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Operator Algebras and Index Theorems in Quantum Field Theory

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1. *Jones Index and Local von Neumann Algebras*
2. *The Structure and Classification of Conformal Nets*
3. *Topological Sectors, QFT Index Theorems*

Jones Index and Local von Neumann Algebras

Basic notions. $\mathcal{H} =$ (complex) Hilbert space

A linear operator $A : \mathcal{H} \to \mathcal{H}$ is continuous w.r.t. the norm topology iff A is *bounded*, namely

$$||A|| \equiv \sup_{||\xi|| \leq 1} ||A\xi|| < \infty$$

$B(\mathcal{H}) =$ algebra of all bounded linear operators on $\mathcal{H}$.

Algebraic structure:

$\alpha A + \beta B$ linear structure

AB multiplication

(+ distributive, associative laws) $\longrightarrow$ $\underline{B(\mathcal{H}) \text{ is an algebra}}$

$A \mapsto A^*$ involution: $(A\xi, \eta) = (\xi, A^*\eta)$ $\longrightarrow$ $B(\mathcal{H})$ is a *algebra

Order structure:

$A \geq 0 : (A\xi, \xi) \geq 0$

$A \geq 0 \Leftrightarrow A = B^*B$: algebraic structure determines order structure

Metric structure:

$A_i \to A$ (in norm): $||A - A_i|| \to 0$. $B(\mathcal{H})$ is a Banach algebra: $||AB|| \leq ||A||||B||$

$||A||^2 = \inf\{\lambda > 0 : A^*A \leq \lambda I\}$: algebraic structure determines metric structure

C^* property of the norm:

$||A^*A|| = ||A||^2$. *$B(\mathcal{H})$ is a C^*algebra*

Other topologies.

$A_i \to A$ strongly: $||A\xi - A_i\xi|| \to 0$

$A_i \to A$ weakly: $(A_i\xi, \eta) \to (A\xi, \eta)$. $B(\mathcal{H})$ *is a weakly/strongly closed, it is a von Neumann algebra*

Def. A *von Neumann algebra* $\mathcal{M}$ is a weakly closed non-degenerate *-subalgebra of $B(\mathcal{H})$.

Example 1. $L^\infty(X, \mu)$ ess. bounded function on a measure space:

$$f \in L^\infty(X, \mu) \leftrightarrow M_f \in B(L^2(X, \mu)), \quad M_f g = fg.$$

Example 2. $B(\mathcal{H})$.

von Neumann density thm. $\mathfrak{A} \subset B(\mathcal{H})$ non-degenerate *-subalgebra

$$\boxed{\mathfrak{A}^- = \mathfrak{A}''}$$

where $'$ denotes the commutant

$$\mathfrak{A}' = \{T \in B(\mathcal{H}) : TA = AT \ \forall A \in \mathfrak{A}\}$$

weak or strong closure = double commutant: Double aspect, analytical and algebraic.

$$\boxed{\mathcal{M} \text{ abelian vN algebra} \Leftrightarrow \mathcal{M} \simeq L^\infty(X, \mu).}$$

<u>von Neumann algebras = NC measure theory</u>

$\mathcal{M}$ is a *factor* if it center $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$.

$\Omega \in \mathcal{H}$ is *cyclic* if $\overline{\mathcal{M}\Omega} = \mathcal{H}$; *separating* if $x\Omega = 0$, $x \in \mathcal{M} \implies x = 0$. Ω cyclic for $\mathcal{M} \Leftrightarrow \Omega$ separating for $\mathcal{M}'$.

Example 3. G discrete group,

$\mathcal{H} \equiv \ell^2(G) = \{\xi : G \to \mathbb{C} \text{ s.t. } \sum |\xi(g)|^2 < \infty\}$

λ left regular rep. of G: $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$, $\xi \in \mathcal{H}$

$\mathcal{M} = \mathsf{vN}(G) =$ weak closure of lin.span$\{\lambda(g), g \in G\}$

$G = \mathbb{Z} \implies \mathsf{vN}(G) \simeq L^\infty(\mathbb{T})$ (Fouries series)

G ICC group (e.g. $\mathbb{S}_\infty, \mathbb{F}_2$) $\implies \mathcal{M}$ is *factor*

