The emergence of time

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Prologue

To become a mathematician

A simple relation

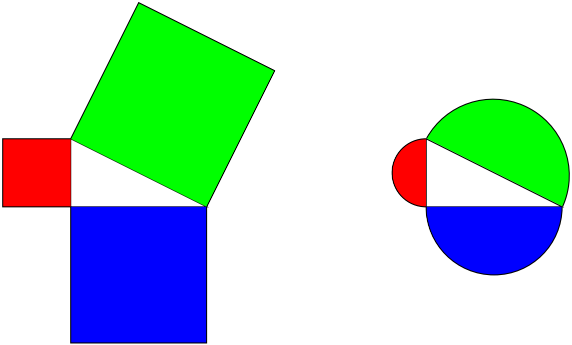
squares	*odds*
$1^2 = 1$	$1 - 0 = 1$
$2^2 = 4$	$4 - 1 = 3$
$3^2 = 9$	$9 - 4 = 5$
$4^2 = 16$	$16 - 9 = 7$
$5^2 = 25$	$25 - 16 = 9$
...	...

difference of consecutive squares = sequence of odd numbers

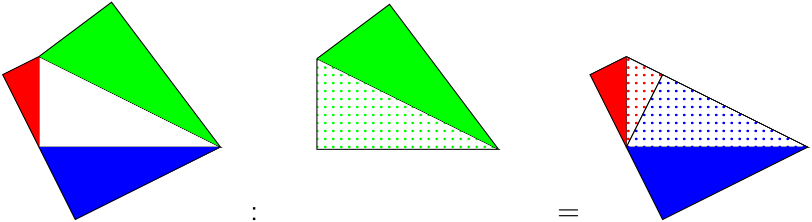
$$\underbrace{(n+1)^2 - n^2}_{\textit{general formula}} = \underbrace{n^2 + 2n + 1 - n^2}_{\textit{proof}} = \underbrace{2n+1}_{\textit{result}}$$

Pythagorean Theorem

Euclid's proof (reconsidered by Polya)

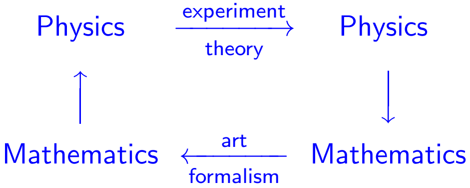


Pythagorean Theorem



The cycle Phys-Math-Math-Phys

"Nessuna humana investigazione si può dimandara vera scienzia s'essa non passa per le matematiche dimonstrazioni".
Leonardo da Vinci, Trattato della Pittura, 1500 circa



"*The Unreasonable Effectiveness of Mathematics in the Natural Sciences.*"
Eugene Wigner, Comm. Pure Appl. Math. 1960

Part I

General introduction

Noncommutativity and modular time

Time as derived quantity

classical static space $\rightarrow$ no time

quantum space $\rightarrow$ quantum fluctuations

no static quantum space may exist

noncommutativity generates time

The arrow of time

The arrow of time is viewed both classically and in quantum physics (Copenhagen interpretation)

thermodynamics $\rightarrow$ positive entropy

quantum mechanics $\rightarrow$ collapse of the wave function

Known question: is there a general frame to encompass both?

Of course, we keep in mind that time is a relative concept as we learnt from Einstein.

$$
\begin{array}{ccc}
 & \textit{Planck} & \\
\swarrow & & \searrow \\
\textit{Schrödinger} & & \textit{Heisenberg} \\
\searrow & & \swarrow \\
 & \textit{von Neumann uniqueness} &
\end{array}
$$

- Schrödinger:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)$$

Differential equations

- Heisenberg:

$$PQ - QP = i\hbar I$$

Linear operators on Hilbert space, **noncommutativity is essential**!

Operator Algebras

$\mathcal{H}$ = Hilbert space,

$B(\mathcal{H})$ = algebra of all bounded linear operators on $\mathcal{H}$.

Algebraic structure: linear structure, multiplication: $B(\mathcal{H})$ is a *algebra

Derived structures:

Order structure: $A \geq 0 \Leftrightarrow A = B^*B$: algebraic structure determines order structure

Metric structure:
$||A||^2 = \inf\{\lambda > 0 : A^*A \leq \lambda I\}$: algebraic structure determines metric structure

C^* property of the norm:
$||A^*A|| = ||A||^2$. $B(\mathcal{H})$ is a C^*-algebra

C^*-algebras = noncommutative topology

Gelfand-Naimark thm. $\exists$ contravariant functor F between category of *commutative* C^*-algebras and category of locally compact topological spaces:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{F} & \mathrm{spec}(\mathfrak{A}) \\ || & & || \\ C(X) & \xleftarrow{F^{-1}} & X \end{array}$$

C^*-algebra = dual of a topological space

Every C^*-algebra is isomorphic to a norm closed *-subalgebras of $B(\mathcal{H})$.

