Lectures on Conformal Nets

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Part II

Nets of von Neumann Algebras

Chapter 6

Möbius covariant nets of von Neumann algebras

We now introduce the main objects of our study, Möbius covariant local nets of factors. We discuss here their main properties in the defining representation, namely on the vacuum Hilbert space. The crucial representation theory will be investigated in later chapters.

6.1 Definition

A *net* $\mathcal{A}$ *of von Neumann algebras on* S^1 is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$, the set of open, non-empty, non-dense intervals of S^1, to the set of von Neumann algebras on a (fixed) Hilbert space $\mathcal{H}$ that verifies the following isotony property:

1. Isotony : *If* I_1, I_2 *are intervals and* $I_1 \subset I_2$*, then*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)\ .$$

For an arbitrary set $E \subset S^1$ we define $\mathcal{A}(E)$ as the von Neumann algebra generated by all the $\mathcal{A}(I)$ with $I \subset E$, $I \in \mathcal{I}$ (setting $\mathcal{A}(E) \equiv \mathbb{C}$ if E has empty interior).

The net $\mathcal{A}$ is said to be *Möbius covariant* if the following properties 2,3 and 4 are satisfied:

2. Möbius invariance: *There is a strongly continuous unitary representation U of* $\mathbf{G}$ *on $\mathcal{H}$ such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)\,, \quad g \in \mathbf{G},\ I \in \mathcal{I}.$$

3. Positivity of the energy : *U is a positive energy representation.*

4. Existence and uniqueness of the vacuum: *There exists a unique (up to a phase) unit U-invariant vector Ω (vacuum vector) and Ω is cyclic for the von Neumann algebra $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$*

The net $\mathcal{A}$ is said to be *local* if the following property holds:

5. Locality : *If I_1 and I_2 are disjoint intervals the von Neumann algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute:*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$$

6.2 First consequences

If $\mathcal{A}$ is local, Möbius covariant net of von Neumann algebras, clearly

$$H(I) \equiv \overline{\mathcal{A}(I)_{\mathrm{sa}}\Omega}$$

is a local, Möbius covariant net of real Hilbert subspaces of $\mathcal{H}$ (see Sect. 2.5). Moreover if $\mathcal{A}$ is local then $\mathcal{H}$ is local too.

Remark 6.2.1. Note that if $\mathcal{A}$ is defined by all the above properties $1-5$ but with $\bar{\mathbf{G}}$ in place of $\mathbf{G}$, then automatically U is indeed a representation of $\mathbf{G}$ due to the one-particle spin-statistics relation, Cor. 3.4.2.

We now discuss a few other consequences of the axioms that follows from Sect. 3.

6.2.1 Reeh-Schlieder theorem

Theorem 6.2.2. *Let $\mathcal{A}$ be a local Möbius covariant net on S^1. For any given $I \in \mathcal{I}$, the vector Ω is cyclic and separating for the von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.*

Proof. Let $\mathcal{H}_0$ be the complex Hilbert subspace of $\mathcal{H}$ generated by all the $H(I) \equiv \overline{\mathcal{A}(I)_{\rm sa}\Omega}$. Clearly H is a local Möbius covariant net of real Hilbert subspaces of $\mathcal{H}_0$. By the Reeh-Schlieder theorem for nets of real linear subspaces, we have $\overline{H(I) + iH(I)} = \mathcal{H}_0$ for every fixed interval I. The orthogonal projection $E = [\mathcal{A}(I)\Omega] \in \mathcal{A}(I)'$ from $\mathcal{H}$ onto $\overline{H(I) + iH(I)}$ is therefore independent of I, so $E \in \cap_I \mathcal{A}(I)' = (\vee_I \mathcal{A}(I))'$. So $E \geq F$ where F is the smallest projection in $(\vee_I \mathcal{A}(I))'$ containg Ω. By the assumed cyclicity of Ω for $\mathcal{A}$ we have $F = 1$, thus $E = 1$. So $\mathcal{H}_0 = \mathcal{H}$ and $H(I)$ is standard, namely, by the equivalence (2.5.2), Ω is cyclic and separating for $\mathcal{A}(I)$ □

