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A geometrical proof of Shiota's theorem on a conjecture of S. P. Novikov

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Abstract. We give a new proof of Shiota's theorem on Novikov's conjecture, which states that the K.P. equation characterizes Jacobians among all indecomposable principally polarized abelian varieties.

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The Kummer variety of a Jacobian has a 4-parameter family of trisecants. Using Riemann's relations, Fay's identity and limit considerations, this property has been translated in a hierarchy of non-linear partial differential equations which is satisfied by the theta function of a Jacobian (see [F], [Mu], [Du], [Kr], [AD3]).

Novikov's conjecture stated that if a theta function associated with an indecomposable principally polarized abelian variety $(X, [\Theta])$ satisfies the K.P. equation, the first equation of the hierarchy, then $(X, [\Theta])$ is the Jacobian of a complete irreducible smooth curve. Shiota originally proved the conjecture in [S] by the use of hard techniques from the theory of non-linear partial differential equations. His proof was later simplified by Arbarello and De Concini (see [AD2]). We give a proof of the theorem which is more geometrical in character; in particular we avoid a technical point, namely Shiota's Lemma 7, which is instrumental in both Shiota's and Arbarello–De Concini's proofs. For our proof, we follow Arbarello and De Concini algebro–geometrical attempt to solve the problem (see [A] and [AD3]) and we go further. First, let us recall that in order to prove Novikov's conjecture, it suffices to recover the whole K.P. hierarchy from its first equation (because of Welters' version of Gunning's criterion). The key point in Arbarello and De Concini geometrical approach is that, no matter what are the parameters in the equations in the K.P. hierarchy, it turns out that the terms to be equated to zero form a sequence of sections of the line bundle $\mathcal{O}(2\Theta)$. One needs to find parameters that make this sequence into the identically zero sequence. The difficulty comes from the fact that the theta divisor may have, a priori, a difficult geometry. The key object in the approach in [A] and [AD3] is the subscheme $D_1\Theta$ of Θ defined by the zeroes of the section of $\mathcal{O}_\Theta(\Theta)$ associated with $D_1\theta$, where D_1 is an invariant vector field that appears in the expression of the first equation of the hierarchy, and θ is the theta function associated with $\mathcal{O}_X(\Theta)$. As it was pointed out in [A] and [AD3], the reduced components of $D_1\Theta$ do not create much trouble. They provide a geometrical proof of the conjecture under the additional hypotheses that the singular locus of the theta divisor has codimension at least 2, and that the scheme $D_1\Theta$ does not contain components which are invariant under the D_1-flow. We remove Arbarello–De Concini's additional hypotheses by proving the following. If the K.P. equation holds, the components of codimension one of the singular locus of the theta divisor are invariant under the D_1-flow (for this we make use of a result of Kollár about the singularities of the theta divisor). Therefore every component of $D_1\Theta$ which creates trouble is D_1-invariant and, in particular, it contains a translate of an abelian subvariety of X. We then prove that the theta function of an abelian variety which contain an abelian subvariety as above, is not a solution of the K.P. equation. For this we combine an algebraic computation which was discovered by Shiota (namely his Lemmas A and B, which we restate and reprove for the convenience of the reader), and a technical lemma on the obstructions to recover the K.P. hierarchy (Lemma 3.11).

For the discrete analogue to Novikov's conjecture see [De]. For further discussions see [AD3], [Do], [GG], [Ma].

1. Introduction

Let C be a smooth complex curve, $J(C)$ its Jacobian, $\text{Pic}^d(C)$ the *Picard group* of line bundles of degree d on C and Γ the image of C via the Abel–Jacobi embedding associated with an element of $\text{Pic}^{-1}(C)$.

Let $(X, [\Theta])$ be an i.p.p.a.v. (indecomposable, principally polarized, abelian variety) of dimension n, and let Θ be a symmetric representative of the polarization. We shall denote by θ a theta function associated with $\mathcal{O}_X(\Theta)$; in particular, θ is naturally a nonzero section of $\mathcal{O}_X(\Theta)$.

The image of the morphism

$$K : X \to |2\Theta|^*$$

associated with the base-point-free linear system $|2\Theta|$ is a projective variety which is called the Kummer variety of $(X, [\Theta])$.

The Kummer variety of $J(C)$ has a rich geometry in terms of trisecants and flexes which is a consequence of the equality

$$W^0_{g-1} \cap (W^0_{g-1} + p - q) = (W^1_g - q) \cup (W^0_{g-2} + p) \quad \forall\, p, q \in C, \quad p \neq q,$$

where $W^r_d = \{|D| \in \text{Pic}^d(C) \mid \dim|D| \geqslant r\}$. Indeed, the inclusion

$$\Theta_\beta \cap \Theta_\gamma \subset \Theta_\alpha \cup \Theta_{\beta+\gamma-\delta}, \quad \forall\, \alpha, \beta, \gamma, \delta \in \Gamma, \quad \beta \neq \gamma,$$

(where $\Theta_p := \Theta + p$), the linear dependence of the sections

$$\theta(z-\alpha)\cdot\theta(z-\beta-\gamma+\delta), \qquad \theta(z-\beta)\cdot\theta(z-\alpha-\gamma+\delta),$$

$$\theta(z-\gamma)\cdot\theta(z-\alpha-\beta+\delta),$$

and the collinearity in the projective space $|2\Theta|^*$ of the points

$$K(\xi+\alpha), \qquad K(\xi+\beta),$$
$$K(\xi+\gamma), \quad \forall\,\alpha,\beta,\gamma,\delta\in\Gamma, \quad \forall\,\xi\in\tfrac{1}{2}(\delta-\alpha-\beta-\gamma),$$

are all different translations (via Abel and Riemann's theorems) of the previous equality. In particular, once distinct points α,β,γ are fixed, one has a family of trisecants parametrized by $\frac{1}{2}\Gamma$. Considering the limit situation where β and γ tend to α one obtain a family of flexes parametrized by $\frac{1}{2}\Gamma$.

This property has been used to characterize Jacobians among all principally polarized abelian varieties (see [G], [W]). Welters' improvement of Gunning's theorem states that an i.p.p.a.v. (X,Θ) is a Jacobian if and only if there exists an Artinian subscheme Y of X of length 3, such that the algebraic subset $V=\{2\xi \mid \xi+Y\subset K^{-1}(l)$ for some line $l\subset|2\Theta|^*\}$ has positive dimension at some point (if this is the case it turns out that V is isomorphic to the curve C). In particular one has:

PROPOSITION 1.1 [AD1]. *Let $(X,[\Theta])$ be an i.p.p.a.v.. The following statements are equivalent:*

(a) *the i.p.p.a.v. $(X,[\Theta])$ is isomorphic to the Jacobian of a curve;*
(b) *there exist invariant vector fields $D_1\neq 0, D_2,\ldots,$ on X such that*

$$\dim\{\xi\in X\mid K(\xi)\wedge D_1K(\xi)\wedge(D_1^2+D_2)K(\xi)=0\}\geqslant 1;$$

(b′) *there exist invariant vector fields $D_1\neq 0, D_2,\ldots,$ on X and constants $d_4,d_5,\ldots,$ such that*

$$P_m\theta(z):=\left[\Delta_mD_1-\Delta_{m-1}(D_2+D_1^2)+\sum_{i=3}^{m}d_{i+1}\Delta_{m-i}\right][\theta(z+\zeta)\cdot\theta(z-\zeta)]|_{\zeta=0}=0,$$

for all $m\geqslant 3$, where the D_i operate on the variable ζ, and the Δ_j are defined by

$$\Delta_j=\sum_{i_1+2i_2+\cdots+ji_j=j}\frac{1}{i_1!\cdot i_2!\cdot\,\cdots\,\cdot i_j!}\cdot D_1^{i_1}\cdots D_j^{i_j}.$$

In this case, the image curve Γ is, up to translation, the curve whose parametric expression is

$$\varepsilon\mapsto\sum_{i=1}^{\infty}\varepsilon^i\cdot 2D_i,$$

where $\varepsilon \in \mathbb{C}$, and each D_i is viewed as a point of the universal cover of X via its natural identification with $T_0(X)$.