Noncommutative geometry = *-subalgebras of C^*-algebras + structure (spectral triple), Connes NC geometry.

von Neumann algebras = noncommutative measure theory

$\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra if $\mathcal{M}$ is a $*$-algebra on $\mathcal{H}$ and is weakly closed. Equivalently (von Neumann density theorem)

$$\mathcal{M} = \mathcal{M}''$$

with $\mathcal{M}' = \{T \in B(\mathcal{H}) : TX = XT \quad \forall X \in \mathcal{M}\}$ the commutant.

$$\mathcal{M} \text{ abelian} \Leftrightarrow \mathcal{M} = L^\infty(X, \mu):$$

$(\mathcal{M} = \{M_f : g \in L^2 \mapsto fg \in L^2\})$

von Neumann algebra = dual of a measure space

Physics: *Observables* are selfadjoint elements X of $\mathcal{M}$, *states* are normalised positive linear functionals φ,

$\varphi(X)$ = expected value of the observable X in the state φ

Operator Algebras

Classical Commutative	Quantum Noncommutative
Manifold X $C^\infty(X)$	$*$-algebra A
Topological space X $C(X)$	C^*-algebra $\mathfrak{A}$
Measure space X $L^\infty(X, \mu)$	von Neumann algebra $\mathcal{A}$

Quantum calculus with infinitely many degrees of freedom

CLASSICAL	Classical variables Differential forms Chern classes	Variational calculus Infinite dimensional manifolds Functions spaces Wiener measure
QUANTUM	Quantum geometry Fredholm operators Index Cyclic cohomology	Subfactors Bimodules, Endomorphisms Multiplicative index Supersymmetric QFT, $(\mathfrak{A}, \mathcal{H}, Q)$

Thermal equilibrium states

A primary role in thermodynamics is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

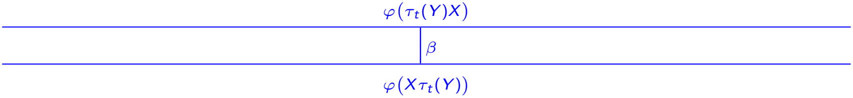
KMS states (HHW, Baton Rouge conference 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$

(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X \to X\Omega}\big\downarrow & & \big\downarrow{\scriptstyle X \to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$S = \bar{S}_0$, $\Delta = S^*S > 0$ positive selfadjoint

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

intrinisic dynamics associated with φ (*modular automorphisms*).

Modular theory and temperature

By a remarkable historical accident, Tomita announced the theorem at the 1967 Baton Rouge conference. Soon later Takesaki completed the theory and charcterised the modular group by the KMS condition.

• σ^φ is a purely noncommutative object (trivial in the commutative case)

• it is a thermal equilibrium evolution If $\varphi(X) = \text{Tr}(\rho X)$ (type I case) then $\sigma_t^\varphi(X) = \rho^{it} X \rho^{-it}$

• arrow of modular time is thermodynamical KMS condition at inverse temperature $\beta = -1$

• modular time is intrinsic modulo scaling the rescaled group $t \mapsto \sigma^\varphi_{-t/\beta}$ is physical, β^{-1} KMS temperature

Time as thermodynamical effect

quantum physics

$\updownarrow$

KMS with positive temperature

$\updownarrow$

modular time arrow

$\updownarrow$

positive entropy

$\updownarrow$

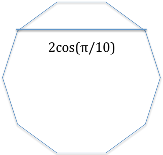
thermodynamical arrow

If time is the modular time, then the time arrow is associated both with positive entropy and with quantum structure!

Factors (von Neumann algebras with trivial center) are "very infinite-dimensional" objects. For an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ the Jones index $[\mathcal{M} : \mathcal{N}]$ measure the relative size of $\mathcal{N}$ in $\mathcal{M}$. Surprisingly, the index values are quantised:

$$[\mathcal{M} : \mathcal{N}] = 4\cos^2\left(\frac{\pi}{n}\right) \ , \quad n = 3, 4, \ldots \quad \text{or} \quad [\mathcal{M} : \mathcal{N}] \geq 4$$

Jones index appears in many places in math and in physics.