$\tau(x) \equiv (x\Omega, \Omega)$, $x \in \mathsf{vN}(G)$, $\Omega = \delta_{g,e}$

τ is a trace: $\tau(xy) = \tau(yx)$

Proof: $\tau(\lambda(g)\lambda(h)) = (\lambda(gh)\Omega, \Omega) = \delta_{gh,e} = \delta_{hg,e} = (\lambda(hg)\Omega, \Omega) = \tau(\lambda(h)\lambda(g))$

Note: Ω is cyclic and separating and the anti-unitary involution $J : \xi(g) \mapsto \overline{\xi(g^{-1})}$ satisfies

$$J\mathcal{M}J = \mathcal{M}'$$

where $\mathcal{M}' = \rho(G)''$, $\rho =$ right regular representation.

There are factor with no (even unbounded) trace, factor of type III.

Def. A C^*-*algebra* is a Banach algebra $\mathfrak{A}$ with an anti-automorphism involution $a \to a^*$ satisfying $||a^*a|| = ||a||^2$.

Example 1. $C(X)$ = continuos functions on a cmpt space X ($||f|| = \max_{x \in X} |f(x)|$, $f^* = \bar{f}$).

Example 2. Norm closed *-subalgebras of $B(\mathcal{H})$.

Gelfand-Naimark thm. $\exists$ contravariant functor F between category of (unital) abelian C^*-algebras and category of cmpt topological spaces:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{F} & \text{spec}(\mathfrak{A}) \\ || & & || \\ C(X) & \xleftarrow{F^{-1}} & X \end{array}$$

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\text{homomor.}} & \mathfrak{B} \\ \updownarrow & & \updownarrow \\ X & \xleftarrow{\text{cont. map}} & Y \end{array}$$

<u>C^*-algebras = noncommutative topology</u>

A *state* ω on a unital C^*-algebra $\mathfrak{A}$ is a positive linear functional on $\mathfrak{A}$, $\omega(1) = 1$ (noncommutative probability measure).

A *representation* π of $\mathfrak{A}$ is a homomorphism $\pi : \mathfrak{A} \to B(\mathcal{H})$.

GNS construction. ω state $\longrightarrow (\mathcal{H}, \pi, \Omega)$

$$\omega(x) = (\pi(x)\Omega, \Omega), \quad x \in \mathfrak{A},$$

$\overline{\pi(\mathfrak{A})\Omega} = \mathcal{H}$, i.e. Ω cyclic.

Every C^*-algebra is isomorphic to a norm closed *-subalgebras of $B(\mathcal{H})$.

Representation theory for C^*-algebras is crucial (NC Radon measures)

A state (or representation) ω on a von Neumann algebra is *normal* if it is σ-weakly continuous; equiv. $x_i \nearrow x \implies \omega(x_i) \to \omega(x)$ (Lebesgue monotone convergence thm. holds).

(Normal) representation theory of vN algebras is only multiplicity.

NC geometry = *-subalgebras of C^*-algebras + structure. cf. Connes NC geometry.

Example 3. $A = \mathrm{Mat}_2(\mathbb{C})$,

$$\mathfrak{A} = \overline{A \otimes A \otimes A \otimes \cdots} \quad \text{(norm completion)}$$

$t \in (0,1)$, φ_t the state on A

$$\varphi_t\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ta + (1-t)d$$

then $\varphi_t \otimes \varphi_t \otimes \cdots$ is a state on $\mathfrak{A}$ with GNS rep. π_t

$t = 1/2 \implies \pi_t(\mathfrak{A})''$ finite factor

$t \neq 1/2 \implies \pi_t(\mathfrak{A})''$ type III factor (Powers factors).

Amenable factors are classified by Connes and Connes-Haagerup (III_1-case).

Tomita-Takesaki theory. $\mathcal{M}$ a von Neumann algebra on $\mathcal{H}$.

$\Omega \in \mathcal{H}$ cyclic for $\mathcal{M} \Leftrightarrow \Omega$ separating for $\mathcal{M}'$.