2. Shiota's theorem

First, we observe that

$$\begin{aligned} P_3(D_1, D_2, D_3; d)\theta &= [-\tfrac{1}{3}D_1^4 - D_2^2 + D_1 D_3 + d][\theta(z+\zeta)\cdot\theta(z-\zeta)]|_{\zeta=0} \\ &= -\tfrac{2}{3}D_1^4\theta\cdot\theta + \tfrac{8}{3}D_1^3\theta\cdot D_1\theta - 2D_1^2\theta\cdot D_1^2\theta \\ &\quad +2D_2\theta\cdot D_2\theta - 2D_2^2\theta\cdot\theta \\ &\quad +2D_1D_3\theta\cdot\theta - 2D_3\theta\cdot D_1\theta + d\theta\cdot\theta. \end{aligned} \tag{2.0}$$

THEOREM 2.1 (Shiota [S], conjectured by Novikov). *The first non-trivial equation of the K.P. hierarchy characterizes Jacobians: an i.p.p.a.v. $(X, [\Theta])$ is a Jacobian if and only if there exist invariant vector fields $D_1 \neq 0, D_2, D_3$ and a constant d such that*

$$P_3(D_1, D_2, D_3; d)\theta = 0.$$

As we already mentioned, our proof consists in recovering the vanishing of the whole K.P. hierarchy from the equation $P_3\theta = 0$, i.e. in recovering the curve Γ from its third order approximation. We observe that $P_i(\ldots)\theta$ is a section of $\mathcal{O}_X(2\Theta)$, for all $D_1, \ldots, D_i$ and $d_4, \ldots, d_{i+1}$. Indeed, if $\mathcal{D}$ is any differential operator, because of Riemann's quadratic identity, we have that

$$\mathcal{D}[\theta(z+\zeta)\cdot\theta(z-\zeta)]|_{\zeta=0} = \sum_{\nu\in Z^g/2Z^g} \mathcal{D}\theta_\nu(0)\cdot\theta_\nu(z) \in H^0(X, 2\Theta),$$

where $\{\theta_\nu\}$ is the basis of $H^0(X, \mathcal{O}(2\Theta))$ having the property that Riemann's identity $\theta(z+\zeta)\cdot\theta(z-\zeta) = \Sigma_\nu\theta_\nu(z)\cdot\theta_\nu(\zeta)$ holds. Assuming by induction that there exist invariant vector fields $D_1, \ldots, D_{m-1}$ and constants $d_4, \ldots, d_m$ such that

$$P_i(D_1, \ldots, D_i; d_4, \ldots, d_{i+1})\theta = 0, \quad \forall\, i \leqslant m-1,$$

one needs to find D_m and d_{m+1} such that $P_m(\ldots)\theta = 0$.

We recall that the vector space $H^0(\Theta, \mathcal{O}(\Theta)|_\Theta)$ is the vector space of derivatives $T\theta$, with $T \in T_0(X)$. We denote by $D\Theta$ the scheme associated with the section $D\theta \in H^0(\Theta, \mathcal{O}(\Theta)|_\Theta)$, i.e. $D\Theta = \Theta \cap \{D\theta = 0\}$. We shall use the following remark.

REMARK 2.2 [AD3] (private communication from G. Welters to E. Arbarello). Whenever a section $S \in H^0(X, \mathcal{O}(2\Theta))$ vanishes on $D\Theta$, there exists an invariant vector field E and a constant d such that

$$S + ED\theta\cdot\theta - E\theta\cdot D\theta + d\theta\cdot\theta = 0 \in H^0(X, \mathcal{O}(2\Theta)).$$

As a consequence of this remark , Shiota's theorem can be stated as follows.

THEOREM 2.3. *An i.p.p.a.v.* $(X, [\Theta])$ *is a Jacobian if and only if there exist invariant vector fields* $D_1 \neq 0$ *and* D_2 *such that* $P_3(D_1, D_2, 0; 0)\theta|_{D_1\Theta} = 0$.

REMARK 2.4. We work with the K.P. differential equation for a theta function, which is an automorphic form θ associated with the polarization. If $\theta(z)$ and $\tilde{\theta}(z)$ are automorphic forms associated with the same polarization, there exists a point z_0 in V, where V is the universal cover of the abelian variety X, and a nowhere-vanishing holomorphic function $g(z)$ on V, such that $\tilde{\theta}(z+z_0) = g(z)\cdot\theta(z)$. One might have $P_3\theta = 0$ and $P_3\tilde{\theta} \neq 0$ but, since $P_3(g\cdot\theta) = g^2\cdot P_3\theta + \theta^2\cdot P_3 g - d\cdot g^2\cdot\theta^2 - 8(D_1^2 g\cdot g - D_1 g\cdot D_1 g)\cdot(D_1^2\theta\cdot\theta - D_1\theta\cdot D_1\theta)$, one has $P_3\tilde{\theta}|_{D_1\Theta} = g^2\cdot P_3\theta|_{D_1\Theta}$ (so that formulation 2.3 of Shiota's theorem is independent of the theta function representing the polarization). In view of Remark 2.2, there exist D_1, D_2, D_3, d such that $P_3(D_1, D_2, D_3; d)\theta = 0$ if and only if there exist $D_1, D_2, \tilde{D}_3, \tilde{d}$ such that $P_3(D_1, D_2, \tilde{D}_3; \tilde{d})\tilde{\theta} = 0$.

TWO FORMULAS 2.5. We have the general formulas (they can be proved by a direct computation)

$$\begin{aligned}
&\left(P_s + \sum_{i=1}^{s-3}\Delta_i P_{s-i}\right)\theta \\
&\quad = (D_1^2\theta - D_2\theta)\cdot(-\tilde{\Delta}_{s-1}\theta) \\
&\qquad + \theta\cdot\left[D_1\tilde{\Delta}_s - (D_1^2 + D_2)\tilde{\Delta}_{s-1} + \sum_{i=3}^{s} d_{i+1}\tilde{\Delta}_{s-i}\right]\theta \\
&\qquad + D_1\theta\cdot(-\tilde{\Delta}_s + 2D_1\tilde{\Delta}_{s-1})\theta,
\end{aligned}$$

$$\begin{aligned}
&\left(P_s + \sum_{i=1}^{s-3}\Delta_i^- P_{s-i}\right)\theta \\
&\quad = (D_1^2\theta + D_2\theta)\cdot(-\tilde{\Delta}_{s-1}^-\theta) \\
&\qquad + \theta\cdot\left[-D_1\tilde{\Delta}_s^- - (D_1^2 - D_2)\tilde{\Delta}_{s-1}^- + \sum_{i=3}^{s} d_{i+1}\tilde{\Delta}_{s-i}^-\right]\theta \\
&\qquad - D_1\theta\cdot(-\tilde{\Delta}_s^- - 2D_1\tilde{\Delta}_{s-1}^-)\theta,
\end{aligned}$$

where $\Delta_i^-(D_1,\ldots,D_i)=\Delta_i(-D_1,\ldots,-D_i)$, $\tilde{\Delta}_i(D_1,\ldots,D_i)=\Delta_i(2D_1,\ldots,2D_i)$, $\tilde{\Delta}_i^-(D_1,\ldots,D_i)=\Delta_i(-2D_1,\ldots,-2D_i)$.

REMARK 2.6 [AD3]. The restriction $P_m\theta|_{D_1\Theta}$ does not depend on D_m, d_{m+1}. In fact

$$\begin{aligned}&P_m(D_1,\ldots,D_m;d_4,\ldots,d_{m+1})\theta\\&\quad=P_m(D_1,\ldots,D_{m-1},0;d_4,\ldots,d_m,0)\theta+2D_mD_1\theta\cdot\theta\\&\qquad-2D_m\theta\cdot D_1\theta+d_{m+1}\theta^2.\end{aligned}$$

This equality leads to a crucial point of Arbarello–De Concini's argument: by Remark 2.2, there exist a D_m and a d_{m+1} which make $P_m\theta$ equal to zero if and only if $P_m\theta$ vanishes on $D_1\Theta$.

From the formulas in 2.5 and the previous remark, assuming by induction that $P_i\theta=0$ for $i<m$, it follows that the only obstruction to find a D_m and a d_{m+1} which make $P_m\theta$ equal to zero is given by those components of $D_1\Theta$ where neither $(D_1^2+D_2)\theta$ nor $(D_1^2-D_2)\theta$ vanish. Since $P_3\theta$ equals $(D_1^2+D_2)\theta\cdot(D_1^2-D_2)\theta$, $\operatorname{mod}(\theta,D_1\theta)$, and since, by hypothesis, $P_3\theta=0$, we have that $(D_1^2+D_2)\theta\cdot(D_1^2-D_2)\theta$ vanishes on $D_1\Theta$. Therefore a component of $D_1\Theta$ where neither $(D_1^2+D_2)\theta$ nor $(D_1^2-D_2)\theta$ vanish must be non-reduced.

In the next section we shall deal with such components. We show that if $\mathcal{W}$ is a component of $D_1\Theta$ then, assuming by induction that $P_3\theta=\cdots=P_{m-1}\theta=0$, only two cases may occur: either $P_m\theta$ vanishes on $\mathcal{W}$, or the reduced scheme underlying $\mathcal{W}$, denoted by $\mathcal{W}_{\text{red}}$, is invariant under the $\langle D_1,D_2\rangle$-flow. Moreover, if Θ is singular along $\mathcal{W}_{\text{red}}$ then the second case occur (Theorems 3.1 and 3.2).

