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Algebraic Cycles on Abelian Varieties and their Decomposition.

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Sunto. – *In questo lavoro consideriamo una varietà abeliana X ed il suo anello di Chow $CH^{\bullet}(X)$ dei cicli algebrici modulo equivalenza razionale. Tramite la decomposizione di Künneth della diagonale $\Delta \subset X \times X$ è possibile ottenere delle formule esplicite per i proiettori associati alla decomposizione di Beauville* (1) *di $CH^{\bullet}(X)$, tali formule sono espresse in termini delle immagini dirette e inverse dei morfismi di moltiplicazione per un intero m. Il teorema* (4) *fornisce delle drastiche semplificazioni di tali formule, la Proposizione* (9) *ed il Corollario* (10) *forniscono alcuni risultati ad esse correlati.*

Summary. – *For an Abelian Variety X, the Künneth decomposition of the rational equivalence class of the diagonal $\Delta \subset X \times X$ gives rise to explicit formulas for the projectors associated to Beauville's decomposition* (1) *of the Chow ring $CH^{\bullet}(X)$, in terms of push-forward and pull-back of m-multiplication. We obtain a few simplifications of such formulas, see theorem* (4) *below, and some related results, see proposition* (9) *below.*

0. – Introduction.

Let X be an abelian variety of dimension n and denote by $CH_{\bullet}(X)$ its Chow group of algebraic cycles modulo rational equivalence. In our notation, $CH_d(X)$ is the subgroup of d-dimensional cycles and $CH^p(X) := CH_{n-d}(X)$ is the subgroup of p-codimensional cycles. For $m \in \mathbb{Z}$, let mult(m) denote the multiplication map $X \to X$, $x \mapsto mx$. By the use of Fourier-Mukai transform for abelian varieties (see [M] and [Be]), Beauville has established a decomposition

$$CH_d(X)_{\mathbb{Q}} = \bigoplus_{s=-d}^{n-d} [CH_d(X)_{\mathbb{Q}}]_s \tag{1}$$

where, by definition, $CH_d(X)_{\mathbb{Q}} = CH_d(X) \otimes \mathbb{Q}$ is the Chow group with $\mathbb{Q}$-

coefficients and the right-hand-side subgroups are defined as follows:

$$(2)\quad \begin{aligned}[CH_d(X)_{\mathbb{Q}}]_s &:= \{W \in CH_d(X)_{\mathbb{Q}} \mid \mathrm{mult}(m)_\star W = m^{2d+s} W,\ \forall m \in \mathbb{Z}\} \\ &= \{W \in CH^p(X)_{\mathbb{Q}} \mid \mathrm{mult}(m)^\star W = m^{2p-s} W,\ \forall m \in \mathbb{Z}\},\end{aligned}$$

where $p = n - d$ is the codimension of W.

This decomposition is a tool to understand cycles and rational equivalence on abelian varieties and it would give a beautiful answer to many questions concerning the Chow groups of abelian varieties (see [Be], [Bl], [J], [Ku] and [S]), provided that Beauville's vanishing conjecture [Be] holds. This conjecture states that the factors of $CH_d(X)$ with $s < 0$ vanish (see B.C. below). As pointed out in the abstract, by the use of Deninger-Murre projectors δ_i, (see [DM], [Ku]), the projections $CH_d(X) \to [CH_d(X)]_s$ with respect to Beauville's decomposition (1) can be written as linear forms of $\mathrm{mult}(m)_\star$ and $\mathrm{mult}(m)^\star$. Theorem (4) simplifies such explicit descriptions. A further simplification is given for the case where one works modulo a piece of the decomposition, see proposition (9); see corollary (10) for a reformulation of Beauville's conjecture.

1. – The algebraic set up.

