SUSY in the Conformal World

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Based on a joint work with S. Carpi and Y. Kawahigashi

Things to discuss

- Prelude on black hole entropy
- Symmetry and supersymmetry
- Local conformal nets
- Modularity and asymptotic formulae
- Fermi and superconformal nets
- Neveu-Schwarz and Ramond representations
- Fredholm index and Jones index
- Noncommutative geometrization (in progress)

Prelude. Black hole entropy

Bekenstein: The entropy S of a black hole is proportional to the area A of its horizon

$$\boxed{S = A/4}$$

- S is proportional to the *area*, not to the volume as a naive microscopic interpretation of entropy would suggest (logarithmic counting of possible states).
- This dimensional reduction has led to the *holographic principle* by t'Hooft, Susskind, ...
- The horizon is not a physical boundary, but a submanifold where coordinates pick critical values $\longrightarrow$ *conformal symmetries*
- The proportionality factor $1/4$ is fixed by Hawking temperature (*quantum* effect).

Black hole entropy

Discretization of the horizon (Bekenstein): horizon is made of cells or area ℓ^2 and k degrees of freedom ($\ell =$ Planck length):

$$A = n\ell^2,$$
$$\text{Degrees of freedom} = k^n,$$
$$S = Cn \log k = C\frac{A}{\ell^2}\log k,$$
$$\mathrm{d}S = C \log k$$

Conclusion.

Black hole entropy

$\downarrow$

Two-dimensional conformal quantum field theory

with a "fuzzy" point of view

Legenda: Fuzzy = *noncommutative geometrical*

Symmetries in Physics

Spacetime symmetries
Lorentz, Poincaré,...

Internal symmetries
Gauge, ...

SUSY
Bose-Fermi

SUSY: $H = Q^2$, Q odd operator, $[\cdot, Q]$ graded super-derivation interchanging Boson and Fermions

Among consequences: Cancellation of some Higgs boson divergence

Conformal and superconformal

- Low dimension, conformal $\to$ infinite dim. symmetry
- Low dimension, conformal $+$ SUSY $\to$ Superconformal symmetry (very stringent)

Three approaches to CFT

$$
\begin{tikzcd}
\text{Vertex Algebras (algebraic)} \arrow[rr, leftrightarrow, "Kac"] \arrow[dr, leftrightarrow, "Carpi-Weiner"'] & & \text{Wightman fields (analytic)} \arrow[dl, leftrightarrow, "Fredenhagen-Jorss"] \\
& \text{Operator Algebras (algebraic \& analytic)} &
\end{tikzcd}
$$

partial relations known

von Neumann algebras

$\mathcal{H}$ Hilbert space, $B(\mathcal{H})$ *algebra of all bounded linear operators on $\mathcal{H}$.

weak topology. $A_i \to A$ weakly: $(A_i\xi, \eta) \to (A\xi, \eta)$.

Def. A *von Neumann algebra* M is a weakly closed non-degenerate *-subalgebra of $B(\mathcal{H})$.

• von Neumann density thm. $\mathfrak{A} \subset B(\mathcal{H})$ non-degenerate *-subalgebra

$$\mathfrak{A}^- = \mathfrak{A}''$$

where $'$ denotes the commutant

$$\mathfrak{A}' = \{T \in B(\mathcal{H}) : TA = AT \ \forall A \in \mathfrak{A}\}$$

Double aspect, analytical and algebraic

M is a factor if its center $M \cap M' = \mathbb{C}$.

The tensor category End(M)

M an infinite factor $\rightarrow$ End(M) is a *tensor C^*-category*:

- *Objects*: End(M)
- *Arrows*: $\mathrm{Hom}(\rho, \rho') \equiv \{t \in M : t\rho(x) = \rho'(x)t \ \forall x \in M\}$
- *Tensor product of objects*: $\rho \otimes \rho' = \rho\rho'$
- *Tensor product of arrows*: $\sigma, \sigma' \in \mathrm{End}(M)$, $t \in \mathrm{Hom}(\rho, \rho')$, $s \in \mathrm{Hom}(\sigma, \sigma')$,

$$t \otimes s \equiv t\rho(s) = \rho'(s)t \in \mathrm{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma')$$

- *Conjugation*: $\exists$ isometries $v \in \mathrm{Hom}(\iota, \rho\bar{\rho})$ and $\bar{v} \in \mathrm{Hom}(\iota, \bar{\rho}\rho)$ such that

$$(\bar{v}^* \otimes 1_{\bar{\rho}}) \cdot (1_{\bar{\rho}} \otimes v) \equiv \bar{v}^* \bar{\rho}(v) = \frac{1}{d}$$

$$(v^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \bar{v}) \equiv v^* \rho(\bar{v}) = \frac{1}{d}$$

for some $d > 0$.