3. The $\langle D_1,D_2\rangle$-invariance

To begin, we observe that we can always assume $D_2\neq 0$, as well as $D_3\neq 0$. Indeed, for all complex numbers b, we have

$$P_3(D_1,D_2,D_3;d_4)=P_3(D_1,D_2+bD_1,D_3+2bD_2+b^2D_1;d_4). \tag{3.0}$$

Let $\mathcal{W}$ be a component of $D_1\Theta$. We assume first that Θ is smooth at a generic point of $\mathcal{W}_{\text{red}}$. We prove the following.

THEOREM 3.1. *Let $(X,[\Theta])$ be an i.p.p.a.v. of dimension n and assume that $P_3\theta=\cdots=P_{m-1}\theta=0$, where $m\geqslant 4$. Let $\mathcal{W}$ be a component of the scheme $D_1\Theta$ and assume that Θ is non-singular at a generic point of $\mathcal{W}_{\text{red}}$. Then either $P_m\theta$ vanishes on $\mathcal{W}$, or $\mathcal{W}_{\text{red}}$ is invariant under the $\langle D_1,D_2\rangle$-flow.*

Proof. Let p be a generic point of $\mathcal{W}_{\text{red}}$. If $\mathcal{W}$ is reduced, $P_m\theta$ vanishes on $\mathcal{W}$. Assume that $\mathcal{W}$ is non-reduced. Since p is a smooth point of Θ, there exist an irreducible element $h \in \mathcal{O}_{X,p}$, an integer $a \geqslant 2$, integers b, c, invertible elements $\varepsilon_2, \varepsilon_3 \in \mathcal{O}_{X,p}$ and elements $g_1, g_2, g_3 \in \mathcal{O}_{X,p}$ such that the ideal of $\mathcal{W}$ in $\mathcal{O}_{X,p}$ is of the form (h^a, θ), and moreover

$$\begin{aligned} D_1\theta &= h^a + g_1 \cdot \theta \\ D_2\theta &= \varepsilon_2 \cdot h^b + g_2 \cdot \theta \\ D_3\theta &= \varepsilon_3 \cdot h^c + g_3 \cdot \theta \\ D_1^2\theta &= a \cdot h^{a-1} \cdot D_1 h + g_1 \cdot h^a + [g_1^2 + D_1 g_1] \cdot \theta. \end{aligned} \tag{3.1.1}$$

We have $b \geqslant 1$, because $P_3\theta$, hence $(D_1^2\theta + D_2\theta) \cdot (D_1^2\theta - D_2\theta)$, vanishes on $\mathcal{W}_{\text{red}}$. If h does not divide $D_1 h$, we prove as in [A] that $P_m\theta$ vanishes on $\mathcal{W}$: by substituting the formulas above in the expression of $P_3\theta$ and $D_1P_3\theta$, one sees that a has to equal 2 and by substituting in the expression of $P_{m-1}\theta$ (which is zero by inductive hypothesis), one sees that $\Delta_{m-1}\theta$ belongs to (h, θ); hence $P_m\theta \in (h^2, \theta)$, that is $P_m\theta|_{\mathcal{W}} = 0$.

If h divides $D_1 h$, the variety $\mathcal{W}_{\text{red}}$ is invariant under the D_1-flow. Under this assumption, the $\langle D_1, D_2\rangle$-invariance of $\mathcal{W}_{\text{red}}$ is a consequence of Lemma 3.5 and Lemma 3.8 below. □

Let us now turn to the case

$$\dim \Theta_{\text{sing}} = n - 2, \quad \Theta \text{ is singular along } \mathcal{W}_{\text{red}}.$$

(During the revision of the manuscript the preprint by Ein and Lazrsfeld [EL] appeared proving that the case $\Theta_{\text{sing}} = n - 2$ does not actually occur. Therefore, Theorem 3.2, Lemma 3.6 and Lemma 3.7 below are no longer strictly necessary for the present proof). We want to prove the following.

THEOREM 3.2. *Let $(X, [\Theta])$ be an i.p.p.a.v. of dimension n. Suppose the divisor Θ is singular along a reduced subvariety Z of codimension 1, and assume that the K.P. equation $P_3\theta = 0$ holds. Then Z is invariant under the $\langle D_1, D_2\rangle$-flow.*

This theorem is consequence of Lemma 3.5, Lemma 3.7 and Lemma 3.8 below; it will be proved later.

REMARK 3.3. We will make a strong use of the fact that Z has codimension 2 in X. It is clearly in general false that, if the K.P. equation holds, Θ is D_1-invariant in its singular points.

In view of the following general fact proved by J. Kollár in [Ko] the theta divisor cannot be 'too singular' along Z.

THEOREM 3.4 (Kollár). *Let $(X, [\Theta])$ be an i.p.p.a.v.. If Θ is singular along an* *irreducible hypersurface Z, it has a local normal crossing singularity at a generic point of Z.*

LEMMA 3.5. *Let $(X, [\Theta])$ be an i.p.p.a.v. of dimension n, let Z be a reduced subvariety of Θ of dimension $n-2$, and let D be an invariant vector field on X. If Θ is D-invariant along Z, then Z is D-invariant.*

Proof. If Z were not D-invariant, the D-span of Z would be contained in Θ. This span would have dimension $n-1$, therefore it would be a D-invariant component of Θ. This is impossible because of the ampleness and the irreducibility of Θ. □

LEMMA 3.6. *Suppose the divisor Θ is singular along Z, and assume that the K.P. equation $P_3\theta = 0$ holds. Let p be a smooth point of Z and $T_p(Z)$ the tangent space to Z at p. Then $D_1, D_2, T_p(Z)$ are not in general position, i.e.*

$$\dim(\langle D_1, D_2, T_p(Z)\rangle) \leqslant n-1.$$

Proof. Since Θ is singular along Z, we have that $\theta|_Z = D_1\theta|_Z = D_2\theta|_Z = D_3\theta|_Z = 0$. It follows that $P_3\theta|_Z = (D_1^2\theta)^2|_Z$, therefore $D_1^2\theta|_Z = 0$. By 2.0 we get $D_2P_3\theta|_Z = [\frac{8}{3}D_1D_2\theta \cdot D_1^3\theta]|_Z$ and $D_1^2P_3\theta|_Z = [\frac{-4}{3}(D_1^3\theta)^2 + 4(D_1D_2\theta)^2]|_Z$. Since $P_3\theta$ is zero, $D_2P_3\theta$ and $D_1^2P_3\theta$ are also zero, and therefore we obtain

$$D_1D_2\theta|_Z = D_1^3\theta|_Z = 0. \tag{3.6.1}$$

We now proceed by contradiction. Suppose there exists $p_0 \in Z_{\text{smooth}}$ such that

$$\langle D_1, D_2, T_{p_0}(Z)\rangle = T_{p_0}(X),$$

then the same equality must hold for every p in a neighborhood U of p_0 in Z. Let $p \in U$. For every $E \in T_p(X)$, there exist λ, μ such that $E = S + \lambda D_1 + \mu D_2$, where $S \in T_p(Z)$. As $D_1\theta|_Z = 0$ and $S \in T_p(Z)$ we have $ED_1\theta(p) = (S + \lambda D_1 + \mu D_2)D_1\theta(p) = 0$. Therefore $ED_1\theta|_Z = 0$, for every $E \in T_0(X)$. The assumption that $\langle D_1, D_2, T_p(Z)\rangle = T_p(X)$ implies that $D_1 \notin T_p(Z)$. By Theorem 3.4, the tangent cone to Θ at p is a pair of distinct hyperplanes whose intersection is $T_p(Z)$. Therefore, for a generic $E \in T_p(X)$, we have that $ED_1\theta|_Z \neq 0$. This is a contradiction. □

LEMMA 3.7. *Suppose the divisor Θ is singular along Z, and assume that the K.P. equation $P_3\theta = 0$ holds. The divisor Θ is D_1-invariant at each point of Z.*

Proof. From the previous lemma, there exist functions λ and μ on Z_{smooth} not simultaneously vanishing and such that

$$\lambda(p) \cdot D_1 + \mu(p) \cdot D_2 \in T_p(Z),$$

for all p in Z_{smooth}. If $\mu \equiv 0$ then Z is D_1-invariant. Assume $\mu \not\equiv 0$; by induction on $\alpha+\beta$, we prove that $D_1^\alpha D_2^\beta\theta|_Z = 0$, for all integers α, β. Let us assume that $D_1^\alpha D_2^\beta\theta|_Z = 0$, for all $\alpha+\beta \leqslant \nu_0$. We need only to show that $D_1^{\nu_0+1}\theta$ vanishes on Z. In fact, since $\lambda(p)\cdot D_1 + \mu(p)\cdot D_2$ is in $T_p(Z)$, and since μ is not identically zero, the vector D_2 is a combination of D_1 and a vector in $T_p(Z)$, for p generic in Z; as $D_1^\alpha\theta$ vanishes on Z for all $\alpha \leqslant \nu_0+1$, we have that $D_1^\alpha D_2^\beta\theta|_Z = 0$, for all $\alpha+\beta \leqslant \nu_0+1$. By 3.6.1, $D_1^3\theta|_Z = 0$; hence we are done if $\nu_0 \leqslant 2$. Assume $\nu_0 \geqslant 3$. We distinguish two cases:

(a) $D_3 \in \langle D_1, D_2, T_p(Z)\rangle$, for all p in Z;

(b) $D_3 \notin \langle D_1, D_2, T_p(Z)\rangle$, for p generic in Z.