We denote by $\omega(z)$ the series expansion of $\log(z+1)$. Namely,

$$\omega(z) := z - \frac{1}{2}z^2 + \frac{1}{3}z^3 \dots .$$

Furthermore, for k and j non-negative integers we define constants $a_{k,j}$ via the formal equality

$$\sum_{j=0}^{\infty} a_{k,j} z^j = \frac{1}{k!}\omega(z)^k$$

Let $A_r \in M_{r+1,r+1}(\mathbb{Q})$ be the matrix $(a_{k,j})$, where k and j run in $[0, \dots, r]$. Let $B_r \in M_{r+1,r+1}(\mathbb{Z})$ be the matrix $(b_{j,h})$, where j and h run in $[0, \dots, r]$ and where, by definition, $b_{j,h} = (-1)^{j-h}\binom{j}{h}$. It is understood that $\binom{j}{h} = 0$ provided that $h > j$. For $k = 0, 1, \dots, r$ we define linear forms $L_k^{(r)}(x_0, \dots, x_r)$ by the following equality:

$$\begin{pmatrix} L_0^{(r)} \\ \vdots \\ L_r^{(r)} \end{pmatrix} = A_r B_r \begin{pmatrix} x_0 \\ \vdots \\ x_r \end{pmatrix},$$

namely we define (observe that $a_{k,j}=0$, if $j<k$ and $b_{j,h}=0$, if $h>j$)

$$L_k^{(r)}(x_0, \dots, x_r) = \sum_{j=k}^{r} \sum_{h=0}^{j} a_{k,j}(-1)^{j-h}\binom{j}{h} x_h,$$

and for $k>r$ we define $L_k^{(r)}=0$.

We now introduce a numerical lemma, the proof of which is very straightforward (and omitted).

LEMMA 3. – *Let $j \geq 1$ and $\sigma \geq 0$ be integers. Then*

$$\sum_{h=0}^{j}(-1)^{j-h}\binom{j}{h} h^{\sigma} = \begin{cases} 0 & \text{if } \sigma<j \\ \sigma! & \text{if } \sigma=j\,. \end{cases}$$

2. – Projections of cycles.

Next, using linear forms $L_k^{(r)}$, we give a criterium to identify components (with respect to Beauville's decomposition 1) of the algebraic cycles. In the sequel, X denotes an abelian variety of dimension n; $W \in CH_d(X)_{\mathbb{Q}}$ denotes a rational algebraic cycle of dimension d and $p=n-d$ its codimension; furthermore, W_s denotes a component of W with respect to Beauville's decomposition (1), in particular s is an integer in the range $[-d, n-d]$. We also consider linear forms $L_k^{(r)}$ as introduced in the previous section. The interpretation, in terms of push-forward and pull-back of multiplication maps, of the decomposition of the diagonal $\Delta \in CH_n(X \times X)$ (see [DM], [Ku]) gives

$$W_s = \big([\log(\Delta)]^{\star_{\mathrm{rel}} 2d+s} \circ W\big)/(2d+s)! = \big({}^t[\log(\Delta)]^{\star_{\mathrm{rel}} 2n-2d-s} \circ W\big)/(2n-2d-s)!,$$

where $\star_{\mathrm{rel}}$ denotes the relative Pontryagin product on $CH_\bullet(X \times X)$ with respect to projection on the first factor and where, for $\alpha \in CH_\bullet(X \times X)$, ${}^t\alpha$ denotes its transpose. This equality in turn, in terms of our $L_k^{(r)}$ gives

$$\begin{aligned} W_s &= L_{2d+s}^{(r)}(\mathrm{mult}(0)_\star, \dots, \mathrm{mult}(r)_\star)W \\ &= L_{2p-s}^{(r)}\big(\mathrm{mult}(0)^\star, \dots, \mathrm{mult}(r)^\star\big)W, \quad \forall r \geq 2n\,. \end{aligned}$$

It is worthwhile to stress that the linear forms $L_k^{(r)}$ enter in a natural way (for $r=2n$) as an explicit version of Deninger-Murre-Künnemann projectors in terms of push-forward and pull-back of multiplication maps. The following theorem (4) goes further, it says that such equalities hold for r that takes smaller values (see (4_a) and (4_b) below). We also want to stress that linear forms $L_k^{(r)}$ have an increasing length with respect to r (see the list at the next page).