Dimension

The *minimal* d is the *dimension* $d(\rho)$

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definition of the Jones index)

$$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$$
$$d(\rho_1 \rho_2) = d(\rho_1) d(\rho_2)$$
$$d(\bar{\rho}) = d(\rho)$$

End(M) is a "universal" tensor category

Local conformal nets

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \ I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

First consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I),\Omega)$, are

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it},\ t\in\mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Guido-L., Frölich-Gabbiani)
- *Haag duality*: $\mathcal{A}(I)' = \mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor
- *Additivity*: $I\subset\cup_i I_i \implies \mathcal{A}(I)\subset\vee_i\mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Local conformal nets

$\mathrm{Diff}(S^1) \equiv$ group of orientation-preserving smooth diffeomorphisms of S^1

$\mathrm{Diff}_I(S^1) \equiv \{g \in \mathrm{Diff}(S^1) : g(t) = t \ \forall t \in I'\}$.

A local conformal net $\mathcal{A}$ is a Möbius covariant net s.t.

F. *Conformal covariance*. $\exists$ a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb s.t.

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}_{I'}(S^1), \end{aligned}$$

$\longrightarrow$ unitary representation of the *Virasoro algebra*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$[L_n, c] = 0,\ L_n^* = L_{-n}$.

Representations

A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$I \in \mathcal{I} \mapsto \pi_I, \text{ normal rep. of } \mathcal{A}(I) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}$$

π is automatically diffeomorphism *covariant*: $\exists$ a projective, pos. energy, unitary rep. U_π of $\mathrm{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$, $g \in \mathrm{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

DHR argument: given I, there is an endomorphism of $\mathcal{A}$ localized in I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$.

- $\mathrm{Rep}(\mathcal{A})$ is a *braided tensor category* (DHR, FRS, L.)

Index-statistics theorem

$$\text{DHR dimension } d(\rho) = \sqrt{\text{Jones index } \mathrm{Ind}(\rho)}$$

$$\text{tensor category } \mathrm{Rep}_I(\mathcal{A}) \xrightarrow[\text{restriction}]{\text{full functor}} \text{tensor category } \mathrm{End}(\mathcal{A}(I))$$

Complete rationality

I_1, I_2 intervals $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, $E \equiv I_1 \cup I_2$.

$$\mu\textit{-index}: \quad \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')' : \mathcal{A}(E)]$$

(Jones index). $\mathcal{A}$ conformal:

$$\mathcal{A} \textit{ completely rational} \stackrel{\text{def}}{=} \mathcal{A} \text{ split } \& \ \mu_{\mathcal{A}} < \infty$$

Thm. (Y. Kawahigashi, M. Müger, R.L.) $\mathcal{A}$ completely rational: then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum over all irreducible sectors. (F. Xu in $SU(N)$ models);

• $\mathcal{A}(E) \subset \mathcal{A}(E')' \sim$ LR inclusion (quantum double);

• Representations form a modular tensor category (i.e. non-degenerate braiding).

Weyl's theorem

M compact oriented Riemann manifold, Δ Laplace operator on $L^2(M)$.

Theorem (Weyl)

Heat kernel expansion as $t \to 0^+$:

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots)}$$

The *spectral invariants* n and $a_0, a_1, \ldots$ encode geometric information and in particular

$$a_0 = \mathrm{vol}(M), \qquad a_1 = \frac{1}{6}\int_M \kappa(m) d\mathrm{vol}(m),$$

κ scalar curvature. $n = 2$: a_1 is proportional to the Euler characteristic $= \frac{1}{2\pi}\int_M \kappa(m) d\mathrm{vol}(m)$ by Gauss-Bonnet theorem.