Let us start with (a). Since D_3 is a combination of D_1, D_2 and a vector in $T_p(Z)$, it follows that $D_1^\alpha D_2^\beta D_3^\gamma\theta|_Z = 0$, for $\alpha+\beta+\gamma \leqslant \nu_0$. Therefore, the only nonzero terms in the restriction to Z of a derivative of $P_3\theta$ are products of derivatives of θ of order at least ν_0+1; as $P_3\theta = -\frac{2}{3}D_1^4\theta\cdot\theta + \frac{8}{3}D_1^3\theta\cdot D_1\theta - 2D_1^2\theta\cdot D_1^2\theta +$ 'lower order terms' we obtain that the only nonzero term of $D_1^{2\nu_0-2}P_3\theta|_Z$ is $D_1^{\nu_0+1}\theta\cdot D_1^{\nu_0+1}\theta$, with coefficient $-2\binom{2\nu_0-2}{\nu_0-1} + \frac{8}{3}\binom{2\nu_0-2}{\nu_0-2} - \frac{2}{3}\binom{2\nu_0-2}{\nu_0-3}$ (which is easily seen to be nonzero). Therefore, as $D_1^{2\nu_0-2}P_3\theta|_Z = 0$, we must have $D_1^{\nu_0+1}\theta|_Z = 0$.

Let us deal with case (b). Since $D_1^\alpha D_2^\beta\theta|_Z = 0$ for all $\alpha+\beta \leqslant 3 \leqslant \nu_0$, we have

$$0 = D_1^4 P_3\theta|_Z = (-2D_1^4\theta\cdot D_1^4\theta - 6D_1^4\theta\cdot D_1D_3\theta)|_Z,$$

$$0 = D_1D_3P_3\theta|_Z = (2D_1^4\theta\cdot D_1D_3\theta)|_Z.$$

It follows that $D_1^4\theta|_Z = 0$, and we may assume $\nu_0 \geqslant 4$. We want to compute $D_1^{\nu_0+1}P_3\theta|_Z$. Since any term of $P_3\theta$ is a product of derivatives of θ of order i and j, where $i+j \leqslant 4$, any term of $D_1^{\nu_0+1}P_3\theta|_Z$ is a product of derivatives of θ of order i and j, where $i+j \leqslant \nu_0+5 < 2\nu_0+2$. Thus, since $D_1^\alpha D_2^\beta\theta|_Z = 0$ for all $\alpha+\beta \leqslant \nu_0$, any contribution to the restriction to Z of $D_1^{\nu_0+1}P_3\theta$ must involve a D_3; therefore, by 2.0, $D_1^{\nu_0+1}P_3\theta|_Z = D_1^{\nu_0+1}[2D_1D_3\theta\cdot\theta - 2D_3\theta\cdot D_1\theta]|_Z = [-2(\nu_0+1)+2]D_1^{\nu_0+1}\theta\cdot D_1D_3\theta|_Z$, where the last equality follows because $D_3\theta|_Z = 0$, $D_1^\alpha\theta|_Z = 0$ for all $\alpha \leqslant \nu_0$. Hence, if $D_1D_3\theta|_Z \not\equiv 0$, then $D_1^{\nu_0+1}\theta|_Z = 0$ and we are done. It only remains to consider the case where $D_1D_3\theta|_Z = 0$. If D_1 is in $T_p(Z)$ for p generic in Z, the variety Z is D_1-invariant, Θ is D_1-invariant along Z, and we are done; so we assume that, for p generic in Z, the vector D_1 is not in $T_p(Z)$. Then, for dimensional reasons, $T_p(X) = \langle D_1, D_2, D_3, T_p(Z)\rangle$. Since $D_1^2\theta$, $D_1D_2\theta$ and $D_1D_3\theta$ all vanish on Z, we have $D_1E\theta|_Z = 0$ for all $E \in T_0(X)$. By Theorem 3.4, the tangent cone to Θ at p is a pair of distinct hyperplanes. Therefore, for a generic $E \in T_p(X)$, we have that $ED_1\theta|_Z \neq 0$. This is a contradiction. □

LEMMA 3.8 (Shiota [S], Lemma A, p. 359). *Let τ be a solution of the equation $P_3\tau = 0$ in a neighborhood of a point p_0 in $\mathbb{C}^n$. If $D_1^\alpha\tau(p_0) = 0$ for all integers α, then $D_1^\alpha D_2^\beta\tau(p_0) = 0$ for all integers α and β.*

Proof. Let us denote by L_1 the (local) D_1-integral complex line through p_0. By hypothesis, $D_1^\alpha\tau(p_0) = 0$, for all α, thus $\tau|_{L_1} = 0$. We proceed by contradiction, i.e. we assume that there exists $b > 0$ such that $D_2^b\tau|_{L_1} \not\equiv 0$. Let

$$\begin{aligned} \beta_\gamma &= \min\{\beta \mid D_2^\beta D_3^\gamma\tau|_{L_1} \not\equiv 0\},\\ c &= \min\{\gamma \mid \beta_\gamma = 0\},\\ w &= \min\{\beta_\gamma + 2\gamma\},\\ \sigma &= \max\{\gamma|\beta_\gamma + 2\gamma = w\}, \end{aligned} \tag{3.8.1}$$

where β_γ and c are allowed to be infinite. Note that $w \leqslant \beta_0 \leqslant b < \infty$, $\sigma \leqslant \frac{1}{2}w < \infty$, $w = \beta_\sigma + 2\sigma$ and $w \leqslant \beta_\gamma + 2\gamma$ for all γ. As $\tau|_{L_1} \equiv 0$ we have $\beta_0 > 0$ and $c \geqslant 1$. Moreover $D_1^\alpha D_2^\beta D_3^\gamma\tau|_{L_1} = 0$, for all $\alpha, \beta < \beta_\gamma$. It follows that

$$\begin{aligned} &\text{if } \beta + 2\gamma < w, \quad \text{then } D_1^\alpha D_2^\beta D_3^\gamma\tau|_{L_1} = 0,\\ &\text{if } \gamma > \sigma \quad \text{and} \quad \beta + 2\gamma \leqslant w, \quad \text{then } D_1^\alpha D_2^\beta D_3^\gamma\tau|_{L_1} = 0. \end{aligned} \tag{3.8.2}$$

First, we prove that $\sigma = c$ (in particular $c < \infty$). It is clear that $\sigma \leqslant c$. If $\sigma < c$ then $\beta_\sigma \geqslant 1$, thus $2\beta_\sigma - 2 \geqslant 0$. Let us set $A_0 = d_4\theta \cdot \theta$, $A_1 = -\frac{2}{3}D_1^4\theta \cdot \theta + \frac{8}{3}D_1^3\theta \cdot D_1\theta - 2D_1^2\theta \cdot D_1^2\theta$, $A_2 = -2D_2^2\theta \cdot \theta + 2D_2\theta \cdot D_2\theta$, $A_3 = 2D_1D_3\theta\cdot\theta - 2D_3\theta\cdot D_1\theta$; so that $P_3\tau = A_0 + A_1 + A_2 + A_3$. By 3.8.2 we have $D_2^{2\beta_\sigma-2}D_3^{2\sigma}[A_0]|_{L_1} = D_2^{2\beta_\sigma-2}D_3^{2\sigma}[A_1]|_{L_1} = D_2^{2\beta_\sigma-2}D_3^{2\sigma}[A_3]|_{L_1} = 0$. Therefore $0 = D_2^{2\beta_\sigma-2}D_3^{2\sigma}P_3\tau|_{L_1} = D_2^{2\beta_\sigma-2}D_3^{2\sigma}[A_2]|_{L_1} = \binom{2\sigma}{\sigma} \cdot \left[2\binom{2\beta_\sigma-2}{\beta_\sigma-1} - 2\binom{2\beta_\sigma-2}{\beta_\sigma-2}\right] \cdot (D_2^{\beta_\sigma}D_3^\sigma\tau)^2|_{L_1}$, where the last equality follows by 3.8.2; this is a contradiction because $D_2^{\beta_\sigma}D_3^\sigma\tau|_{L_1} \not\equiv 0$. Note that $\sigma = c$ implies $w = 2c$ and $\beta_\gamma \geqslant w - 2\gamma = 2c - 2\gamma \geqslant 2$, for all $\gamma \leqslant c - 1$. Let

$$\begin{aligned} \tilde{w} &= \min\{\beta_\gamma + 2\gamma \mid \gamma < c\},\\ \gamma_0 &= \max\{\gamma \mid \gamma < c, \beta_\gamma + 2\gamma = \tilde{w}\}. \end{aligned} \tag{3.8.3}$$

