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A geometrical argument for a theorem of G. E. Welters

Giambattista Marini

Dipartimento di matematica, II Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Roma, Italy (e-mail: marini@axp.mat.uniroma2.it)

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The existence of a one parameter family of trisecants to the Kummer variety of an indecomposable principally polarized abelian variety characterizes Jacobians. This result was first proved by Gunning in [G] under additional hypotheses. Then Welters removed the additional hypotheses and considered the degenerate cases, see [W1] and [W2]. In this note we provide a short geometrical argument for the inflectionary case.

Theorem (Welters). *Let* $(X, [\Theta])$ *be an indecomposable principally polarized abelian variety. Let* Y *be the Artinian subscheme of* X *of length 3 supported at 0 defined as the image of* $Spec\,\mathbb{C}[\epsilon]/(\epsilon^3)$ *in* X *via the immersion* $\epsilon \mapsto \epsilon 2D_1 + \epsilon^2 2D_2$ *, where* $D_1 \neq 0$*. Denote by* $K : X \to |2\Theta|$ *the Kummer morphism. Assume that the algebraic subset*

$$V \quad = \quad \{\, 2u \in X \mid u + Y \subset K^{-1}(l) \text{ for some line } l \subset |2\Theta| \,\}\,,$$

has positive dimension. Then V *is a smooth irreducible curve through zero, and* $(X, [\Theta])$ *is the polarized Jacobian of* V*.*

The proof of this theorem is naturally divided in two steps. First one proves that the endomorphism δ associated with the pair $(\mathcal{C}, \Theta)$ is a positive multiple of the identity; here $\mathcal{C}$ is any curve contained in V. Then one proves that δ is the identity over the flow generated by D_1 (thus δ is the identity and Matsusaka's criterion applies, moreover $\mathcal{C} = V$ by curve theory). For the first step we go back to Welters' homological argument in [W1]. For the second one we present a geometrical argument. We recall that $\delta(x)$ is, by definition, the sum in X of the points of the intersection $\mathcal{C}.\Theta_x$, opportunely translated in order to have $\delta(0) = 0$ (namely $\delta(x) = \text{sum}\, \mathcal{C}.(\Theta_x - \Theta)\,$). We shall use the following remark.

1 Remark. Assume that V and W are subvarieties of X of complementary dimensions and assume that they intersect transversally at distinct points $p_1, \ldots, p_r$. If a point x is in a small analytic neighborhood of zero, we are allowed to number the points of the intersection $V.W_x$ according to the numbering of the p_i's. Thus, we can single out the contribution of p_i to the differential $d\,\delta$. We shall denote this contribution by $d\,\delta_{p_i}$. Let D be a vector in $T_0(X)$, or better an invariant vector field on X. We claim that, for $d\,\delta_{p_i}(D)$, which is the i^{th}-contribution to $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\text{sum}\,\big(V.(W_{\epsilon D}-W)\big)$, the following holds. If D is tangent to V at p_i, then $d\,\delta_{p_i}(D) = D$. If D is tangent to W at p_i, then $d\,\delta_{p_i}(D) = 0$. Our claim is a trivial consequence of the fact that since V and W intersect transversally at p_i, for the computation of the contribution δ_{p_i} up to its first order, we are allowed to work with tangent spaces.

Let us go back to the hypothesis of the theorem. The image under K of the germ of curve $u+Y = u+2\epsilon D_1+2\epsilon^2 D_2$ is the germ of curve $\vec{\vartheta}(u+2\epsilon D_1+2\epsilon^2 D_2) = \vec{\vartheta}(u)+2\epsilon D_1\vec{\vartheta}(u)+2\epsilon^2(D_1^2+D_2)\vec{\vartheta}(u) \bmod(\epsilon^3)$, where $\vec{\vartheta} := (\ldots, \vartheta_\sigma, \ldots)$, $\{\vartheta_\sigma\}$ is a basis of $H^0(X, 2\Theta)$. The image germ above, viewed in $\mathbb{P}H^0(X,2\Theta)$, is contained in a straight line if and only if the three terms of degrre 0, 1 and 2 are dependent, thus

$$2u \in V \quad \text{if and only if} \quad \operatorname{rank}\big(\vec{\vartheta}(u)\;\; D_1\vec{\vartheta}(u)\;\; (D_1^2+D_2)\vec{\vartheta}(u)\big) \leq 2\,.$$

Observe that V parametrizes inflectionary points of the Kummer variety. Also, we recall that by a well know degeneration of Fay's trisecant formula (e.g. see [W1]),

$$(\star)\;\; x \in V\,,\; x \neq 0 \qquad \text{if and only if} \qquad \Theta \cap \Theta_x \subset D_1\Theta \cup D_1\Theta_x\,,$$

where $D_1\Theta := \Theta \cap \{D_1\theta = 0\}$, and θ is a generator of $H^0(X,\Theta)$.