Modularity

With ρ rep. of $\mathcal{A}$, set $L_{0,\rho}$ conf. Hamiltonian of ρ,

$$\chi_\rho(\tau) = \mathrm{Tr}\left(e^{2\pi i\tau(L_{0,\rho}-c/24)}\right) \quad \mathrm{Im}\,\tau > 0.$$

specialized character, c the central charge.
$\mathcal{A}$ is *modular* if $\mu_{\mathcal{A}} < \infty$ and

$$\chi_\rho(-1/\tau) = \sum_\nu S_{\rho,\nu}\chi_\nu(\tau),$$
$$\chi_\rho(\tau+1) = \sum_\nu T_{\rho,\nu}\chi_\nu(\tau).$$

with S, T the (algebraically defined) Kac-Peterson Rehren matrices generating a representation of $SL(2,\mathbb{Z})$. One has:

- Modularity $\implies$ complete rationality
- Modularity holds in all computed rational case, e.g. $SU(N)_k$-models
- $\mathcal{A}$ modular, $\mathcal{B} \supset \mathcal{A}$ irreducible extension $\implies$ $\mathcal{B}$ modular.
- All conformal nets with central charge $c < 1$ are modular.

Asymptotics

$\mathcal{A}$ modular. The following asymptotic formula holds as $t \to 0^+$:

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{\pi c}{12}\frac{1}{t} - \frac{1}{2}\log \mu_{\mathcal{A}} - \frac{\pi c}{12} t$$

In any representation ρ, as $t \to 0^+$:

$$\boxed{\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12}\frac{1}{t} + \frac{1}{2}\log \frac{d(\rho)^2}{\mu_{\mathcal{A}}} - \frac{\pi c}{12} t}$$

Modular nets as NC manifolds (∞ degrees of freedom)

2-dim. cpt manifold M	conformal net $\mathcal{A}$
$\mathrm{supp}(f) \subset I$	$x \in \mathcal{A}(I)$
Laplacian Δ	conf. Hamiltonian L_0
Δ elliptic	L_0 log-elliptic
area $\mathrm{vol}(M)$	NC area $a_0(2\pi L_0)$
Euler charact. $\chi(M)$	NC Euler char. $12a_1$

Entropy. Physics and geometric viewpoints:

Inv.	*Value*	*Geometry*	*Physics*
a_0	$\pi c/12$	NC area	Entropy
a_1	$-\frac{1}{2}\log\mu_{\mathcal{A}}$	NC Euler charact.	1^{st} order entr.
a_2	$-\pi c/12$	2^{nd} spectral invariant	2^{nd} order entr.

Rem. Physical literature: arguments for $2\pi c/12 = A/4$.

Question: What can we say for SUSY? (Dirac operator case)

McKean-Singer formula

Γ be a selfadjoint unitary on a Hilbert space $\mathcal{H}$, thus
$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is graded.

Q selfadjoint *odd* operator: $\Gamma Q \Gamma^{-1} = -Q$ or

$$Q = \begin{bmatrix} 0 & Q_- \\ Q_+ & 0 \end{bmatrix}$$

$\mathrm{Tr_s} = \mathrm{Tr}(\Gamma\,\cdot)$ the *supertrace*.

If e^{-tQ^2} is trace class then $\mathrm{Tr_s}(e^{-tQ^2})$ is an integer independent of t:

$$\mathrm{Tr_s}(e^{-tQ^2}) = \mathrm{ind}(Q_+) \quad \forall t > 0$$

$\mathrm{ind}(Q_+) \equiv \mathrm{Dim}\ker(Q_+) - \mathrm{Dim}\ker(Q_+^*)$ is the Fredholm index of Q_+.

Fermi conformal nets

$\mathcal{A}$ is a Fermi net if locality is replaced by twisted locality:
$\exists$ self-adjoint unitary Γ, $\Gamma\Omega = \Omega$, $\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$; if $I_1 \cap I_2 = \varnothing$

$$[x, y] = 0, \quad x \in \mathcal{A}(I_1),\ y \in \mathcal{A}(I_2)\ .$$

$[x, y]$ is the graded commutator w.r.t. $\gamma = \mathrm{Ad}\Gamma$: if x, y are homogeneous

$$[x, y] \equiv xy - (-1)^{\partial x \cdot \partial y} yx$$

The *Bose subnet* $\mathcal{A}_b \equiv \mathcal{A}^\gamma$ of degree zero elements is local. Setting

$$Z \equiv \frac{1 - i\Gamma}{1 - i}$$

then the unitary Z fixes Ω and

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*$$

(indeed $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$ *twisted duality*). Spin-statistics:

$$U(2\pi) = \Gamma\ .$$

Therefore, in the Fermi case, U is representation of $\mathrm{Diff}^{(2)}(S^1)$.