Note that, as $c \geqslant 1$ we have $\tilde{w} \leqslant \beta_0 < \infty$. Thus, $\tilde{w} = \beta_{\gamma_0} + 2\gamma_0$. Moreover, as $\gamma_0 < c$, we have $\beta_{\gamma_0} \geqslant 2$. We want to compute $D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}P_3\tau|_{L_1}$. By 3.8.3 we have

$$\begin{aligned} &\text{if } \gamma < c \quad \text{and} \quad \beta + 2\gamma < \tilde{w}, \quad \text{then } D_1^\alpha D_2^\beta D_3^\gamma\tau|_{L_1} = 0,\\ &\text{if } \gamma_0 < \gamma < c \quad \text{and} \quad \beta + 2\gamma \leqslant \tilde{w}, \quad \text{then } D_1^\alpha D_2^\beta D_3^\gamma\tau|_{L_1} = 0. \end{aligned} \tag{3.8.4}$$

By 3.8.2 and 3.8.4 we get $D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}[A_0+A_1]|_{L_1}=0$, $D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}[A_2]|_{L_1}=-2\binom{c+\gamma_0}{c}(D_2^{\beta_{\gamma_0}}D_3^{\gamma_0}\tau)\cdot(D_3^c\tau)|_{L_1}$; if $\gamma_0<c-1$, then $D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}[A_3]|_{L_1}=0$; if $\gamma_0=c-1$, then $D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}[A_3]|_{L_1}=D_2^{\beta_{\gamma_0}-2}[(2\binom{c+\gamma_0}{c}-2\binom{c+\gamma_0}{\gamma_0})D_1D_3^c\tau\cdot D_3^c\tau]|_{L_1}+D_2^{\beta_{\gamma_0}-2}[\Sigma_{i+j=2c,\,i\neq c}(\ldots)D_1D_3^i\tau\cdot D_3^j\tau]|_{L_1}=0$ (in fact, the coefficient $2\binom{c+\gamma_0}{c}-2\binom{c+\gamma_0}{\gamma_0}$ is zero). Therefore, $0=D_2^{\beta_{\gamma_0}-2}D_3^{c+\gamma_0}P_3\tau|_{L_1}=-2\binom{c+\gamma_0}{c}(D_2^{\beta_{\gamma_0}}D_3^{\gamma_0}\tau)\cdot(D_3^c\tau)|_{L_1}$. On the other hand, by 3.8.1 and 3.8.3, $(D_2^{\beta_{\gamma_0}}D_3^{\gamma_0}\tau)\cdot(D_3^c\tau)|_{L_1}\not\equiv 0$, thus a contradiction. □

Proof. (of Theorem 3.2) By Lemma 3.7, Θ is D_1-invariant along Z; then, by Lemma 3.5, Z is D_1-invariant. Hence, by Lemma 3.8, Θ is $\langle D_1,D_2\rangle$-invariant along Z; so, by Lemma 3.5, Z is $\langle D_1,D_2\rangle$-invariant. □

We shall use the following algebraic computation about the possible series expansion of a solution of the K.P. equation. The following lemma is Lemma B from Shiota, restated in a way that is more convenient to our purpose.

LEMMA 3.9 (Shiota [S], Lemma B, p. 359). *Let $(S,\mathcal{L})$ be a polarized abelian variety, $D_1\neq 0, D_2, \tilde D_3\in T_0(S)$. Assume that S is generated by $\langle D_1,D_2\rangle$. Let Y be a 2-dimensional disk with analytic coordinates t and λ. Let τ be a nonzero section of $\mathcal{O}_Y\otimes H^0(S,\mathcal{L})$ and assume that*

(i) $P_3(D_1,D_2,\tilde D_3+\partial_t;d)\tau=0,$

(ii) $\displaystyle \tau(t,\lambda,x)=\sum_{i,j\geqslant 0}\tau_{i,j}(x)\cdot t^i\lambda^j,$

where $x\in S$ (observe that $\tau_{i,j}\in H^0(S,\mathcal{L})$ for all i and j). Also assume $\tau_{0,\rho}=0$, where $\rho:=\min\{j\mid\exists\, i:\tau_{i,j}(\cdot)\not\equiv 0\}$. Then there exist local sections at zero of $\mathcal{O}_Y$ and $\mathcal{O}_Y\otimes H^0(S,\mathcal{L})$, f and ψ, such that

$$\tau(t,\lambda,x)=\lambda^\rho\cdot f(t,\lambda)\cdot\psi(t,\lambda,x),$$

where $\psi(0,0,\cdot)\not\equiv 0$, $f(0,0)=0$ and $f(\cdot,0)\not\equiv 0$.

Proof. Step I (Shiota), *we look for formal power series in t and λ, f and ψ as in the lemma.* Since $P_3(\lambda^\rho\cdot[\ldots])=\lambda^{2\rho}\cdot P_3(\ldots)$, we can assume $\rho=0$. Let $\nu=\max\{i\mid\tau_{i,0}(\cdot)\equiv 0\}$, $f_0=t^\nu$ and $\bar\tau_0(t,x)=\Sigma_{i\geqslant\nu}\tau_{i,0}(x)\cdot t^{i-\nu}$, so that $\tau=f_0\cdot\bar\tau_0 \bmod(\lambda)$. Note that $P_3(\bar\tau_0)=0$, in fact $0=P_3(\tau)=t^{2\nu}\cdot P_3(\bar\tau_0)\bmod(\lambda)$. Note also that $\bar\tau_0(0,x)=\tau_{\nu,0}(x)\not\equiv 0$. It suffices to find constants and sections

$$c_{i,j},0\leqslant i\leqslant\nu-1,1\leqslant j,\qquad g_{i,j}(x)\in H^0(S,\mathcal{L}),i\geqslant\nu,j\geqslant 1,$$

such that

$$\tau(t,\lambda,x)=\left(f_0+\sum_{j\geqslant 1}f_j(t)\cdot\lambda^j\right)\cdot\left(\bar\tau_0(t,x)+\sum_{j\geqslant 1}\bar\tau_j(t,x)\cdot\lambda^j\right),\quad(3.9.1)$$

where, for $j \geqslant 1$, we define

$$f_j(t) = \sum_{i=0}^{\nu-1} c_{i,j} \cdot t^i, \qquad \bar{\tau}_j(t,x) = \sum_{i \geqslant \nu} g_{i,j}(x) \cdot t^{i-\nu}. \tag{3.9.2}$$

We now proceed by induction: let l be a positive integer, and assume that we found constants $c_{i,j}$, for all $1 \leqslant j \leqslant l-1, i \leqslant \nu - 1$, and sections $g_{i,j}(x)$, for all $1 \leqslant j \leqslant l-1, i \geqslant \nu$, such that 3.9.1 holds modulo (λ^l). Define $\tau'(t,x)$ by

$$\begin{aligned} \tau(t,\lambda,x) \;=\; & \left(f_0 + \sum_{j=1}^{l-1} f_j(t)\cdot\lambda^j\right)\cdot\left(\bar{\tau}_0(t,x) + \sum_{j=1}^{l-1}\bar{\tau}_j(t,x)\cdot\lambda^j\right) \\ & + \lambda^l \cdot \tau'(t,x) \quad \mathrm{mod}(\lambda^{l+1}). \end{aligned} \tag{3.9.3}$$

We need to prove that there exist constants $c_{i,l}, i \leqslant \nu - 1$, and sections $g_{i,l}, i \geqslant \nu$, such that

$$\tau'(t,x) = \sum_{i=0}^{\nu-1} c_{i,l}\cdot t^i \cdot \bar{\tau}_0(t,x) + \sum_{i\geqslant\nu} g_{i,l}(x)\cdot t^i.$$

In fact, defining $f_l, \bar{\tau}_l$ as 3.9.2 requires, it is clear that 3.9.1 holds modulo (λ^{l+1}). We define $\tilde{P}_3(r,s) = \frac{1}{2}[P_3(r+s) - P_3(r) - P_3(s)]$. By substitution in 2.0 we get

$$\begin{aligned} & \tilde{P}_3(D_1, D_2, \tilde{D}_3 + \partial_t; d)(r,s) \\ & = -\tfrac{1}{3}(D_1^4 r\cdot s + D_1^4 s\cdot r) + \tfrac{4}{3}(D_1^3 r\cdot D_1 s + D_1^3 s\cdot D_1 r) \\ & \quad - 2D_1^2 r\cdot D_1^2 s - (D_2^2 s\cdot r + D_2^2 r\cdot s) + 2D_2 r\cdot D_2 s + d\cdot r\cdot s \\ & \quad + (D_1\tilde{D}_3 r\cdot s + D_1\tilde{D}_3 s\cdot r) - (\tilde{D}_3 r\cdot D_1 s + \tilde{D}_3 s\cdot D_1 r) \\ & \quad + (D_1\partial_t r\cdot s + D_1\partial_t s\cdot r) - (\partial_t r\cdot D_1 s + \partial_t s\cdot D_1 r). \end{aligned} \tag{3.9.4}$$