THEOREM 4. – *Let X, W and W_s be as above. Then*

$$(4_a)\qquad W_s = L^{(r)}_{2d+s}(\mathrm{mult}(0)_\star, \dots, \mathrm{mult}(r)_\star)W, \qquad \forall r \geqslant n+d;$$

$$(4_b)\qquad W_s = L^{(r)}_{2p-s}(\mathrm{mult}(0)^\star, \dots, \mathrm{mult}(r)^\star)W, \qquad \forall r \geqslant n+p.$$

Formulas (4_a) and (4_b) are obtained by using lemma (7) below. We shall also see that (4_b) can be refined: the equality there also holds for $r \geqslant n+p-\min\{d, 2\}$. A similar achievement does not hold for (4_a). As an explicit example we want to point out that for a 4-dimensional abelian variety and a 2-cycle W the known formula for projectors would give

$$W_1 = 8W - 14\ \mathrm{mult}(2)^\star W + \frac{56}{3}\mathrm{mult}(3)^\star W - \frac{35}{2}\mathrm{mult}(4)^\star W + \frac{56}{5}\mathrm{mult}(5)^\star W - \frac{14}{3}\mathrm{mult}(6)^\star W + \frac{8}{7}\mathrm{mult}(7)^\star W - \frac{1}{8}\mathrm{mult}(8)^\star W$$

meanwhile, by theorem (4), or better by remark (8), one has the simpler expression $W_1 = 4W - 3\ \mathrm{mult}(2)^\star W + \frac{4}{3}\mathrm{mult}(3)^\star W - \frac{1}{4}\mathrm{mult}(4)^\star W$.

REMARK. – Beauville's conjecture (see [Be]) states that

$$(\mathrm{B.C.})\qquad [CH_d(X)_{\mathbb{Q}}]_s = 0, \qquad \text{if } s<0.$$

As a consequence of theorem (4), proving the conjecture is equivalent to proving that either

$$L^{(n+d)}_{2d+s}(\mathrm{mult}(0)_\star, \dots, \mathrm{mult}(n+d)_\star) \quad \text{or} \quad L^{(n+p)}_{2p-s}(\mathrm{mult}(0)^\star, \dots, \mathrm{mult}(n+p)^\star)$$

acts trivially on $CH_d(X)_{\mathbb{Q}}$, for $s<0$. Another equivalent formulation for Beauville's conjecture (B.C.) is that the property (4_b) holds also for $r \geqslant 2p$ (this is trivial: since $L^{(2p)}_{2p-s} = 0$ for $s<0$, if (4_b) holds for $r = 2p$, B.C. holds as well; it is straightforward to check that the converse implication follows from the proof of theorem 4).

REMARK. – Let us look at (4_a) and (4_b). The operators

$$L^{(r)}_{2d+s}(\mathrm{mult}(0)_\star, \dots, \mathrm{mult}(r)_\star)$$

are non-trivial for $r \geqslant n+d$ and the operators $L^{(r)}_{2p-s}(\mathrm{mult}(0)^\star, \dots, \mathrm{mult}(r)^\star)$ are non-trivial for $r \geqslant n+p$. Infact, since $-d \leqslant s \leqslant n-d$, then $2d+s \leqslant n+d$ as well as $2p-s \leqslant n+p$.