Nets on a cover of S^1

A conformal net $\mathcal{A}$ on $S^{1(n)}$ is a isotone map

$$I \in \mathcal{I}^{(n)} \mapsto \mathcal{A}(I) \subset B(\mathcal{H})$$

with a projective unitary, positive energy representation U of $\mathrm{Diff}^{(\infty)}(S^1)$ on $\mathcal{H}$ with

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(\dot{g}I), \quad I \in \mathcal{I}^{(n)},\ g \in \mathrm{Diff}^{(\infty)}(S^1)$$

conformal net $\mathcal{A}$ on $S^1 \xrightarrow{\text{promotion}}$ conformal net $\mathcal{A}^{(n)}$ on $S^{1(n)}$

Representations of a Fermi net

Let $\mathcal{A}$ be a Fermi net on S^1. A general representation λ of $\mathcal{A}$ is a representation the cover net of $\mathcal{A}^{(\infty)}$ such that $\lambda|_{\mathcal{A}_b}$ is a DHR representation $\mathcal{A}_b$.

λ is indeed a representation of $\mathcal{A}^{(2)}$. The following alternative holds:

(a) λ is a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is not a scalar.

(b) λ is the restriction of a representation of $\mathcal{A}^{(2)}$ and λ is not a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is a scalar.

Case (a): Neveu-Schwarz representation
Case (b): Ramond representation

Representations of the Bose subnets

ρ DHR representation of $\mathcal{A}_b$: we have
$m(\sigma,\rho) \equiv \varepsilon(\rho,\sigma)\varepsilon(\sigma,\rho) = \pm 1$.
ρ is σ-Bose if $m(\sigma,\rho) = 1$, σ-Fermi if $m(\sigma,\rho) = -1$.

$id|_{\mathcal{A}_b} \equiv id \oplus \sigma$, ν DHR irreducible rep. of $\mathcal{A}_b$:

ν is σ-Bose $\Leftrightarrow \alpha_\nu$ is Neveu-Schwarz

ν is σ-Fermi $\Leftrightarrow \alpha_\nu$ is Ramond

Fermi nets and modularity

λ graded irreducible general rep. of the Fermi modular conformal net $\mathcal{A}$.

$\mathcal{H}_\lambda = \mathcal{H}_{\lambda,+} \oplus \mathcal{H}_{\lambda,-}$ graded by $\Gamma_\lambda$and

$$H_\lambda \simeq L_{0,\rho} \oplus L_{0,\rho'}$$

where $\lambda_{\mathcal{A}_b} = \rho \oplus \rho'$

$$\mathrm{Tr}_s(e^{-tH_\lambda}) = \mathrm{Tr}(e^{-tL_{0,\rho}}) - \mathrm{Tr}(e^{-tL_{0,\rho'}})$$

We also set

$$\tilde{H}_\lambda \equiv H_\lambda - c/24, \quad \tilde{L}_{0,\rho} \equiv L_{0,\rho} - c/24 \ldots$$

Then $S_{\rho,\nu} = \pm S_{\rho',\nu}$ according ν is s-Bose/Fermi.

$$\begin{aligned}\mathrm{Tr}_s(e^{-2\pi t\tilde{H}_\lambda}) &= \sum_\nu S_{\rho,\nu}\,\mathrm{Tr}(e^{-2\pi\tilde{L}_{\rho,\nu}/t}) - \sum_\nu S_{\rho',\nu}\,\mathrm{Tr}(e^{-2\pi\tilde{L}_{\rho',\nu}/t}) \\ &= \sum_\nu (S_{\rho,\nu} - S_{\rho',\nu})\,\mathrm{Tr}(e^{-2\pi\tilde{L}_{0,\nu}/t}) \\ &= 2\sum_{\nu\ \mathrm{Ramond}} S_{\rho,\nu}\,\mathrm{Tr}(e^{-2\pi\tilde{L}_{0,\nu}/t})\end{aligned}$$

Super-Virasoro algebra

The super-Virasoro algebra governs the superconformal invariance:

local conformal $\leftrightarrow$ Virasoro

superconformal $\leftrightarrow$ super-Virasoro

Two super-Virasoro algebras: They are the super-Lie algebras generated by L_n, $n \in \mathbb{Z}$ (even), G_r (odd), and c (central):

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$$

$$[G_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

Neveu-Schwarz case: $r \in \mathbb{Z} + 1/2$, Ramond case: $r \in \mathbb{Z}$.