6.2.2 Bisognano-Wichmann property and Haag duality

Theorem 6.2.3. *Let $I \in \mathcal{I}$ and Δ_I, J_I the modular operator and the modular conjugation of $(\mathcal{A}(I), \Omega)$. Then we have:*
(i)*:* Bisognano-Wichmann property*:*

$$\Delta_I^{it} = U(\delta_I(-2\pi t)), \quad t \in \mathbb{R}\,; \tag{6.2.1}$$

(i')*: U extends to an (anti-)unitary representation of $\mathbf{G}_2$ determined by*

$$U(r_I) = J_I, \quad I \in \mathcal{I}, \tag{6.2.2}$$

acting covariantly on $\mathcal{A}$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \quad g \in \mathbf{G}_2,\ I \in \mathcal{I}\,.$$

(ii)*:* Haag duality*: For every interval I*

$$\mathcal{A}(I') = \mathcal{A}(I)'\,.$$

Proof. The results for nets of standard subspaces in Sect. 3 immediately imply (i) and that U extends to an (anti-)unitary representation of $\mathbf{G}_2$ determined by (6.2.2). Haag duality then follows by Prop. 2.5.1 because $H(I')$ is the symplectic complement of $H(I)$. Then (i') follows because $\mathrm{Ad}J_I = \mathrm{Ad}U(r_I)$ acts geometrically and $\mathbf{G}_2 = \mathbf{G} \cdot r_I$. □

Remark 6.2.4. If in the real line picture I is the half-line $(0, \infty)$, then J_I is an anti-unitary involution corresponding to the symmetry $t \mapsto -t$. This is the *PCT symmetry* and the relation (6.2.2) is in particular a general, algebraic form of the PCT theorem in the conformal setting.

Remark 6.2.5. It follows easily by Möbius covariance (or by Haag duality) that $\mathcal{A}(I_0) = \cap_{I \supset \bar{I}_0} \mathcal{A}(I)$ where $I_0, I \in \mathcal{I}$. Therefore the net could be equivalently defined on closed intervals, namely it would follow from any definition that $\mathcal{A}(I) = \mathcal{A}(\bar{I})$ for any open inetval I.

Corollary 6.2.6 (Additivity)**.** *Let I and I_i be intervals with $I \subset \cup_i I_i$. Then $\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$.*

Proof. Replacing I with $I \cap I_i$ we may assume that $I_i \subset I$. With $M \equiv \vee_i \mathcal{A}(I_i)$ we then have $M \subset \mathcal{A}(I)$. Moreover $\overline{M_{\mathrm{sa}}\Omega}$ contains the closed real linear span of the $H(I_i)$'s, which is equal to $H(I) \equiv \overline{\mathcal{A}(I)_{\mathrm{sa}}\Omega}$ by Cor. 3.3.3. So $M = \mathcal{A}(I)$ by Prop. 2.5.1. □

Remark 6.2.7. As we have seen in Sect. 3, the representation U of $\mathbf{G}$ is *unique*, due to formula (6.2.1).

We can then define an isomorphism of nets in different ways.

Proposition 6.2.8. *Let $\mathcal{A}_k$ be local, Möbius covariant nets of von Neumann algebras on the Hilbert space $\mathcal{H}_k$, $k = 1, 2$, and Φ a family $\{\Phi_I\}_{I \in \mathcal{I}}$ with Φ_I an isomorphism between $\mathcal{A}_1(I)$ and $\mathcal{A}_2(I)$ such that $\Phi_{\tilde{I}}|_I = \Phi_I$ if $I \subset \tilde{I}$. Then (with obvoius notations) the following are equivalent:*

(i)*: Φ preserves the vacuum state, namely $(\Phi_I(x)\Omega_2, \Omega_2) = (x\Omega_1, \Omega_1)$, $\forall x \in \mathcal{A}_1(I)$, $\forall I \in \mathcal{I}$;*

(i')*: There exists a unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $V\Omega_1 = \Omega_2$ and $\Phi_I = \mathrm{Ad}V|_{\mathcal{A}_1(I)}$, $\forall I \in \mathcal{I}$;*

(ii)*: Φ intertwines the Möbius group actions, namely $\Phi_I \cdot \mathrm{Ad}U_1(g) = \mathrm{Ad}U_2(g) \cdot \Phi_I$, $\forall I \in \mathcal{I}$;*

(ii')*: There exists a unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\Phi_I = \mathrm{Ad}V|_{\mathcal{A}(I)}$, $\forall I \in \mathcal{I}$, and $VU_1(g) = U_2(g)V$, $\forall g \in \mathbf{G}$.*

Proof. (i) ⇒ (i'): Given $I \in \mathcal{I}$, the unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$ determined by $Vx\Omega_1 \equiv \Phi_I(x)\Omega_2$, $x \in \mathcal{A}(I)$, is independent of I by the Reeh-Schlieder theorem and satisfies (i). The implication (ii) ⇒ (i) is obvious.