ω normal faithful state, i.e. $\omega(x^*x) > 0 \; \forall x \neq 0$. We may assume $\omega = (\cdot\Omega, \Omega)$ with Ω cyclic and separating ($\mathcal{M}$ acts standardly). Set

$$L^\infty(\mathcal{M}) \equiv \mathcal{M}, \quad L^2(\mathcal{M}) = \mathcal{H} \quad L^1(\mathcal{M}) = \mathcal{M}_*,$$

where $\mathcal{M}_*$ is the predual of $\mathcal{M}$ (normal lin. functionals), $(\mathcal{M}_*)^* = \mathcal{M}$.

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "x \mapsto x*", "\text{isometric}"'] \arrow[d, "x \to x\Omega"'] & \mathcal{M} \arrow[d, "x \to x\Omega"] \\
L^2(\mathcal{M}) \arrow[r, "S_0 : x\Omega \mapsto x^*\Omega", "\text{non isometric}"'] & L^2(\mathcal{M})
\end{tikzcd}
$$

S the closure of the anti-linear operator S_0, $S = J\Delta^{1/2}$ polar decomposition, thus $\Delta = S^*S > 0$ positive selfadjoint, J anti-unitary involution:

$$
\boxed{\begin{array}{c} \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \\ J\mathcal{M}J = \mathcal{M}' \end{array}}
$$

$\omega \to \sigma_t^\omega = \mathrm{Ad}\Delta^{it}$ canonical "evolution" associated with ω (*modular automorphisms*).

<u>NC measure theory is non-trivial and rich</u>; in the abelian case only one standard, non-atomic Borel space!

Example. $\mathcal{M} = \mathrm{Mat}_n(\mathbb{C})$,

$\mathcal{H} = \mathcal{M}$ with scalar product $(A, B) = \mathrm{Tr}(B^* A)$

ω faithful state, $\mathrm{Tr}(T\cdot)$ (Riesz lemma), $T \geq 0$

GNS: $\pi : \mathcal{M} \to B(\mathcal{H})$, $\pi(A)B = AB$, $\Omega = T^{1/2} \in \mathcal{H}$, $\omega(A) = (\pi(A)\Omega, \Omega)$.

$S : AT^{1/2} \mapsto A^* T^{1/2}$, $\Delta : A \mapsto TAT^{-1}$,

$$\sigma_t^\omega(A) = T^{it} A T^{-it}, \quad J\pi(A)J = \pi'(A^*)$$

$\pi'(A)B = BA$

σ measures the deviation of ω from being a trace. σ inner iff $\exists$ a (bounded or unbounded) trace.

σ^ω is characterized by the KMS condition

$$\omega(yx) = \underset{t \to -i}{\text{anal.cont.}}\, \omega(\sigma_t^\omega(x)y), \quad x, y \in \mathcal{M},$$

that characterizes thermal equilibrium states in Quantum Statistical Mechanics (Haag, Hugenholtz and Winnik).

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann. A *conditional expectation* $\varepsilon : \mathcal{M} \to \mathcal{N}$ is a positive linear map with $\varepsilon \restriction \mathcal{N} = \iota$.

Takesaki thm. ω faithful normal state of $\mathcal{M}$. $\exists$ ω- preserving expectation $\varepsilon : \mathcal{M} \to \mathcal{N}$ (i.e. $\omega \cdot \varepsilon = \omega$) $\Leftrightarrow \sigma_t^\omega(\mathcal{N}) = \mathcal{N}$.

$$\begin{tikzcd}
L^\infty(\mathcal{M}) \arrow[r, "\varepsilon", "\text{expect.}"'] \arrow[d, "x \to x\Omega"'] & L^\infty(\mathcal{N}) \arrow[d, "x \to x\Omega"] \\
L^2(\mathcal{M}) \arrow[r, "e", "\text{orth. proj.}"'] & L^2(\mathcal{N}) = \overline{\mathcal{N}\Omega}
\end{tikzcd}$$

Jones theory. $\mathcal{N} \subset \mathcal{M}$ inclusion of factors. $\mathcal{M}$ to be finite, namely there exists a (unique) tracial state $\omega = (\cdot\Omega, \Omega)$ on $\mathcal{M}$. With e the projection onto $\overline{\mathcal{N}\Omega}$, the von Neumann algebra generated by $\mathcal{M}$ and e

$$\mathcal{M}_1 = \langle \mathcal{M}, e \rangle = J_{\mathcal{M}} \mathcal{N}' J_{\mathcal{M}}$$

is a semifinite factor ($\exists$ unbounded trace).