2 Remark (see [W2]). If V has positive dimension at 0, then V is smooth, one-dimensional at 0. Furthermore $T_0(V) = \langle D_1\rangle$. To prove this claim, first observe that the condition

$$\operatorname{rank}\big(\vec{\vartheta}(u)\;\; D_1\vec{\vartheta}(u)\;\; (D_1^2+D_2)\vec{\vartheta}(u)\big) \leq 2$$

gives rise to natural determinantal equations defining V as a scheme. These equations are the determinants equated to zero of the minors of order 3 of the matrix above. For computational purpose it is convenient to work with $\Lambda^3 H^0(X,2\Theta)$, so that

$$V = \Big\{ 2u \;\Big|\; f(u) := \vec{\vartheta}(u) \wedge D_1\vec{\vartheta}(u) \wedge (D_1^2+D_2)\vec{\vartheta}(u) = 0 \Big\}\,.$$

Furthermore, once for all we recall that $\mathcal{D}\vec{\vartheta}(0) = 0$ whenever $\mathcal{D}$ is a derivative of odd order. Note that, in particular, $f(0) = 0$, i.e. $0 \in V$.

As $E \in T_0(V)$ if and only if $Ef(0) = 0$, to prove the claim it suffices to show that

$$Ef(0) = 0 \qquad \text{if and only if} \qquad E \in \langle D_1 \rangle .$$

We have $Ef(0) = E\big(\vec{\vartheta} \wedge D_1\vec{\vartheta} \wedge (D_1^2 + D_2)\vec{\vartheta}\big)(0) = \vec{\vartheta}(0) \wedge ED_1\vec{\vartheta}(0) \wedge D_1^2\vec{\vartheta}(0)$ which equals zero if and only if $E \in \langle D_1 \rangle$ by Wirtinger's theorem stating that the matrix $\big(\vec{\vartheta}(0), ..., E_iE_j\vec{\vartheta}(0), ...\big)$ has maximal rank $1 + \frac{1}{2}n(n+1)$, where $n := \dim(X)$ and $\{E_1, ..., E_n\}$ is a basis of $T_0(X)$, provided that $(X, [\Theta])$ is indecomposable. Thus we are done. We now come to the proof of the theorem.

Proof (of the Theorem). Let $\mathcal{C}$ be an irreducible (and reduced) curve contained in V. We want to prove that the endomorphism δ is the identity. By hypothesis, the following inclusions hold

$$(\star) \qquad \Theta \cap \Theta_x \quad \subset \quad D_1\Theta \cup D_1\Theta_x , \qquad \forall x \in V,\ x \neq 0 .$$

Step 1. The endomorphism δ is a multiple of the identity.
This follows by Welters' homological argument (see [W1], p. 503), which we now recall for the sake of completeness. By the inclusions above there exist algebraic cycles of codimension 2 in X, M' and M, such that

$$\Theta \cap \Theta_x \quad = \quad M' + M_x , \qquad \forall\, x \in \mathcal{C},\ x \neq 0 .$$

The M_x 's cover Θ as x moves in $\mathcal{C}$. It follows that the Pontryagin product of homology classes $[M] \star [\mathcal{C}]$ is $c \cdot [\Theta]$, where c is a constant. On the other hand, since $\Theta_{-x} \cap \Theta = M'_{-x} + M$ and since the cycle $-\mathcal{C}$ is homologically equivalent to the cycle $\mathcal{C}$, there exists a constant c' such that the Pontryagin product $[M'] \star [\mathcal{C}]$ equals $c' \cdot [\Theta]$. Thus $[\Theta^2] \star \mathcal{C} = [M' + M] \star [\mathcal{C}] = (c' + c) \cdot [\Theta] = const[\Theta^2] \star [\Theta^{n-1}]/(n-1)!$. As Θ is ample, the map $\big([\Theta^2] \star\big) : H_2(X) \longrightarrow H_{2n-2}(X)$ is an isomorphism. It follows that $[\mathcal{C}] = const[\Theta^{n-1}]/(n-1)!$, so that the endomorphism δ is a multiple of the identity as claimed. This multiple must be positive because $\mathcal{C}$ is effective and Θ is ample.