Clearly, one has

$$\operatorname{mult}(0)_\star W = \begin{cases} 0 & \text{if } d = \dim W > 0\,; \\ \deg W \cdot o & \text{if } W \text{ is a } 0\text{-cycle, where } o \text{ is the origin of } X. \end{cases}$$

$$\operatorname{mult}(1)_\star W = W$$

For $n + d$ that takes the indicated value, the operators $L_k = L_k^{(n+d)}(\dots, \operatorname{mult}(i)_\star, \dots)$ act as follows.

n+d=1

$L_0 W = \operatorname{mult}(0)_\star W$
$L_1 W = -\operatorname{mult}(0)_\star W + W$

n+d=2

$L_0 W = \operatorname{mult}(0)_\star W$
$L_1 W = -\frac{3}{2}\operatorname{mult}(0)_\star W + 2W - \frac{1}{2}\operatorname{mult}(2)_\star W$
$L_2 W = \frac{1}{2}\operatorname{mult}(0)_\star W - W + \frac{1}{2}\operatorname{mult}(2)_\star W$

n+d=3

$L_0 W = \operatorname{mult}(0)_\star W$
$L_1 W = -\frac{11}{6}\operatorname{mult}(0)_\star W + 3W - \frac{3}{2}\operatorname{mult}(2)_\star W + \frac{1}{3}\operatorname{mult}(3)_\star W$
$L_2 W = \operatorname{mult}(0)_\star W - \frac{5}{2}W + 2\operatorname{mult}(2)_\star W - \frac{1}{2}\operatorname{mult}(3)_\star W$
$L_3 W = -\frac{1}{6}\operatorname{mult}(0)_\star W + \frac{1}{2}W - \frac{1}{2}\operatorname{mult}(2)_\star W + \frac{1}{6}\operatorname{mult}(3)_\star W$

n+d=4

$L_0 W = \operatorname{mult}(0)_\star W$
$L_1 W = -\frac{25}{12}\operatorname{mult}(0)_\star W + 4W - 3\operatorname{mult}(2)_\star W + \frac{4}{3}\operatorname{mult}(3)_\star W - \frac{1}{4}\operatorname{mult}(4)_\star W$
$L_2 W = \frac{35}{24}\operatorname{mult}(0)_\star W - \frac{13}{3}W + \frac{19}{4}\operatorname{mult}(2)_\star W - \frac{7}{3}\operatorname{mult}(3)_\star W + \frac{11}{24}\operatorname{mult}(4)_\star W$
$L_3 W = -\frac{5}{12}\operatorname{mult}(0)_\star W + \frac{3}{2}W - 2\operatorname{mult}(2)_\star W + \frac{7}{6}\operatorname{mult}(3)_\star W - \frac{1}{4}\operatorname{mult}(4)_\star W$
$L_4 W = \frac{1}{24}\operatorname{mult}(0)_\star W - \frac{1}{6}W + \frac{1}{4}\operatorname{mult}(2)_\star W - \frac{1}{6}\operatorname{mult}(3)_\star W + \frac{1}{24}\operatorname{mult}(4)_\star W$

From Beauville's conjecture point of view the first interesting case is $W_{-1} = L_5^{(8)}(\dots, \operatorname{mult}(i)_\star, \dots) = L_5^{(7)}\big(\dots, \operatorname{mult}(i)^\star, \dots\big)$, for $W \in CH^2(X)_{\mathbb{Q}}$ and $\dim X = 5$, see [Be]. Indeed, we have also $W_{-1} = L_5^{(r)}\big(\dots, \operatorname{mult}(i)^\star, \dots\big)$, for $r \geqslant 5 = n + p - \min\{d, 2\}$.

Next we prove theorem (4) and some related results. First, we recall that the Chow group of an abelian variety has two ring structures: the first one is given by the intersection product, the second one is given by the Pontryagin product, which we shall always denote by $\bigstar$. Consider the ring $CH_\bullet(X\times X)$ with the natural sum of cycles and the relative Pontryagin product with respect to projection on the first factor $X\times X\rightarrow X$ (in other terms, we consider Pontryagin product on $X\times X$ regarded as an abelian scheme over X via the first-factor-projection). Let $\Delta\in CH_n(X\times X)$ be the diagonal and let $E=X\times\{o\}\in CH_n(X\times X)$ be the unit of $CH_\bullet(X\times X)$ with respect to the product above, where o is the origin of X. The projectors $\delta_0,\dots,\delta_{2n}$ are defined by (see [Ku], pag. 200)