Quantum Field Theory

In QFT we have a quantum system with infinitely many degrees of freedom. The system is relativistic and there is particle creation and annihilation.

No mathematically rigorous QFT model with interaction still exists in 3+1 dimensions!

Haag local QFT:

$$O \text{ spacetime regions} \mapsto \text{von Neumann algebras } \mathcal{A}(O)$$

to each region one associates the "noncommutative functions" with support in O.

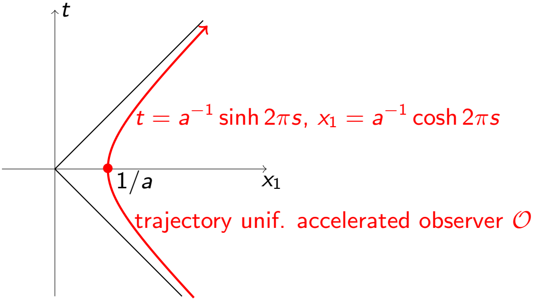
Local QFT nets

Local net $\mathcal{A}$ on spacetime M: map $O \subset M \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony*, $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality*, O_1, O_2 spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance* (conformal, diffeomorphism) .
- *Positive energy and vacuum vector*.

$O \mapsto \mathcal{A}(O)$: "Noncommutative chart" in QFT

Rindler spacetime (wedge $x_1 > 0$), vacuum modular group



a uniform acceleration of $\mathcal{O}$
s/a proper time of $\mathcal{O}$
$\beta = 2\pi/a$ inverse KMS temperature of $\mathcal{O}$

Hawking-Unruh effect!
Time is geodesic, quantum gravitational effect!

Representations

A (DHR) *representation* ρ of local net $\mathcal{A}$ maps $\mathcal{A}(O)$ on a different Hilbert space $\mathcal{H}$ s.t. but $\rho|_{\mathcal{A}(O')}$ is equivalent to the vacuum rep.

Index-statistics theorem (R.L. 1988):

$$d(\rho) = \Big[\mathcal{A}(O) : \rho\big(\mathcal{A}(O)\big)\Big]^{\frac{1}{2}}$$

DHR dimension $= \sqrt{\text{Jones index}}$

$$\nearrow \qquad\qquad\qquad \nwarrow$$

$$\textit{Physical index} \qquad\qquad \textit{Anal. index}$$

(basis for a QFT index theorem).

Part II

Applications

Intrinsic bounds on entropy

Bekenstein's bound

For decades, modular theory has played a central role in the operator algebraic approach to QFT, very recently several physical papers in other QFT settings are dealing with the modular group, although often in a heuristic (yet powerful) way!

I will discuss the Bekenstein bound, a universal limit on the entropy that can be contained in a physical system with given size and given total energy

If R is the radius of a sphere that can enclose our system, while E is its total energy including any rest masses, then its entropy S is bounded by

$$S \leq \lambda RE$$

The constant λ is often proposed $\lambda = 2\pi$ (natural units).

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exists; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

CP maps, quantum channels and entropy

$\mathcal{N}, \mathcal{M}$ vN algebras. A linear map $\alpha : \mathcal{N} \to \mathcal{M}$ is completely positive if

$$\alpha \otimes \mathrm{id}_n : \mathcal{N} \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{M} \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive $\forall n$ (quantum operation)

ω faithful normal state of $\mathcal{M}$ and $\alpha : \mathcal{N} \to \mathcal{M}$ CP map as above. Set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.

The conditional entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$ by the monotonicity of the relative entropy .

α is a quantum channel if its conditional entropy $\mathrm{H}(\alpha)$ is finite.

Generalisation of Stinespring dilation

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, completely positive unital map between the vN algebras $\mathcal{N}$, $\mathcal{M}$. A pair (ρ, v) $\rho : \mathcal{N} \to \mathcal{M}$ a homomorphism, $v \in \mathcal{M}$ an isometry s.t.

$$\alpha(n) = v^* \rho(n) v \ , \quad n \in \mathcal{N} \ .$$

(ρ, v) is *minimal* if the left support of $\rho(\mathcal{N}) v \mathcal{H}$ is equal to 1.