Note that $\tilde{P}_3$ is a symmetric $\mathbb{C}[\lambda]$-bilinear operator and that $P_3(r) = \tilde{P}_3(r,r)$. If $g = g(t,\lambda)$ does not depend on x, by a straightforward computation we obtain

$$\begin{aligned} & \tilde{P}_3(g\cdot r, g\cdot s) = g^2\cdot\tilde{P}_3(r,s) \\ & \tilde{P}_3(t^i\cdot r, t^j\cdot s) = t^{i+j}\cdot\tilde{P}_3(r,s) + (i-j)t^{i+j-1}\cdot(D_1 r\cdot s - D_1 s\cdot r) \end{aligned} \tag{3.9.5}$$

We define $g = g(t,\lambda) = \Sigma_{j=0}^{l-1} f_j(t)\cdot\lambda^j$ and $\phi(t,\lambda,x) = \Sigma_{j=0}^{l-1}\bar{\tau}_j(t,x)\cdot\lambda^j$, so that $\tau = g\cdot\phi + \lambda^l\cdot\tau'$ mod(λ^{l+1}). Thus, by 3.9.5 the following equalities hold modulo (λ^{l+1}): $0 = P_3(\tau) = P_3(g\cdot\phi + \lambda^l\cdot\tau') = P_3(g\cdot\phi) + 2\tilde{P}_3(g\cdot\phi, \lambda^l\cdot\tau') = g^2\cdot P_3(\phi) + 2\lambda^l\tilde{P}_3(g\cdot\phi,\tau') = g^2\cdot P_3(\phi) + 2\lambda^l\tilde{P}_3(t^\nu\cdot\bar{\tau}_0,\tau')$. In particular we get $g^2 \cdot P_3(\phi) = 0 \mod (\lambda^l)$. Since $g^2(t,\lambda) = t^{2\nu} \mod (\lambda)$ is nonzero, we get $P_3(\phi) = 0 \mod (\lambda^l)$. Since $g^2 \cdot P_3(\phi) + 2\lambda^l \tilde{P}_3(t^\nu \cdot \bar{\tau}_0, \tau') = 0 \mod (\lambda^{l+1})$ and (again) $g^2(t,\lambda) = t^{2\nu} \mod (\lambda)$ we get

$$\tilde{P}_3(t^\nu \cdot \bar{\tau}_0, \tau') = 0 \mod (t^{2\nu}). \tag{3.9.6}$$

We now proceed by induction on i: assume that $\tau'(t,x) = \Sigma_{i=0}^{i_0-1} c_{i,l} \cdot t^i \cdot \bar{\tau}_0(t,x) + \eta(x) \cdot t^{i_0}, \mod(t^{i_0+1})$, where $0 \leqslant i_0 \leqslant \nu - 1$. Since $\tilde{P}_3(\bar{\tau}_0, \bar{\tau}_0) = P_3(\bar{\tau}_0) = 0$, by 3.9.5 we get $\tilde{P}_3(t^\nu \cdot \bar{\tau}_0, t^i \cdot \bar{\tau}_0) = 0$. Thus, by substitution in 3.9.6 and (again) by 3.9.5 we get that the following equalities hold modulo $(t^{\nu+i_0})$: $0 = \tilde{P}_3(t^\nu \cdot \bar{\tau}_0, \Sigma_{i=0}^{i_0-1} c_{i,l} \cdot \bar{\tau}_0 \cdot t^i + \eta(x) \cdot t^{i_0}) = \tilde{P}_3(t^\nu \cdot \bar{\tau}_0, \eta(x) \cdot t^{i_0}) = \tilde{P}_3(t^\nu \cdot \bar{\tau}_0(0,x), \eta(x) \cdot t^{i_0}) = (\nu - i_0) \cdot t^{\nu+i_0-1} \cdot [D_1\bar{\tau}_0(0,x) \cdot \eta(x) - D_1\eta(x) \cdot \bar{\tau}_0(0,x)] = -(\nu - i_0) \cdot t^{\nu+i_0-1} \cdot [\bar{\tau}_0(0,x)]^2 \cdot D_1(\eta(x)/\bar{\tau}_0(0,x))$. It follows that $\eta(x)/\bar{\tau}_0(0,x)$ is D_1-invariant; on the other hand, the zeroes of $\bar{\tau}_0(0,\cdot)$ do not contain D_1-integral curves, otherwise, by 3.8 (applied to $\bar{\tau}_0$), we would have $\tau_{\nu,0}(x) = \bar{\tau}_0(0,x) = 0$. Thus $\eta(x) = c_{j_0,l} \cdot \bar{\tau}_0(0,x)$. It follows that $\tau'(t,x) = \Sigma_{i=0}^{i_0} c_{i,l} \cdot t^i \cdot \bar{\tau}_0(t,x) \mod(t^{i_0+1})$, and we are done.

Step II, *we prove that both f and ψ can be assumed to be regular functions.* As $\psi(0,0,\cdot) \not\equiv 0$ we are allowed to fix an x_0 such that $\psi(0,0,x_0) \neq 0$ and consider the formal power series $q(t,\lambda)$ such that $\psi(t,\lambda,x_0) \cdot q(t,\lambda) = 1$. Consider $\tilde{f}(t,\lambda) := f(t,\lambda) \cdot \psi(t,\lambda,x_0)$ and $\tilde{\psi}(t,\lambda,x) := \psi(t,\lambda,x) \cdot q(t,\lambda)$. It is clear that $\tau(t,\lambda,x) = \lambda^\rho \cdot \tilde{f}(t,\lambda) \cdot \tilde{\psi}(t,\lambda,x)$. As $\tilde{\psi}(t,\lambda,x_0) = 1$ and $\tau(t,\lambda,x_0)$ are both convergent, $\tilde{f}(t,\lambda)$ is also convergent. Since $\tau(t,\lambda,x)$ and $\tilde{f}(t,\lambda)$ are convergent, $\tilde{\psi}(t,\lambda,x)$ is also convergent. Note that t^ν divides $\tilde{f}(t,0) = 0, \tilde{f}(t,0) \not\equiv 0$ and $\tilde{\psi}(0,0,\cdot) \not\equiv 0$, i.e. the properties of f and ψ we need still hold for $\tilde{f}$ and $\tilde{\psi}$. □

LEMMA 3.10. *As usual, assume that $P_i\theta = 0$, for all $i \leqslant m - 1$. Let $\mathcal{W}$ be a component of the scheme $D_1\Theta$ and let p be a generic point of $\mathcal{W}_{\text{red}}$. Either $P_m\theta$ vanishes on $\mathcal{W}$, or there exist irreducible elements h, k of $\mathcal{O}_{X,p}$ such that*

(i) *the ideal of $\mathcal{W}_{\text{red}}$ at p is (h,k);*
(ii) *the hypersurfaces $\{h = 0\}, \{k = 0\}$ are smooth at p;*
(iii) *there exists an integer l such that $D_3\theta \notin (k, h^l)$ and $D_1^\alpha D_2^\beta \theta \in (k, h^l)$, for all $\alpha, \beta \geqslant 0$.*