Note: $G_0^2 = 2L_0 - c/12$ in Ramond sectors

FQS: admissible values for central charge c and lowest weight h

Either $c \geq 3/2$, $h \geq 0$ ($h \geq c/24$ in the Ramond case) or

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right), \; m = 2, 3, \ldots$$

and

$$h = h_{p,q}(c) \equiv \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{\varepsilon}{8}$$

where $p = 1, 2, \ldots, m-1$, $q = 1, 2, \ldots, m+1$ and $p - q$ is even or odd corresponding to the Neveu-Schwarz case ($\varepsilon = 0$) or Ramond case ($\varepsilon = 1/2$).

Neveu-Schwarz algebra has a vacuum representation, the Ramond algebra has no vacuum representation.

Super-Virasoro nets

c an admissible value, $h = 0$. Bose and Fermi stress-energy tensors:

$$T_B(z) = \sum_n z^{-n-2} L_n$$
$$T_F(z) = \frac{1}{2} \sum_r z^{-r-3/2} G_r$$

in any Neveu-Schwarz/Ramond rep. and we have:

$$[T_F(z_1), T_F(z_2)] = \frac{1}{2} z_1^{-1} T_F(z_1)\delta(w) + z_1^{-3} w^{-\frac{3}{2}} \frac{c}{12}\left(w^2\delta''(w) + \frac{3}{4}\delta(w)\right)$$

$(w \equiv z_2/z_1)$. In the Neveu-Schwarz vacuum rep. define:

$SVir(I) \equiv \{e^{iT_B(f_1)}, e^{iT_F(f_2)} : f_1, f_2 \in C^\infty(S^1)\ \mathrm{real}, \mathrm{supp} f_1, \mathrm{supp} f_2 \subset I\}''$

Neveu-Schwarz rep. of $SVir$ net $\longleftrightarrow$ rep. of Neveu-Schwarz algebra
Ramond rep. of $SVir$ net $\longleftrightarrow$ rep. of Ramond algebra

$SVir_b$ is modular (F. Xu)
$SVir_b = \left(SU(2)_{N+2}\right)' \cap \left(SU(2)_2 \otimes SU(2)_N\right)$ (GKO)

Supersymmetric representations

A general representation λ of the Fermi conformal net $\mathcal{A}$ is *supersymmetric* if λ is graded

$$\lambda(\gamma(x)) = \Gamma_\lambda \lambda(x) \Gamma_\lambda^*$$

and the conformal Hamiltonian H_λ satisfies

$$\tilde{H}_\lambda \equiv H_\lambda - c/24 = Q_\lambda^2$$

where Q_λ is a selfadjoint odd w.r.t. Γ_λ.

Then

$$H_\lambda \geq c/24$$

McKean-Singer lemma:

$$\mathrm{Tr}_s(e^{-t(H_\lambda - c/24)}) = \dim\ker(H_\lambda - c/24) ,$$

the multiplicity of the lowest eigenvalue $c/24$ of H_λ.

Super-Virasoro net:

λ supersymmetric $\Rightarrow$ λ Ramond (irr. iff $h = c/24$ i.e. minimal)

SUSY, Fredholm and Jones index

Assume $\mathcal{A}_b$ modular $\lambda|_{\mathcal{A}_b} = \rho \oplus \rho'$.

$$\mathrm{Tr_s}(e^{-2\pi t \tilde{H}_\lambda}) = 2 \sum_{\nu\,\mathrm{Ramond}} S_{\rho,\nu}\, \mathrm{Tr}(e^{-2\pi \tilde{L}_{0,\nu}/t}) \ .$$

If λ is supersymmetric then

$$\mathrm{Tr_s}(e^{-2\pi t \tilde{H}_\lambda}) = 2 \sum_{\nu\,\mathrm{Ramond}} S_{\rho,\nu} \mathrm{null}(\nu, c/24)$$

on the other hand

$$\mathrm{Tr_s}(e^{-2\pi t \tilde{H}_\lambda}) = \mathrm{ind}(Q_{\lambda+}) \ .$$

Therefore we have

$$\mathrm{ind}(Q_{\lambda+}) = 2 \sum_{\nu\,\mathrm{Ramond}} S_{\rho,\nu} \mathrm{null}(\nu, c/24)$$

then, writing Rehren definition of the S matrix, we have

$$\mathrm{ind}(Q_{\lambda+}) = \frac{d(\rho)}{\sqrt{\mu_{\mathcal{A}}}} \sum_{\nu\,\mathrm{Ramond}} K(\rho,\nu) d(\nu) \mathrm{null}(\nu, c/24)$$