Thm Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, CP unital map with $\mathcal{N}$, $\mathcal{M}$ properly infinite. There exists a minimal dilation pair (ρ, v) for α. If (ρ_1, v_1) is another minimal pair, $\exists!$ unitary $u \in \mathcal{M}$ such that

$$u\rho(n) = \rho_1(n) u \, , \quad v_1 = uv , \qquad n \in \mathcal{N}$$

We have

$$H(\alpha) = \log Ind(\alpha) \qquad \text{(minimal index)}$$

Bimodules and CP maps

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$

$\exists!$ $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\rm out}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\rm in}(m)\,,$$

with $\omega_{\rm in} \equiv \omega$, $\omega_{\rm out} \equiv \omega_{\rm in} \cdot \alpha$. Converse is true.

CP map $\alpha \longleftrightarrow$ cyclic bimodule $\mathcal{H}_\alpha$

We have

$$H(\alpha) = \log Ind(\mathcal{H}_\alpha) \qquad \text{(Jones' index)}$$

Promoting modular theory to the bimodule setting

$\mathcal{H}$ an $\mathcal{N} - \mathcal{M}$-bimodule with finite Jones' index $\mathrm{Ind}(\mathcal{H})$

Given faithful, normal, states φ, ψ on $\mathcal{N}$ and $\mathcal{M}$, I define the modular operator $\Delta_{\mathcal{H}}(\varphi|\psi)$ of $\mathcal{H}$ with respect to φ, ψ as

$$\Delta_{\mathcal{H}}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon) \ ,$$

Connes' spatial derivative, $\varepsilon : \ell(\mathcal{N})' \to r(\mathcal{M})$ is the minimal conditional expectation

$\log \Delta_{\mathcal{H}}(\varphi|\psi)$ is called the modular Hamiltonian of the bimodule $\mathcal{H}$, or of the quantum channel α if $\mathcal{H}$ is associated with α.

Properties of the modular Hamiltonian

If $\mathcal{N}$, $\mathcal{M}$ factors

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)\ell(n)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = \ell\big(\sigma^{\varphi}_{t}(n)\big)$$

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)r(m)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = r\big(\sigma^{\psi}_{t}(m)\big)$$

(implements the dynamics)

$$\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) \otimes \Delta^{it}_{\mathcal{K}}(\varphi_2|\varphi_3) = \Delta^{it}_{\mathcal{H}\otimes\mathcal{K}}(\varphi_1|\varphi_3)$$

(additivity of the energy)

$$\Delta^{it}_{\bar{\mathcal{H}}}(\varphi_2|\varphi_1) = d_{\mathcal{H}}^{-i2t}\,\overline{\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2)}$$

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$T\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) = (d_{\mathcal{H}'}/d_{\mathcal{H}})^{it}\Delta^{it}_{\mathcal{H}'}(\varphi_1|\varphi_2)T$$

Connes's bimodule tensor product w.r.t. φ_2; $d_{\mathcal{H}} = \sqrt{Ind(\mathcal{H})}$

Physical Hamiltonian

We may modify the modular Hamiltonian in order to fulfil the right physical requirements (additivity of energy, invariance under charge conjugation,...)

$$K(\varphi_1|\varphi_2) = -\log\Delta_{\mathcal{H}}(\varphi_1|\varphi_2) - \log d$$

is the physical Hamiltonian (at inverse temperature 1).

The physical Hamiltonian at inverse temperature $\beta > 0$ is given by

$$-\beta^{-1}\log\Delta - \beta^{-1}\log d$$

From the modular Hamiltonian to the physical Hamiltonian:

$$-\log\Delta \xrightarrow{\text{shifting}} -\log\Delta - \log d \xrightarrow{\text{scaling}} \beta^{-1}\big(-\log\Delta - \log d\big)$$

The shifting is intrinsic, the scaling is to be determined by the context!

Thermodynamical quantities

The entropy $S \equiv S_{\alpha,\omega_{\rm in}}$ of α is

$$S = -(\hat{\xi}, \log \Delta_\alpha \hat{\xi})$$

where $\hat{\xi}$ is a vector representative of the state $\omega_{\rm in} \cdot r^{-1} \cdot \varepsilon$ in $\mathcal{H}_\alpha$.

The quantity

$$E = (\hat{\xi}, K\hat{\xi})$$

is the relative energy w.r.t. the states $\omega_{\rm in}$ and $\omega_{\rm out}$.