$\mathcal{N} \subset \mathcal{M}$ has finite *index* $\overset{\text{def}}{=}$ $\mathcal{M}_1$ is finite. The index is defined as

$$[\mathcal{M} : \mathcal{N}] = \omega(e)^{-1}$$

with ω also denoting the trace of $\mathcal{M}_1$.

Jones thm.

$$\boxed{[\mathcal{M} : \mathcal{N}] \in \left\{ 4\cos^2 \frac{\pi}{n}, n \geq 3 \right\} \cup [4, \infty] \,.}$$

A *probabilistic definition* of the index was given by Pimsner and Popa through the inequality

$$\varepsilon(x) \geq \lambda x, \quad x \in \mathcal{M}^+,$$

$\lambda = [\mathcal{M} : \mathcal{N}]^{-1}$ where $\varepsilon : \mathcal{M} \to \mathcal{N}$ is the trace preserving conditional expectation.

$\mathcal{N} \subset \mathcal{M}$ any inclusion of factors, $\varepsilon : \mathcal{M} \to \mathcal{N}$ normal expectation:

$[\mathcal{M} : \mathcal{N}]_\varepsilon$ defined by Popa, Kosaki (e.g. by Pimsner-Popa inequality)

Minimal index (Hiai, L.)

$$[\mathcal{M} : \mathcal{N}] = \inf_\varepsilon [\mathcal{M} : \mathcal{N}]_\varepsilon = [\mathcal{M} : \mathcal{N}]_{\varepsilon_0}$$

where ε_0 is the unique *minimal conditional expectation*.

Jones tower. One can iterate Jones construction

$$\mathcal{M} \subset \mathcal{M}_1 = \langle \mathcal{M}, e_0 \rangle \subset \mathcal{M}_2 \subset \langle \mathcal{M}_1, e_1 \rangle \cdots$$

the projections e_i's satisfy

$$e_i e_j = e_j e_i \text{ if } |i - j| \geq 2,$$
$$e_{i\pm 1} e_i e_{i\pm 1} = \lambda e_i$$

If $\lambda^{-1} = [\mathcal{M} : \mathcal{N}] < 4$ then $g_i = qe_i - (1 - e_i)$ gives a representation of Artin braid group $\mathbb{B}$, $q + q^{-1} + 2 = \lambda$:

$$g_j g_i = g_i g_j \text{ if } |i - j| \geq 2,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

Evaluating and rescaling with ω an element $\alpha \in \mathbb{B}_n$ with exponent sum l gives *Jones polynomial* invariant for knots and links:

$$\boxed{V_L(q) = \left(-\frac{q+1}{\sqrt{q}}\right)^{n-1} (\sqrt{q})^l \omega(\alpha)}$$

Joint modular structure. Sectors. $\mathcal{N} \subset \mathcal{M}$ type III factors. $J_{\mathcal{N}}$ and $J_{\mathcal{M}}$ modular conjugations of $\mathcal{N}$ and $\mathcal{M}$.

The unitary $\Gamma = J_{\mathcal{N}} J_{\mathcal{M}}$ implements a *canonical endomorphism* of $\mathcal{M}$ into $\mathcal{N}$

$$\boxed{\gamma(x) = \Gamma x \Gamma^*, \qquad x \in \mathcal{M}.}$$

Proof. $\Gamma \mathcal{M} \Gamma = J_{\mathcal{N}} J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} J_{\mathcal{N}} = J_{\mathcal{N}} \mathcal{M}' J_{\mathcal{N}} \subset J_{\mathcal{N}} \mathcal{N}' J_{\mathcal{N}} = \mathcal{N}$.

γ depends on $J_{\mathcal{N}}$ and $J_{\mathcal{M}}$ only up to inners of $\mathcal{N}$; γ is canonical as a sector of $\mathcal{M}$:

The *sectors of* $\mathcal{M}$ are

$$\mathrm{Sect}(\mathcal{M}) = \mathrm{End}(\mathcal{M})/\mathrm{Inn}(\mathcal{M})$$

$\rho, \rho' \in \mathrm{End}(\mathcal{M})$, $\rho \sim \rho'$ iff there is a unitary $u \in \mathcal{M}$ such that $\rho'(x) = u\rho(x)u^*$ for all $x \in \mathcal{M}$.