$$\begin{aligned}\delta_j &= \frac{1}{(2n-j)!}[\log(\Delta)]^{\bigstar_{\mathrm{rel}}2n-j}\\ &= \frac{1}{(2n-j)!}\left[(\Delta-E)-\frac{1}{2}(\Delta-E)^{\bigstar_{\mathrm{rel}}2}+\frac{1}{3}(\Delta-E)^{\bigstar_{\mathrm{rel}}3}\dots\right]^{\bigstar_{\mathrm{rel}}2n-j}.\end{aligned}$$

Since $(\Delta-E)^{\bigstar_{\mathrm{rel}}2n+1}=0$ (see [Ku]), the series above are infact finite sums. Now let Δ_m denote the graph of $\mathrm{mult}(m)$. By Deninger, Murre and Künnemann theorem (see [DM], [Ku]) we have

$$\begin{aligned}&[{}^t\Delta_m]\circ\delta_j=m^j\delta_j,\quad \forall m\in\mathbb{Z},\ 0\leqslant j\leqslant 2n;\\ &{}^t\delta_j=\delta_{2n-j},\quad \forall\, 0\leqslant j\leqslant 2n;\end{aligned}\tag{5}$$

where the composition above is the composition of correspondences and where, for $\sigma\in\mathrm{Corr}(A,B)$, ${}^t\sigma\in\mathrm{Corr}(B,A)$ denotes its transpose. As a consequence, for $W\in CH_d(X)_{\mathbb{Q}}$ and $0\leqslant j\leqslant 2n$, one has

$$\begin{aligned}\mathrm{mult}(m)^\star(\delta_j\circ W)&=[{}^t\Delta_m]\circ(\delta_j\circ W)\\ &=m^j(\delta_j\circ W),\qquad \forall m\in\mathbb{Z}.\end{aligned}$$

Clearly, one identifies $CH_\bullet(X)$ with $\mathrm{Corr}(\mathrm{Spec}\,\mathbb{C},X)=CH_\bullet(\mathrm{Spec}\,\mathbb{C}\times X)$. Thus, by the definition (2) one has

$$\delta_j\circ W\in[CH_d(X)_{\mathbb{Q}}]_s,\qquad s:=2n-2d-j.\tag{5'}$$

Since $\sum\delta_j=\Delta$ acts as the identity map, (5) and (5′) give

$$W_s=\delta_{2n-2d-s}\circ W={}^t\delta_{2d+s}\circ W\tag{5''}$$

where, as usual, W_s denotes the component of W with respect to Beauville's decomposition (1).

For the proof of theorem (4) we need the following.

LEMMA 6. – *Let W be as in the theorem. Then*

$$[(\Delta - E)^{\bigstar_{\mathrm{rel}} j}] \circ W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)_\star W$$

$$^t[(\Delta - E)^{\bigstar_{\mathrm{rel}} j}] \circ W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^\star W$$

PROOF. – Since E is the unit for relative Pontryagin product and since $\Delta^{\bigstar_{\mathrm{rel}} h} \circ W = \operatorname{mult}(h)_\star W$ as well as $^t[\Delta^{\bigstar_{\mathrm{rel}} h}] \circ W = \operatorname{mult}(h)^\star W$, the two equalities follow by a straightforward computation. ■

LEMMA 7. – *Let W be as in the theorem. Then*

$$(7_a) \qquad [(\Delta - E)^{\bigstar_{\mathrm{rel}} j}] \circ W = 0, \qquad \forall j \geqslant n + d + 1\,;$$

$$(7_b) \qquad ^t[(\Delta - E)^{\bigstar_{\mathrm{rel}} j}] \circ W = 0, \qquad \forall j \geqslant n + p + 1\,.$$