The free energy F is now defined by the relative partition function

$$F = -\beta^{-1} \log(\hat{\xi}, e^{-\beta K}\hat{\xi})$$

F satisfies the thermodynamical relation

$$F = E - TS$$

A form of Bekenstein bound

As $F = \frac{1}{2}\beta^{-1}\mathrm{H}(\alpha)$, we have

$$F \geq 0 \quad \text{(positivity of the free energy)}$$

because

$$\mathrm{H}(\alpha) \geq 0 \quad \text{(monotonicity of the entropy)}$$

So the above thermodynamical relation

$$F = E - \beta^{-1}S$$

entails the following general, rigorous version of the Bekenstein bound

$$S \leq \beta E$$

To determine β we have to plug this general formula in a physical context

Schwarzschild black hole

Schwarzschild-Kruskal spacetime of mass $M > 0$, namely the region inside the event horizon, and $\mathcal{N} \equiv \mathcal{A}(O)$ the local von Neumann algebra associated with O on the underlying Hilbert space $\mathcal{H}$, O Schwarzschild black hole region, ω vacuum state

$\mathcal{H}$ is a $\mathcal{N} - \mathcal{N}$ bimodule, indeed the identity $\mathcal{N} - \mathcal{N}$ bimodule $L^2(\mathcal{N})$ associated with ω.

The modular group of $\mathcal{A}(O)$ associated with ω is geometric and corresponds to the geodesic flow. KMS Hawking temperature is

$$T = 1/8\pi M = 1/4\pi R$$

with $R = 2M$ the Schwarzschild radius, then

$$S \leq 4\pi RE$$

with S the entropy associated with the state transfer of ω by a quantum channel, and E the corresponding relative energy.

Landauer's bound for infinite systems

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel between quantum systems $\mathcal{N}$, $\mathcal{M}$. If α is irreversible, then

$$F_\alpha \geq \frac{1}{2} kT \log 2$$

The original lower bound for the incremental free energy is $F_\alpha \geq kT \log 2$, it remains true for finite-dimensional systems $\mathcal{N}$, $\mathcal{M}$.

Energy conditions in QFT

Energy conditions play classically an important role in general relativity.

In QFT, energy may locally have negative density states (Epstein, Glaser, Jaffe), although certain energy lower bounds may occur. E.g., in conformal QFT, there are local lower bounds (Fewster, Hollands; Wiener)

Bousso, Fisher, Liechenauer, and Wall proposed the QNEC: For null direction deformation

$$\langle T_{uu} \rangle \geq \frac{1}{2\pi} S''_A(\lambda) \ ,$$

T stress-energy tensor, S_A is the entropy relative to the region A on one side of the deformation and S''_A is the second derivative of S_A w.r.t. the deformation parameter λ.

Entropy of localised states: $U(1)$-current model

Case of $U(1)$-current j: ℓ real function in $S(\mathbb{R})$ and $t \in \mathbb{R}$. We have

$$S(t) = \pi \int_t^{+\infty} (x - t)\ell^2(x)\mathrm{d}x \ ,$$

$S(t)$ vacuum relative entropy of excited state by $j \mapsto j + \ell$, so

$$S'(t) = -\pi \int_t^{+\infty} \ell^2(x)\mathrm{d}x \leq 0 \ ,$$

$$S''(t) = \pi \ell^2(t) \geq 0$$

positivity of S''

Quantum Null Energy Condition

The vacuum energy density is $E(t) = \frac{1}{2}\ell^2(t)$ so we have here the QNEC:

$$E(t) = \frac{1}{2\pi}S''(t) \geq 0$$

QNEC is not saturated in every point of positive energy density.

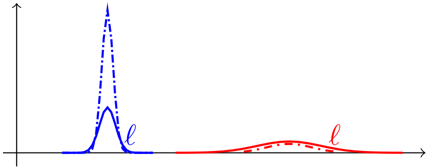


Figure: Two distributions, blue and red, for the same charge $q = \int \ell$. The dashed lines plot the corresponding entropy density rate $S''(t)$: blue high entropy, red low entropy.

QNEC for coherent states (L., Ciolli, Ruzzi & L.)

Let Φ be a real Klein-Gordon wave, namely $(\Box + m^2)\Phi = 0$. The relative entropy of Φ w.r.t. the vacuum is

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda, x^1\geq\lambda} (x^1 - \lambda) T_{00}(x) dx \ ,$$

where $T_{\mu\nu}$ is the stress-energy tensor in the wave Φ w.r.t. the wedge region $x^1 - \lambda > |x^0 - \lambda|$.
In particular, the QNEC inequality

$$\frac{d^2}{d\lambda^2} S_\Phi(\lambda) \geq 0$$

holds true for the coherent state associated with Φ.