$\mathrm{Sect}(\mathcal{M})$ is a *-*semiring*

Addition (direct sum): Let $\rho_1, \rho_2 \in \mathrm{End}(\mathcal{M})$; then $\rho \equiv \rho_1 \oplus \rho_2$

$$\rho : x \in \mathcal{M} \to \begin{bmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{M}) \simeq \mathcal{M}$$

naturally up to inners, thus in $\mathrm{Sect}(\mathcal{M})$.

Composition (monoidal product). Usual composition of maps

$$\rho_1 \cdot \rho_2(x) = \rho_1(\rho_2(x)), \qquad x \in \mathcal{M}$$

passes to the quotient Sect($\mathcal{M}$).

Conjugation. With $\rho \in$ End($\mathcal{M}$), choose a canonical endomorphism $\gamma_\rho : \mathcal{M} \to \rho(\mathcal{M})$. Then

$$\bar{\rho} = \rho^{-1} \cdot \gamma_\rho$$

well-defines a conjugation in Sect($\mathcal{M}$). Thus have

$$\boxed{\gamma_\rho = \rho \cdot \bar{\rho}}$$

Connes bimodules and sectors. $L^2(\mathcal{M})$ is a normal bimodule for $\mathcal{M}$

$$x, y \in \mathcal{M},\ \xi \in L^2(\mathcal{M}) \mapsto x\xi y \equiv xJy^*J\xi$$

If $\rho \in$ End($\mathcal{M}$) the bimodule $L^2_\rho(\mathcal{M})$ is $L^2(\mathcal{M})$ with left-rigth actions

$$x, y \in \mathcal{M},\ \xi \in L^2(\mathcal{M}) \mapsto \rho(x)\xi y \equiv xJy^*J\xi$$

All normal bimodules on $\mathcal{M}$ arise in this way up to unitary equivalence (Connes). <u>Representation concepts make sense</u>.

$$\text{Bimod}(\mathcal{M})_{/\sim} = \text{Sect}(\mathcal{M})$$

$\mathrm{Ind}(\rho) \equiv [\mathcal{M}; \rho(\mathcal{M})]$.

Prop. $\rho \in \mathrm{End}(\mathcal{M})$ irreducible.

$$\mathrm{Ind}(\rho) < \infty \Leftrightarrow \rho\bar{\rho} \succ \iota \;\&\; \bar{\rho}\rho \succ \iota$$

Analytic def. of conjugate = algebraic def. of conjugate

One may represent objects with non-integral dimension $d(\rho) = \sqrt{\mathrm{Ind}(\rho)}$ as quantum groups, loop groups, infinite-dimensional Lie algebras, superselection sectors, . . .

The tensor category End(M).

Tensor category = category equipped with monoidal product (internal tensor product) on objects and arrows (+ natural compatibility conditions).

Tensor C^**-category* = tensor category + arrows form a Banach space with an involution reversing $C*$ property $||T^* \circ T|| = ||T||^2$ (Doplicher, Roberts).

$\mathcal{M}$ an infinite factor $\rightarrow$ End(M) is a *tensor* C^*-*category*:

Objects: $=$ End(M)

Hom$(\rho, \rho') \equiv \{a \in M : a\rho(x) = \rho'(x)a \ \forall x \in M\}$

Composition of intertwiners (arrows): operator product

C^* *property*: obvious

Tensor product of objects: $\rho \otimes \rho' = \rho\rho'$

Tensor product of arrows: $\sigma, \sigma' \in$ End(M), $t \in$ Hom(ρ, ρ'), $s \in$ Hom(σ, σ'),

$$t \otimes s \equiv t\rho(s) = \rho'(s)t \in \text{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma') \ .$$

If ρ is irreducible (i.e. $\rho(M)' \cap M = \mathbb{C}$) and has finite index, then $\bar{\rho}$ is the unique sector such that $\rho\bar{\rho}$ contains the identity sector.