PROOF. – We prove (7_b), the proof of (7_a) is very similar. By lemma (6), we have to show that for $j \geqslant n + p + 1$ one has

$$\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^\star W = 0\,.$$

By linearity of the left-hand-side operator we are free to assume that W belongs to one of the factors from Beauville decomposition (1), namely we are free to assume that $W \in [CH_d(X)_{\mathbb{Q}}]_s$ for some $s \in [-d, n-d]$. Thus (see 2), we assume that $\operatorname{mult}(m)^\star W = m^{2p-s} W$, $\forall m \in \mathbb{Z}$. It follows

$$\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^\star W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} h^{2p-s} W\,.$$

For s in the range above, the range for $2p - s$ is $[p, n+p]$; in particular, we have $2p - s < j$. By lemma (3), the coefficient $\sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} h^{2p-s}$ vanishes. Then we are done. ■

PROOF (of theorem 4). – We start with formula (4_a). Let $k = 2d + s$. Then, we have

$$W_s = \delta_{2n-2d-s} \circ W = \frac{1}{(2d+s)!} [\log(\Delta)^{\bigstar_{\mathrm{rel}} 2d+s}] \circ W$$

$$= \sum_{j=k}^{2n} a_{k,j} (\Delta - E)^{\bigstar_{\mathrm{rel}} j} \circ W\,.$$

Now observe that by lemma (7), we have $(\Delta - E)^{\bigstar_{\mathrm{rel}} j} \circ W = 0$ for $j \geqslant n + d + 1$.

Thus, the summation above can be taken up to r, provided that $r \geq n + d$. It follows that

$$W_s = \sum_{j=k}^{r} a_{k,j} (\Delta - E)^{\star_{\mathrm{rel}} j} \circ W, \qquad \forall r \geq n + d.$$

Looking at the definition of the operators $L_k^{(r)}$ it is then clear that (4_a) follows from the first equality from lemma (6),

$$(\Delta - E)^{\star_{\mathrm{rel}} j} \circ W = \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)_\star W .$$

The proof of formula (4_b) is similar. For $r \geq n + p$ we have

$$\begin{aligned} W_s = {}^t\delta_{2d+s} \circ W &= \frac{1}{(2p-s)!} {}^t[\log(\Delta)^{\star_{\mathrm{rel}} 2p-s}] \circ W \\ &= \sum_{j=2p-s}^{2n} a_{2p-s,j} \, {}^t[(\Delta - E)^{\star_{\mathrm{rel}} j}] \circ W \\ &= \sum_{j=2p-s}^{r} a_{2p-s,j} \, {}^t[(\Delta - E)^{\star_{\mathrm{rel}} j}] \circ W \\ &= \sum_{j=2p-s}^{r} a_{2p-s,j} \sum_{h=0}^{j} (-1)^{j-h} \binom{j}{h} \operatorname{mult}(h)^\star W \\ &= L_{2p-s}^{(r)} (\operatorname{mult}(0)^\star, \ldots, \operatorname{mult}(r)^\star) \, W \end{aligned}$$

where the 4^{th} equality follows by lemma (7), the 5^{th} equality follows by lemma (6) and the 6^{th} equality follows by the definition of the operators $L_k^{(r)}$. ■

REMARK 8. – The equality (7_b) can be improved. We have,

$$(8') \qquad {}^t[(\Delta - E)^{\star_{\mathrm{rel}} j}] \circ W = 0, \qquad \forall j \geq n + p + 1 - \delta$$

where $\delta = \min\{d, 2\}$. Infact, since $[CH_d(X)_{\mathbb{Q}}]_s = 0$ provided that $s \leq \min\{-d+1, -1\}$ (see [Be]), the actual range for s can be shrinked to $\min\{-d+2, 0\} \leq s \leq n - d$. Thus in turn, one obtains $(8')$ by the same proof of (7_b). As a consequence, (4_b) can be refined: the equality there also holds for all $r \geq n + p - \delta$ (where δ is as above).

