j^{*}_{b_i}(\eta_i)$, the third one follows by the commutativity of diagram (6.2) and the last one is trivial.
Going back to our diagram (6.2), the equality $cl(Z) = \gamma\left(\sum \eta_i\right)$ shows that $cl(Z)$ belongs to the image of the push-forward $j_{b\,*} : H_{2r}(X_b; \mathbb{C}) \to H_{2r}(Y; \mathbb{C})$, i.e. there exists $\alpha \in H_{2r}(X_b; \mathbb{C})$ satisfying $cl(Z) = j_{b\,*}(\alpha)$. On the other hand, choosing a smooth X_b, the composition

$$\begin{array}{ccccc} H_{2r}(X_b;\, \mathbb{C}) & \xrightarrow{j_{b\,*}} & H_{2r}(Y;\, \mathbb{C}) & \xrightarrow{j^{\star}_b} & H_{2r-2}(X_b;\, \mathbb{C}) \\ \alpha & \mapsto & cl(Z) & \mapsto & cl([Z]|_{X_b}) = 0 \end{array}$$

is injective by the Hard-Lefschetz theorem. Thus we deduce that $cl(Z) = 0$ as required.

□

So far we have proved **a)** and **b)**. To prove **a′)** and **b′)**, the previous proof can be adapted with minor changes. First, as before, the Theorem follows by the analogue of ($\star$) for $d \geq 3$. Then one observes that a Zariski general pencil ℓ has fibers X_b's that meet all components of our cycle Z properly and its singular fibers are finitely many: those not meeting Y_{sing} having only one ordinary double point, and those meeting Y_{sing} only at one point and generically (so, any of such sections is singular only at one point and its singularity is a general section of a "mild" singularity). Eventually, for such pencil, one proves the analogues of steps 4, 5 and 6.

As for it regards the latter, one has to work a little more: as in step 6, one reduces to prove that for any $i \in \{1, ..., k\}$ the Gysin map $j^\star_{b_i}$ is surjective, where $\{b_1, \dots, b_k\}$ denotes the discriminant locus of the pencil (compare with 6.3 in the proof of step 6). Set $X := X_{b_i}$. If X has only one ordinary double point at a smooth point p of Y, then the same argument we used in the proof of (6.3) applies (here the assumption $d \geq 3$ allows us to say that $\mathcal{O}_{B_p(\mathbb{P}^N)}(\psi^*(dH_{\mathbb{P}}) - 2E_{\mathbb{P}})$ is very ample). It remains to examine the case where X is a general hypersurface section through a singular point p of Y. Denote by $B_p(Y)$ and $B_p(\mathbb{P}^N)$ the blowings-up of Y and $\mathbb{P}^N$ at p, by E_Y and $E_{\mathbb{P}}$ the exceptional divisors, and by $\phi : B_p(Y) \to Y$ and $\psi : B_p(\mathbb{P}^N) \to \mathbb{P}^N$ the natural projections. Let $\widetilde{X}$ be the strict transform of X in $B_p(Y)$. Since the singularity of Y at p is "mild", then $B_p(Y)$ still has only isolated singularities and $\widetilde{X}$ is a smooth Cartier divisor on $B_p(Y)$. Moreover $\widetilde{X}$ is still very ample on $B_p(Y)$ because $\mathcal{O}_{B_p(Y)}(\widetilde{X}) \cong \mathcal{O}_{B_p(Y)}(\phi^*(dH_Y) - E_Y)$, and this line bundle is the restriction on $B_p(Y)$ of $\mathcal{O}_{B_p(\mathbb{P}^N)}(\psi^*(dH_{\mathbb{P}}) - E_{\mathbb{P}})$, which is very ample on $B_p(\mathbb{P}^N)$ because $d \geq 2$. At this point one may prove that $j^\star_{b_i}$ is surjective exactly as in the "tangential" case.

$\square$

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