$\rho, \bar{\rho} \in$ End(M) are conjugate as sectors iff $\exists$ isometries $v \in$ Hom$(\iota, \rho\bar{\rho})$ and $\bar{v} \in$ Hom$(\iota, \bar{\rho}\rho)$ such that

$$\boxed{\begin{aligned} (\bar{v}^* \otimes 1_{\bar{\rho}}) \cdot (1_{\bar{\rho}} \otimes v) \equiv \bar{v}^* \bar{\rho}(v) = \frac{1}{d}, \\ (v^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \bar{v}) \equiv v^* \rho(\bar{v}) = \frac{1}{d}, \end{aligned}}$$

for some $d > 0$.

The *minimal* d is the *dimension* $d(\rho)$; it is related to the minimal index by

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definion of the index)

$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$

$d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)$

$d(\bar{\rho}) = d(\rho)$.

<ins>Every</ins> subset of End(M) having finite-index generate (by composition, subobjects, diret sum) a *C^*-tensor category with conjugates*.

Example 1. (Connes) G discrete (or locally compact) group,

π finite-dimensional unitary rep. of G on $\mathcal{H}$

$\lambda \otimes \pi$ acts on the left on $\ell^2(G) \otimes \mathcal{H}$

$\rho \otimes \iota$ acts on the right on $\ell^2(G) \otimes \mathcal{H}$

$\lambda \otimes \pi \sim \lambda$ (absorbing propery of λ) $\implies$ $\ell^2(G) \otimes \mathcal{H}$ is a vN(G) bimodule with dimension dim$\mathcal{H}$.

Tensor product of reps. $\leftrightarrow$ tensor product of sectors.

Example 2. (Cuntz) Let $V_1, V_2, \ldots V_n$ be isometries with final projections forming a partion of I:

$$V_i^* V_i = I, \quad \sum_{i=1}^{n} V_i V_i^* = I$$

$H =$ Lin.span$\{V_i\}$ is a Hilbert space: $(X, Y)I \equiv X^* Y$

C^*-algebra generated by the V_i's is <u>universal</u>, it depends only on H: the Cuntz algebra O_n.

$U \in O_n$ unitary $\rightarrow \lambda_U \in$ End(O_n), $\lambda_U : V_i \mapsto UV_i$

W multiplicative unitary on $H \otimes H$ (Baaj and Skandalis)

$$W_{12} W_{13} W_{23} = W_{23} W_{12}$$

$\Leftrightarrow$ Hopf algebra (in particular all finite groups arise in this way!)

$R \equiv WF$, F flip symmetry of $H \otimes H$. $R \in H \cdot H \cdot \bar{H} \cdot \bar{H} \to R \in O_n$

(On a weak closure) dim$\lambda_R =$ dim. of Hopf algebra,

tensor category generated by $\lambda_R =$ rep. tensor category of Hopf algebra.

Embedding an abstract tensor C^*-category $\mathcal{T}$. (Roberts, L.)

For each finite-dimensional object ρ there is an associated von Neumann algebra $\mathcal{M}_\rho$

$$\left(\varinjlim_{n,m} \mathrm{Hom}(\rho^n, \rho^m)\right)''$$

and a tensor functor $F : \mathcal{T}_\rho \to \mathrm{End}(\mathcal{M}_\rho)$. F is full if ρ is rational or amenable following Popa:

$$\boxed{\mathrm{End}(\mathcal{M})\ \text{“universal” tensor } C^* \text{ tensor category}}$$

Haag-Kastler nets in QFT. Minkowski space-time: $\mathbb{R}^4$ with metric $\langle \mathbf{x}, \mathbf{y} \rangle = x_0^2 - x_1^2 - x_2^2 - x_3^2$

$\mathcal{K}$ family of regions (say double cones)

$\mathcal{A}(\mathcal{O})$: von Neumann algebra generated by the observables localized in $\mathcal{O}$ in a QFT. The net

$$\mathcal{O} \to \mathcal{A}(\mathcal{O})$$

satisfies :

isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$;

locality: $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$, with $\mathcal{O}' \equiv \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle < 0\ \forall \mathbf{y} \in \mathcal{O}\}$

Haag duality: $\mathcal{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$;

Poincaré covariance: $\exists$ unitary rep. U of Poincaré group $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ with

$$U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}),\ g \in \mathcal{P}_+^\uparrow, \mathcal{O} \in \mathcal{K}.$$

Positive energy. The generator of time translation is positive: $H \geq 0$.