The Fredholm index of the supercharge operator $Q_{\lambda+}$ and the

Some consequences

- An identity for the S matrix:

$$\sum_{\nu \,\text{Ramond}} S_{\rho,\nu} d(\nu) = 0$$

- If $\text{ind}(Q_{\lambda_+}) \neq 0$ there exists a Ramond sector ν such that $c/24$ is an eigenvalue of $L_{0,\nu}$.
- Suppose that ρ is the only Ramond sector with lowest eigenvalue $c/24$ modulo integers. Then

$$S_{\rho,\rho} = \frac{d(\rho)^2}{\sqrt{\mu_{\mathcal{A}_b}}} K(\rho,\rho) = \frac{1}{2}\ .$$

Further structure. Topological embedding of super-Virasoro algebras and nets

Proposition

For each integer $k \in \mathbb{N}$ consider the linear map $\varphi^{(k)}$ determined by

$$L_m \mapsto L_m^{(k)} \equiv \frac{1}{k} L_{km} \; , \quad m \neq 0 \; ,$$

$$L_0 \mapsto L_0^{(k)} \equiv \frac{1}{k} L_0 + \frac{c}{24} \frac{(k^2 - 1)}{k}$$

$$G_r \mapsto G_r^{(k)} \equiv \frac{1}{\sqrt{k}} G_{kr}$$

$$c \mapsto kc$$

If k is even, $\varphi^{(k)}$ is an isomorphic embedding of NS into R and of R into R. If k is odd, $\varphi^{(k)}$ is an isomorphic embedding of NS into NS and of R into R.

Some consequences

- The double cover $\mathrm{SVir}^{(2)}_{2c}$ of SVir_{2c} is isomorphic to $\mathcal{R}_{2c}$ (ramond net).
- Topological sectors give new class of non-trivial supersymmetric representations

$$\tau_f \cdot \rho \otimes \cdots \otimes \rho\big|_{(\mathcal{A}\otimes\cdots\otimes\mathcal{A})^{\mathbb{Z}_k}}$$

τ_f = top. sector (Xu, L.), $\mathcal{A} = \mathrm{SVir}$.

Classification (S. Carpi, Y. Kawahigashi, R. L.)

Complete list of superconformal nets, i.e. Fermi extensions of the super-Virasoro net, with $c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$

1. The super Virasoro net: (A_{m-1}, A_{m+1}).
2. Index 2 extensions of the above: $(A_{4m'-1}, D_{2m'+2})$, $m = 4m'$ and $(D_{2m'+2}, A_{4m'+3})$, $m = 4m' + 2$.
3. Six exceptionals: (A_9, E_6), (E_6, A_{13}), (A_{27}, E_8), (E_8, A_{31}), (D_6, E_6), (E_6, D_8).

Work in progress (S. Carpi, R. Hillier, R.L.)

Relation with the Noncommutative Geometrical framework of A. Connes.

A supersymmetric representation ρ of a Fermi net $\mathcal{A}$ gives rise to a θ-summable spectral triple if the superderivation δ

$$\delta(a) \equiv [a, Q]$$

has a dense domain in the representation ρ.
Then the JLO cocycle (Chern character) on the Bose algebra

$$\tau_n^{\rho}(a_0, a_1, \ldots, a_n) \equiv$$
$$(-1)^{-\frac{n}{2}} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \mathrm{Tr_s}\left(e^{-H_\rho} a_0 \alpha_{it_1}(\delta a_1) \alpha_{it_2}(\delta a_2) \ldots \alpha_{it_n}(\delta a_n)\right) \mathrm{d}t_1 \mathrm{d}t_2 \ldots$$

(n even) is entire cyclic coclycle

Noncommutative geometrization

We want to associate to each supersymmetric sector the above Chern character

$$\rho \to \tau^{\rho}$$

• The supersymmetric Ramond sectors of SVir give rise to θ-summable spectral triple (δ has a dense domain)

For the super-Virasoro net the index map

$$\rho \to \sum \tau_n^{\rho}(1, 1, \dots, 1) = \mathrm{Tr}_s(e^{-tH_{\rho}})$$

for Ramond sectors is given by

$$\mathrm{Index}(\rho_{h=c/24}) = 1, \qquad \mathrm{Index}(\rho_{h\neq c/24}) = 0$$