Proof. If Θ is not singular along $\mathcal{W}_{\text{red}}$ we take $k = \theta$ and we define h as in 3.1.1. We proved that either $P_m\theta|_{\mathcal{W}} = 0$, or h divides D_1h in $\mathcal{O}_{\Theta,p}$. If h divides D_1h in $\mathcal{O}_{\Theta,p}$, by substitution in the expression of $P_3\theta$, we get $2b = a + c$, where the notations are the ones of the formulas 3.1.1. If $b > a$ then $(D_1^2 - D_2)\theta$, thus $P_m\theta$, vanishes on $\mathcal{W}$. It follows that either $P_m\theta|_{\mathcal{W}} = 0$, or $c < b < a$. Therefore the lemma holds with $l = b$. Let us turn to the case where Θ is singular along $\mathcal{W}_{\text{red}}$. By Theorem 3.4, we can write

$$\theta = h \cdot k,$$

where h and k satisfy (i), (ii) and belong to the analytic completion of $\mathcal{O}_{X,p}$. We prove that h, k satisfy (iii). Then, taking $\tilde{h}$ and $\tilde{k}$ approximating h and k to the order j $(j \gg 0)$, one has that (i), (ii) and (iii) hold. Thus, we can assume that $h, k \in \mathcal{O}_{X,p}$. As there are no D_1-invariant components of Θ, the element h does not divide $D_1 h$ in $\mathcal{O}_{X,p}$, likewise k does not divide $D_1 k$ in $\mathcal{O}_{X,p}$, (and similarly for D_2) and we can write

$$\begin{aligned} D_1 h &= \varepsilon_1 \cdot k^a + g_1 \cdot h, \\ D_1 k &= \tilde{\varepsilon}_1 \cdot h^{\tilde{a}} + \tilde{g}_1 \cdot k, \\ D_2 h &= \varepsilon_2 \cdot k^b + g_2 \cdot h, \\ D_2 k &= \tilde{\varepsilon}_2 \cdot h^{\tilde{b}} + \tilde{g}_2 \cdot k, \end{aligned}$$

where $\varepsilon_1, \tilde{\varepsilon}_1, \varepsilon_2, \tilde{\varepsilon}_2$ are invertible, $a, \tilde{a}, b, \tilde{b} \geqslant 1$. Note that, by 3.0, we are allowed to assume $a \geqslant b, \tilde{a} \geqslant \tilde{b}$. Note that $D_1^\alpha D_2^\beta h, D_1^\alpha D_2^\beta k \in (h, k)$, for all α, β, since, by Theorem 3.2, $\mathcal{W}_{\text{red}}$ is $\langle D_1, D_2 \rangle$-invariant. It follows that

$$D_1^\alpha \theta = D_1^\alpha (h \cdot k) \in (h \cdot k, k^{a+1}, h^{\tilde{a}+1}), \quad \forall \alpha \geqslant 0;$$

$$D_1^\alpha D_2^\beta \theta = D_1^\alpha D_2^\beta (h \cdot k) \in (h \cdot k, k^{b+1}, h^{\tilde{b}+1}), \quad \forall \alpha, \beta \geqslant 0.$$

We now claim that either $D_3\theta \notin (h \cdot k, k^{b+1}, h^{\tilde{b}+1})$, or $P_m\theta|_{\mathcal{W}} = 0$. Since $(h \cdot k, k^{b+1}, h^{\tilde{b}+1}) = (h, k^{b+1}) \cap (k, h^{\tilde{b}+1})$ we have that the previous claim (up to interchanging the roles played by h and k) implies the lemma. So, let us prove our claim. First, observe that if $D_3\theta \in (h, k^{b+1}) \cap (k, h^{\tilde{b}+1})$ then by substitution in 2.0 we get $0 = P_3\theta = 2 \cdot \tilde{\gamma}^2 \cdot h^{2\tilde{b}+2}, \text{mod}(k, h^{\tilde{a}+\tilde{b}+2})$. Thus $2\tilde{b} + 2 \geqslant \tilde{a} + \tilde{b} + 2$. Since $\tilde{a} > \tilde{b}$ we get $\tilde{b} = \tilde{a}$. Similarly, computing $P_3\theta$ modulo (h, k^{a+b+2}) we get $a = b$. It follows that $(D_1^2 - D_2)\theta \in (h \cdot k, k^{a+1}, h^{\tilde{a}+1})$. Note that the ideal $I_{D_1\Theta}$ is $(h \cdot k, \varepsilon_1 k^{a+1} + \tilde{\varepsilon}_1 h^{\tilde{a}+1})$, where $\varepsilon_1, \tilde{\varepsilon}_1$ are invertible. Thus $I_{D_1\Theta} \supset (h \cdot k, k^{a+2}, h^{\tilde{a}+2})$. Since $P_3\theta = \cdots = P_{m-1}\theta = 0$ by inductive hypothesis, by the first one of formulas 2.5 (with $s = m$) we get $P_m\theta|_{\mathcal{W}} = -(D_1^2 - D_2)\theta \cdot \tilde{\Delta}_{m-1}\theta$, where we keep the notation of the formulas 2.5. We claim that it suffices to prove that $\tilde{\Delta}_{m-1}\theta \in (h, k)$. Indeed, since $(D_1^2 - D_2)\theta$ is in $(h \cdot k, k^{a+1}, h^{\tilde{a}+1})$, if $\tilde{\Delta}_{m-1}\theta$ is in (h, k), then $P_m\theta|_{\mathcal{W}} = -(D_1^2 - D_2)\theta \cdot \tilde{\Delta}_{m-1}\theta \in (h \cdot k, k^{a+2}, h^{\tilde{a}+2}) \subset I_{D_1\Theta}$, and we are done. By inductive hypothesis, the left-hand side of the first formula 2.5 (with $s = m - 1$) is zero; it follows that the right hand side must be zero, in particular we get

$$-(D_1^2 - D_2)\theta \cdot \tilde{\Delta}_{m-2}\theta - D_1\theta \cdot (\tilde{\Delta}_{m-1} + 2D_1\tilde{\Delta}_{m-2})\theta = 0 \mod(\theta). \quad (3.10.1)$$

It follows that $\tilde{\Delta}_{m-2}\theta \in (h, k)$, otherwise we would have $-(D_1^2 - D_2)\theta \in I_{D_1\Theta}$ and we would be done. We now compute the left-hand side of 3.10.1 modulo the ideal $(h \cdot k, k^{a+2}, h^{\tilde{a}+2})$ (note that this ideal contains $(\theta) = (h \cdot k)$). Since $\tilde{\Delta}_{m-2}\theta \in$

(h,k) and $(D_1^2 - D_2)\theta \in (h \cdot k, k^{a+1}, h^{\tilde{a}+1})$ we have that $(D_1^2 - D_2)\theta \cdot \tilde{\Delta}_{m-2}\theta$ is in $(h \cdot k, k^{a+2}, h^{\tilde{a}+2})$. Since $D_1\theta$ is in $(h \cdot k, k^{a+1}, h^{\tilde{a}+1})$ and $\tilde{\Delta}_{m-2}\theta$, hence $D_1\tilde{\Delta}_{m-2}\theta$, is in (h,k), also $D_1\theta \cdot D_1\tilde{\Delta}_{m-2}\theta$ is in $(h \cdot k, k^{a+2}, h^{\tilde{a}+2})$. Therefore, by 3.10.1

$$-D_1\theta \cdot \tilde{\Delta}_{m-1}\theta \in (h \cdot k, k^{a+2}, h^{\tilde{a}+2}). \tag{3.10.2}$$

Since $D_1\theta = D_1(h \cdot k) = \varepsilon \cdot k^{a+1} + \tilde{\varepsilon} \cdot h^{\tilde{a}+1} \operatorname{mod}(h \cdot k)$ it follows that $D_1\theta$ is not in $(h \cdot k, k^{a+2}, h^{\tilde{a}+2})$. Therefore, by 3.10.2, $\tilde{\Delta}_{m-1}\theta$ is in (h,k) and we are done. □

4. End of the proof

Let us go back to the K.P. hierarchy. We assume, by induction, that we found invariant vector fields $D_1, \ldots, D_{m-1}$, and constants $d_4, \ldots, d_m$ such that

$$P_i(D_1, \ldots, D_i; d_4, \ldots, d_{i+1})\theta = 0, \quad \forall\, i \leqslant m-1.$$

We need to find an invariant vector field D_m and a constant d_{m+1} such that $P_m(D_1, \ldots, D_m; d_4, \ldots, d_{m+1})\theta = 0$.

Let $P_m\theta := P_m(D_1, \ldots, D_{m-1}, 0; d_4, \ldots, d_m, 0)\theta$. Recall that if $P_m\theta|_{D_1\Theta} = 0$ we are done by Remark 2.6. We proved that then the only components of the scheme $D_1\Theta$ where $P_m\theta$ might not vanish are, set-theoretically, $\langle D_1, D_2\rangle$-invariant. In order to conclude our proof of Shiota's Theorem we proceed by contradiction. Let $\mathcal{W}$ be a component of $D_1\Theta$ such that $P_m\theta|_{\mathcal{W}} \neq 0$. Thus, $\mathcal{W}_{\mathrm{red}}$ is $\langle D_1, D_2\rangle$-invariant.

We denote by X' the $\langle D_1, D_2\rangle$-invariant minimal abelian subvariety of X. Since $D_1 \neq 0$ we have $X' \neq 0$, on the other hand $\mathcal{W}$ contains a translate of X', therefore $X' \neq X$. Note that $\mathcal{W}_{\mathrm{red}}$ is $T_0(X')$-invariant. Let X'' be the complement of X' in X, relative to the polarization Θ. This means that X'' is the connected component containing zero of the kernel of the composite map $X \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X')$. Here the first map sends x to the class of $\Theta_x - \Theta$, and the second map is the natural restriction.