Vacuum: $\exists!$ U-invariant vector Ω, cyclic for the *quasi-local C^*-algebra*

$$\mathfrak{A} = \cup_{\mathcal{O}\in\mathcal{K}}\mathcal{A}(\mathcal{O})^-$$

(norm closure)

Representations. A *superselection sector* (Wick, Wightman and Wigner), i.e. a label for quantum "charges", is an equivalence class of physical representations of $\mathfrak{A}$.

What subset of Rep$\mathfrak{A}$ is physical?

Borchers: $\exists$ a positive-energy representation U_ρ of the universal covering group $\tilde{\mathcal{P}}_+^\uparrow$ s.t. $\forall X \in \mathfrak{A}, \quad g \in \tilde{\mathcal{P}}_+^\uparrow$

$$U_\rho(g)\rho(X)U_\rho(g)^{-1} = \rho(U(g)XU(g)^{-1}),$$

DHR: localized representation

Buchholz-Fredenhagen thm.: positive energy $\Longrightarrow$ localization

We shall see a converse with above methods.

Doplicher-Haag-Roberts theory.

$$\pi \text{ DHR rep. of } \mathfrak{A} \overset{\text{def}}{\Leftrightarrow} \pi\restriction\mathfrak{A}(\mathcal{O}') \simeq \iota\restriction\mathfrak{A}(\mathcal{O}')$$

$\forall \mathcal{O} \in \mathcal{K}$.

Lemma. Given $\mathcal{O} \in \mathcal{K}$, $\exists \rho \in \mathrm{End}(\mathfrak{A})$, $\rho \simeq \pi$

$$\rho\restriction\mathfrak{A}(\mathcal{O}') = \mathrm{id}$$

ρ is a <u>localized endomorphism</u> of $\mathfrak{A}$.

Proof. U unitary s.t. $\pi(X) = UXU^*, \forall X \in \mathfrak{A}(\mathcal{O}')$.

$\rho \equiv U^*\pi(\cdot)U$.

$\rho(X) = X$ if $X \in \mathfrak{A}(\mathcal{O}')$.

$Y \in \mathcal{A}(\mathcal{O}), X \in \mathfrak{A}(\mathcal{O}') \implies YX - XY = 0$ thus

$\rho(Y)X - X\rho(Y) = \rho(Y)\rho(X) - \rho(X)\rho(Y) = \rho(YX - XY) = 0$

thus $\rho(Y) \in \mathfrak{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O})$ (Haag duality)

thus $\rho\restriction_{\mathcal{A}(\mathcal{O})} \in \mathrm{End}(\mathcal{A}(\mathcal{O}))$ and $\rho \in \mathrm{End}(\mathfrak{A})$.

<u>DHR endom. form a tensor C^*-category.</u>

Statistics. ρ localized in $\mathcal{O} \in \mathcal{K}$

Choose $\rho_1 \sim \rho$ localized in $\mathcal{O}_1 \subset \mathcal{O}'$: $\rho_1 = u\rho(\cdot)u^*$ with $u \in \mathfrak{A}$.

$\rho\rho_1 = \rho_1\rho$ gives $\epsilon = u^*\rho(u) \in \rho^2(\mathfrak{A})'$

$\epsilon_i \equiv \rho^{i-1}(\epsilon)$, $i \in \mathbb{N}$,

$$\begin{cases} \epsilon_i^2 = 1, \\ \epsilon_i\epsilon_j = \epsilon_j\epsilon_i \quad \text{if } |i-j| \geq 2, \\ \epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1} \end{cases}$$

unitary representation of $\mathbb{S}_\infty$, the *statistics* of ρ.

There is an expectation $\varepsilon : \mathfrak{A} \to \rho(\mathfrak{A})$.

ρ irreducible: *statistics parameter* $\lambda_\rho = \varepsilon(\epsilon)$

$$\lambda_\rho = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$$

and classifies the statistics.

$\lambda_\rho = \kappa_\rho / d_{\mathrm{DHR}}(\rho)$ with $d_{\mathrm{DHR}}(\rho) > 0$ and $\kappa_\rho \in \mathbb{T}$.