Furthermore, for the same reason, if Beauville's conjecture (B.C.) mentioned above holds, then

$${}^t[(\Delta - E)^{\star_{\mathrm{rel}} j}] \circ W = 0, \qquad \forall j \geq 2p + 1$$

In particular, if W is a divisor (hence it satisfies B.C.), then ${}^t[(\Delta - E)^{\star_{\mathrm{rel}} 3}] \circ$

$W=0$, namely $3W-3\ \mathrm{mult}(2)^{\star}W+\mathrm{mult}(3)^{\star}W=0$, which is obvious (in the case of divisors, this kind of computations provide trivial results).

Now fix s, working modulo $\bigoplus_{l\geq s+1}[CH_d(X)_{\mathbb{Q}}]_l$, or rather modulo $\bigoplus_{l\leq s-1}[CH^p(X)_{\mathbb{Q}}]_l$, yields simpler formulas than the ones from theorem (4); furthermore, it can be used to provide a reformulation for Beauville's conjecture (B.C.), see corollary (10) and the example below.

PROPOSITION 9. – *Let W and W_s be as in the theorem. Then*

$$(9_a)\qquad W_s=\frac{1}{(2d+s)!}\sum_{h=0}^{2d+s}(-1)^{2d+s-h}\binom{2d+s}{h}\mathrm{mult}(h)_{\star}W,\qquad \text{modulo } \bigoplus_{l\geq s+1}[CH_d(X)_{\mathbb{Q}}]_l$$

Furthermore,

$$(9_b)\qquad W_s=\frac{1}{(2p-s)!}\sum_{h=0}^{2p-s}(-1)^{2p-s-h}\binom{2p-s}{h}\mathrm{mult}(h)^{\star}W,\qquad \text{modulo } \bigoplus_{l\leq s-1}[CH^p(X)_{\mathbb{Q}}]_l$$

PROOF. – We prove (9_b). Let $K=\frac{1}{(2p-s)!}\sum_{h=0}^{2p-s}(-1)^{2p-s-h}\binom{2p-s}{h}\mathrm{mult}(h)^{\star}$. It suffices to prove that

$$KW=\begin{cases}0 & \text{if } W\in[CH_d(X)_{\mathbb{Q}}]_l,\ l\geq s+1\\ W & \text{if } W\in[CH_d(X)_{\mathbb{Q}}]_s.\end{cases}$$

This is clear by the proof of (7_b); as for the case $W\in[CH_d(X)_{\mathbb{Q}}]_s$, the equality $KW=W$ follows since, by lemma (3), the coefficient $\sum_{h=0}^{\sigma}(-1)^{\sigma-h}\binom{\sigma}{h}h^{\sigma}$ equals $\sigma!$ (here $\sigma=2p-s$). The proof of (9_a) is similar. ■

A straightforward consequence of (9_b) is the following.

COROLLARY 10. – *Let X be as in the theorem. Then, it satisfies Beauville's conjecture for d-dimensional cycles if and only if*

$$\sum_{h=0}^{k}(-1)^{k-h}\binom{k}{h}\mathrm{mult}(h)^{\star}$$

acts trivially on $CH_d(X)_{\mathbb{Q}}$ for $k\geq 2p+1$, where $p=n-d$ as usual.

For 5-dimensional abelian varieties the only *bad* component that might exist is $[CH_3(X)_{\mathbb{Q}}]_{-1}$. Then, by the corollary above it follows that a 5-dimensional abelian variety X satisfies Beauville's conjecture (B.C.) if and only if

$$5W - 10\ \mathrm{mult}(2)^\star W + 10\,\mathrm{mult}(3)^\star W - 5\ \mathrm{mult}(4)^\star W + \mathrm{mult}(5)^\star W = 0,$$

for all $W \in CH_3(X)_{\mathbb{Q}}$.

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