Let $\mathcal{R} := (\mathcal{W} \cap X'')_{\mathrm{red}}$. Note that $\mathcal{W}_{\mathrm{red}}$ is the $T_0(X')$-span of $\mathcal{R}$, i.e. $\mathcal{W}_{\mathrm{red}} = \mathcal{R} + X'$, and that $\mathcal{R}$ has codimension 2 in X''. In the sequel we shall work on $X'' \times X'$. Observe that θ is naturally a theta function also for $\pi^\star\mathcal{O}_X(\Theta)$ via the sum map $\pi\colon X'' \times X' \to X$. In fact, as $T_0(X'') \times T_0(X') \cong T_0(X)$ (canonically), there is a canonical identification of the universal cover of $X'' \times X'$ with the one of X which commutes with the isogeny $\pi\colon X'' \times X' \to X, (x'', x') \mapsto x'' + x'$. In particular, this property allows us to write θ instead of $\pi^\star\theta$ while working on $X'' \times X'$.

Let us fix general points $b \in \mathcal{R}, x' \in X'$, so that $p := (b, x')$ is a general point of $\pi^{-1}(\mathcal{W}_{\mathrm{red}})$. Let us decompose D_3 as $D_3' + D_3''$, where $D_3' \in T_0(X'), D_3'' \in T_0(X'')$. Since X' is generated by the $\langle D_1, D_2\rangle$-flow, D_3'' is nonzero by Lemma 3.10 (iii). Let L be the (analytic) germ at zero of the D_3''-integral line in X'' through zero, let

$\mathcal{C}$ be the germ at b of a smooth curve in X'' meeting $L+b$ transversally only at b, and let Y be the surface $\mathcal{C}+L$ in X''. Let Ω be the subvariety $Y \times X'$ of $X'' \times X'$.

Let λ be a parameter on $\mathcal{C}$ vanishing at b and let t be the coordinate on L (vanishing at zero) with $\partial_t = D_3''$. Thus λ, t are parameters on Y, likewise they are naturally parameters on the product $\Omega = Y \times X'$. Note that $[D_1^\alpha D_2^\beta D_3^\gamma(\ldots)]|_\Omega = D_1^\alpha D_2^\beta D_3^\gamma(\ldots|_\Omega)$. On Ω we write

$$\theta(t,\lambda,x) = \sum_{i,j\geqslant 0} \tau_{i,j}(x)\cdot t^i \cdot \lambda^j, \tag{4.1}$$

where x is in X'. We recall that by the definition of the complement of X' there is an isomorphism $(t_x^*\mathcal{O}(\Theta))|_{X'} \cong \mathcal{O}(\Theta)|_{X'}$ for all $x \in X''$, where t_x denotes the translation $y \mapsto y+x$. Thus the $\theta(t,\lambda,\cdot)$'s are sections of the restriction $\Theta|_{X'}$. Note that $\tau_{i,j}$ depends on the point b and the curve $\mathcal{C}$ chosen, and that $\tau_{i,j} = (1/i!\cdot j!)((\partial^j/\partial\lambda^j)D_3''^i\theta)(0,0,\cdot)$ is in $H^0(X',\Theta|_{X'})$. Indeed, since the $\theta(t,\lambda,\cdot)$'s are sections of the restriction $\Theta|_{X'}$, so are its derivatives with respect to t and λ.

We use Lemmas 3.9 and 3.10 to reach a contradiction. Our analysis is divided naturally in two cases which correspond to whether the variety $\mathcal{R}$ is not D_3''-invariant, or it is D_3''-invariant.

Let us first assume that $\mathcal{R}$ is not D_3''-invariant. Let us choose $\mathcal{C}$ in such a way that it meets $\mathcal{R}$ transversally only at $b, \partial_\lambda \notin \langle T_b(\mathcal{R}), D_3''\rangle$. This is possible because $\mathcal{R}$ has codimension 2 in X''. We have $Y\cap\mathcal{R} = \{\lambda = t = 0\}$, thus $\Omega \cap \pi^{-1}(\mathcal{W}_{\mathrm{red}}) = \{\lambda = t = 0\}\times X'$. It follows that $\tau_{i,0} \not\equiv 0$ for some i, and, moreover, $\tau_{0,0}(x) = 0$ (otherwise we would not have $\theta|_{b+X'} = 0$). Because of Lemma 3.9 we have $\theta = f(t,\lambda)\cdot\psi(t,\lambda,x)$, where $f(0,0) = 0$. We have $\Omega\cap\pi^{-1}\mathcal{W} = \Omega\cap\{\theta = 0\}\cap\{D_1\theta = 0\} \supseteq \Omega\cap\{f = 0\}$. Moreover, since $f(0,0) = 0$, it follows that $\Omega\cap\pi^{-1}\mathcal{W}$ has codimension 1 in Ω. This contradicts $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}}) = \{\lambda = t = 0\}\times X'$.

Let us now assume that $\mathcal{R}$ is D_3''-invariant. Choose $\mathcal{C}$, depending on the point x', in such a way that it meets $\mathcal{R}$ transversally only at b, and $\mathcal{C}\times\{x'\}\subset\{k=0\}$, where k is as in Lemma 3.10. Since the loci $\{h=0\}$ and $\{k=0\}$ are transverse by 3.10 (i) , and $\mathcal{C}$ meets $\mathcal{R}$ transversally at b, we may assume that λ is the restriction of h to $\mathcal{C}\times\{x'\}\cong\mathcal{C}$. We have that $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}}) = \{\lambda = 0\}$. Let $\rho = \min\{j|\exists i : \tau_{i,j}(\cdot)\not\equiv 0\}$. Note that, as $\mathcal{C}$ depends on x', $\tau_{i,j}$ depends on x'. We want to prove that $\tau_{0,\rho} = 0$. For this it suffices to prove that $D_1^\alpha D_2^\beta\tau_{0,\rho}(x') = 0$, for all α and β, since the flow generated by D_1 and D_2 is dense in X'. Since $\mathcal{C} = \{t = 0\}$, by 4.1 we have

$$\begin{aligned} D_1^\alpha D_2^\beta\theta|_{\mathcal{C}\times X'} &= D_1^\alpha D_2^\beta\theta(0,\lambda,\cdot) = \lambda^\rho\cdot D_1^\alpha D_2^\beta\tau_{0,\rho}(\cdot), \quad \mathrm{mod}(\lambda^{\rho+1}),\\ D_3\theta|_{\mathcal{C}\times X'} &= D_3\theta(0,\lambda,\cdot) = 0, \quad \mathrm{mod}(\lambda^\rho). \end{aligned} \tag{4.2}$$

By Lemma 3.10, in the local ring $\mathcal{O}_{\{k=0\},p}$ we have that $D_1^\alpha D_2^\beta\theta\in(h)^l, D_3\theta\notin(h)^l$, for some l, where h is as in 3.10. Since λ is the restriction of h to $\mathcal{C}\times\{x'\}\cong\mathcal{C}$, in the local ring $\mathcal{O}_{\mathcal{C}\times\{x'\},p}$, we have that $D_1^\alpha D_2^\beta\theta \in (\lambda)^l, D_3\theta \notin (\lambda)^l$. We have $l > \rho$ by the second formulas in 4.2. On the other hand, since $l > \rho$ we must have $D_1^\alpha D_2^\beta \tau_{0,\rho}(x') = 0$ for all α and β, by the first formulas in 4.2. Since X' is generated by the $\langle D_1, D_2\rangle$-flow and $D_1^\alpha D_2^\beta \tau_{0,\rho}(x') = 0$ for all α and β, we get $\tau_{0,\rho} = 0$. Hence we can apply Lemma 3.9. It follows that the equality 4.1 takes the form $\theta(t,\lambda,x) = \lambda^\rho \cdot f(t,\lambda)\cdot\psi(t,\lambda,x)$, so that f divides both $\theta|_\Omega$ and $D_1\theta|_\Omega$. Therefore, $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}}) \supset \Omega\cap\{f=0\}$. By Lemma 3.9, $f(0,0) = 0$ and $f(\cdot,0) \not\equiv 0$. As $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}}) \supset \Omega\cap\{f=0\}$, the locus $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}})$ contains (locally at p) a component which is not the component $\{\lambda = 0\}$. This contradicts the fact that, locally at p, $\Omega\cap\pi^{-1}(\mathcal{W}_{\mathrm{red}}) = \{\lambda = 0\}$. □

REMARK 4.3. If one could show that $\mathcal{W}$ (not only its underlying reduced scheme $\mathcal{W}_{\mathrm{red}}$) were $\langle D_1, D_2\rangle$-invariant it would easily follow by the very expression of $P_m\theta$ that $P_m\theta$ vanishes on $\mathcal{W}$. In fact, in this case, $\theta(z+a)$ (where $a \in X'$) would vanish on $\mathcal{W}$. Hence, $D_1^2\theta$ and $D_2\theta$ would vanish on $\mathcal{W}$ as well.

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