$d_{\mathsf{DHR}}(\rho)$ is the *statistical dimension* of ρ;

$$\boxed{d_{\mathsf{DHR}}(\rho) \in \mathbb{N} \cup \infty}$$

($d_{\mathsf{DHR}}(\rho)$ is an "index") and κ_ρ is the *univalence* of ρ.

Index-statistics theorem (L.). Natural connection between the Jones index and QFT

$$\boxed{\mathrm{Ind}(\rho) = d_{\mathsf{DHR}}(\rho)^2.}$$

Here $\mathrm{Ind}(\rho)$ is $\mathrm{Ind}(\rho|_{\mathcal{A}(\mathcal{O})})$, the minimal index ($\mathcal{A}(\mathcal{O})$ is a III-factor for certain regions).

Passing to quotient one obtains a natural embedding

$$\boxed{\text{Superselection sectors} \longrightarrow \mathrm{Sect}(\mathcal{M}).}$$

<u>Subfactor theory contains all local information</u>.

Low dimensional Quantum Field Theory. DHR analysis is not entirely valid if the spacetime dimension $= 2$. Reason: $\mathcal{O}'$ has two connercted components.

Low dimensional statistics was analysed independently by Fredenhagen-Rehren-Schroer and L..

$$\begin{cases} \epsilon_i^2 \neq 1, \\ \epsilon_i \epsilon_j = \epsilon_j \epsilon_i \quad \text{if } |i-j| \geq 2, \\ \epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_{i+1} \epsilon_i \epsilon_{i+1} \end{cases}$$

$$\text{thus} \quad \mathbb{S}_\infty \to \mathbb{B}_\infty$$

<u>braid group statistics</u>.

Index-statistics thm. gives:

$$\boxed{d(\rho) \in \{2\cos\frac{\pi}{n}, n \geq 3\} \cup [2, \infty].}$$

$\rho^2 = \rho_1 \oplus \cdots \oplus \rho_n$ irred. decomposition.

$n \leq 3$, in particular for "small" index, statistics is <u>classified</u> by the braid group representation of Jones or Birman-Wenzl-Murakami, i.e. knot and link polynomial invariants of Jones and Kauffman.

In particular

$$4 < d(\rho)^2 < 6$$
$$\Rightarrow d(\rho)^2 = 5,\ 5.049\ldots, 5.236\ldots, 5.828\ldots$$

(Rehren, L.) while Jones index values $\supset [4, \infty)$!.

Relativistic invariance and the particle-anti-particle symmetry. Reeh-Schlieder thm: Ω is cyclic and separating for any $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$.

Bisognano-Wichmann thm. (in a Wightman frame)

$$\boxed{\Delta_{\mathcal{W}}^{it} = U(\Lambda_{\mathcal{W}}(2\pi t))}$$

$$\Lambda_{\mathcal{W}}(t) = \begin{bmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ *Buchholz-Summer geometric modular action, Guido-L. modular covariance*

Sewell black hole themodynamical interpretation:

Bisognano-Wichmann thm.

$$\updownarrow$$

KMS for boosts

$$\updownarrow$$

temperature for unif. accelerated observer

$$\updownarrow$$

Hawking-Unruh effect

Let ρ be a localized endomorphism. In the above setting (strong additive nets)

Thm. (Guido, L.)

$$\rho \text{ Poincaré covariant} \iff \exists \,\text{conjugate sector}\, \bar{\rho}$$

Proof. Algebraic conjugate $\bar{\rho}$ = analytic conjugate $\bar{\rho}$.

algebraic conjugation is preserved under restriction $\tilde{\mathcal{O}} \supset \mathcal{O}$

$\rightarrow$ consistency relations for analytic conjugate

$\rightarrow$ consistency relations for modular conjugations

$\rightarrow$ symmetries (by geom. meaning of modular objects).

Algebraic spin-statistics theorem. The index-statistics theorem provides a new understanding of the absolute value of λ_ρ, but also

$$\kappa_\rho = \text{phase}(\lambda_\rho)$$

is intrinsic (see above).

Thm. Algebraic version of the spin-statistics theorem (Guido, L.).

$$\boxed{\kappa_\rho = U_\rho(2\pi).}$$