Step 2. The differential $d\,\delta$ sends the vector field D_1 to itself.
Both the inclusions $(\star)$ and the endomorphism δ are independent on the translate of the theta divisor we choose. Denote by $D_1\mathcal{C}$ the set of points p of $\mathcal{C}$ such that D_1 is tangent to $\mathcal{C}$ at p (in particular $D_1\mathcal{C}$ includes singular points of $\mathcal{C}$). Note that $D_1\mathcal{C}$ is finite. Otherwise, the curve $\mathcal{C}$ would be D_1-invariant, thus it would be an elliptic curve, and the image of δ would be contained in the translate of $\mathcal{C}$ through 0, contradicting that δ is a non-zero multiple of the identity. Modulo translations of Θ we can assume the following:

1) $\mathcal{C} \not\subset \Theta$;
2) $0 \in \Theta_{\text{smooth}}$;
3) $D_1 \notin T_0(\Theta)$;
4) $\mathcal{C} \cap \Theta_{\text{sing}}$ is empty;
5) $D_1\mathcal{C} \cap D_1\Theta$ is empty.

Indeed, in order to find a translate $\tilde{\Theta} := \Theta_{-x}$ satisfying 1, 2, 3, 4, 5, it suffices to require that the point x is in

$$\Theta_{\text{smooth}} - \Big(\Delta\left(((\{0\} \cup D_1\mathcal{C}) \times D_1\Theta) \cup (\mathcal{C} \times \Theta_{\text{sing}})\right) \\ \cup \{x \in X \,|\, \mathcal{C} \subset \Theta_{-x}\}\Big),$$

where Δ is the difference map $\mathcal{C}\times\Theta \to X$, $(p.q) \mapsto q-p$. We claim that the set above is not empty. The set $\Delta\left(((\{0\}\cup D_1\mathcal{C})\times D_1\Theta)\cup(\mathcal{C}\times\Theta_{\text{sing}})\right)$ has at most dimension $n-2$, thus it does not contain Θ_{smooth} (here we use Ein-Lazarsfeld's theorem on Arbarello-De Concini's conjecture stating that $\dim\Theta_{\text{sing}} \le n-3$, cf. [EL]). If the set $A := \{x \in X | \mathcal{C} \subset \Theta_{-x}\}$ has dimension $n-1$, the inverse image $\Delta^{-1}(A)$ is $\mathcal{C} \times \Theta$. In fact, the product $\mathcal{C} \times \Theta$ is irreducible and the fiber $\Delta^{-1}(x)$ has positive dimension for all $x \in A$ (as $\mathcal{C} \subset \Theta_{-x}$, $\mathcal{C} + x \subset \Theta$ and $\Delta^{-1}(x)$ is the set of pairs $(p, p+x)_{p\in\mathcal{C}}$). As a consequence, if Θ_{-x} meets $\mathcal{C}$, then it contains $\mathcal{C}$. This leads to a contradiction. In fact, it would imply that the general translate of Θ does not meet $\mathcal{C}$. This concludes the proof of our claim.

By remark 2, if $\mathcal{C}$ contains zero, then it is smooth at zero and $D_1 \in T_0(\mathcal{C})$. It follows that if $0 \in \mathcal{C}$, the theta divisor meets $\mathcal{C}$ transversally at zero by the 3^{th} condition. Set now $\{x_1, \dots, x_r\} := \Theta \cap (\mathcal{C} - \{0\})$. As 0 and x_i are in Θ, the point x_i is in the left hand side of the inclusion $\Theta \cap \Theta_{x_i} \subset D_1\Theta \cup D_1\Theta_{x_i}$, so that it must be in the right hand side. This implies that either D_1 is in the tangent space $T_{x_i}(\Theta)$, or D_1 is in the tangent space $T_0(\Theta)$. The latter case does not occur by the 3^{th} condition, thus D_1 is in the tangent space $T_{x_i}(\Theta)$ for all i. On the other hand, by the 5^{th} condition, the intersection $D_1\mathcal{C}\cap D_1\Theta$ is empty, so that $D_1 \notin T_{x_i}(\mathcal{C})$. In particular, Θ and $\mathcal{C}$ meet transversally at the x_i's, hence they meet transversally (recall that if $0 \in \mathcal{C}$, then Θ and $\mathcal{C}$ meet transversally at zero). We now apply remark 1. Since D_1 is in the tangent space $T_{x_i}(\Theta)$ for all i, then $d\,\delta_{x_i}(D_1) = 0$ for all i ($d\,\delta_{x_i}$ has been introduced in remark 1). If $\mathcal{C}$ does not contain 0, it would follow $d\,\delta(D_1) = 0$. This is impossible because, by the first step, $d\,\delta(D_1)$ is a positive multiple of D_1. On the other hand, again by remark 1, $d\,\delta_0(D_1)$ equals D_1 because D_1 is tangent to $\mathcal{C}$ at zero, and it is transverse to the tangent space $T_0(\Theta)$. Taking the sum over all points of the intersection $\mathcal{C}.\Theta$ we obtain that the image of D_1 under the differential $d\,\delta$ is D_1 as claimed.

q.e.d.

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