(i') ⇒ (ii'): As V implements Φ and $V\Omega_1 = \Omega_2$ we have $VU_1(g) = U_2(g)V$ by the uniqueness of the unitary Möbius representation.

Clearly (ii') ⇒ (ii).

Finally, we show that (ii) ⇒ (i). With $x \in \mathcal{A}_1(I)$ we have

$$\begin{aligned}(\Phi_I(x)\Omega_2, \Omega_2) &= (U_2(\delta_I(s))^*\Phi_I(x)U_2(\delta_I(s))\Omega_2, \Omega_2) \\ &= (\Phi_I(U_1(\delta_I(s))^* x U_1(\delta_I(s)))\Omega_2, \Omega_2) \to (x\Omega_1, \Omega_1)\end{aligned}$$

because, as $s \to \infty$, $U_1(\delta_I(s))^* x U_1(\delta_I(s))$ weakly converges to $(x\Omega_1, \Omega_1)$. □

If the conditions in the above Prop. 6.2.8 hold, we shall say that Φ is an *isomorphism* Φ between $\mathcal{A}_1$ and $\mathcal{A}_2$. Note that V is uniquely determined by Φ and we can define Φ by giving V. An *automorphism* of $\mathcal{A}$ is, of course, an isomorphism of $\mathcal{A}$ with itself.

6.2.3 Irreducibility

We shall say that $\mathcal{A}$ is *irreducible* if the von Neumann algebra $\vee\mathcal{A}(I)$ generated by all local algebras $\mathcal{A}(I)$ coincides with $B(\mathcal{H})$. The irreducibility property is indeed equivalent to several other requirement.

Proposition 6.2.9. *Assume all properies* $1-5$ *for* $\mathcal{A}$ *except for the uniqueness of the vacuum. The following are equivalent:*

(i) $\mathbb{C}\Omega$ *are the only* $U(\mathbf{G})$ *invariant vectors.*

(ii) *The algebras* $\mathcal{A}(I)$, $I \in \mathcal{I}$, *are factors. In particular they are type* III_1 *factors (unless* $\mathcal{H}$ *is one-dimensional).*

(iii) *For any given two points* p_1, p_2 *the algebra* $\mathcal{A}(S^1 \backslash \{p_1, p_2\})$ *is irreducible.*

(iv) *The net* $\mathcal{A}$ *is irreducible.*

(v) *The algebra* $\cap\mathcal{A}(I)$ *given by the intersection of all local algebras coincides with* $\mathbb{C}$.

Proof. $(i) \Rightarrow (ii)$: With $H(I) \equiv \overline{\mathcal{A}(I)_{\mathrm{sa}}\Omega}$, by Prop. 3.4.1 every $H(I) \ominus \mathbb{R}\Omega$ is a factorial standard subspace of $\mathcal{H} \ominus \mathbb{C}\Omega$; as $(\mathcal{A}(I) \cap \mathcal{A}(I'))_{\mathrm{sa}}\Omega \subset H(I) \cap H(I') = \mathbb{C}\Omega$ and Ω is separating, we have $\mathcal{A}(I) \cap \mathcal{A}(I') = \mathbb{C}$. Moreover the modular group of $\mathcal{A}(I)$ is ergodic by the Thm. 1.7.2 on the vanishing of the matrix coefficients, so $\mathcal{A}(I)$ is a factor of type III_1 by Prop. 6.6.5 $(ii) \Rightarrow (iii)$: If $\mathcal{A}(I)$ is a factor then by Haag duality $\mathcal{A}(I) \vee \mathcal{A}(I') = \mathcal{A}(I) \vee \mathcal{A}(I)' = B(\mathcal{H})$ and (iii) follows by taking I with endpoints p_1, p_2.

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Leftrightarrow (v)$: Immediate by Haag duality.

$(v) \Rightarrow (i)$: Fix an interval I. By the vanishing of the matrix coefficients theorem it is sufficient to show that Ω is the unique invariant vector for $U(\delta_I(-2\pi t)) = \Delta_I^{it}$. By the ergodic theorem this is equivalent to the ergodicity of $\mathrm{Ad}\,U(\delta_I(\cdot))$ on $\mathcal{A}(I)$.

So let $x \in \mathcal{A}(I)$ be fixed $\mathrm{Ad}U(\delta_I(\cdot))$. Then $U(\delta_I(-2\pi t))x\Omega = x\Omega$. By the vanishing of the matrix coefficients theorem we have

$$U(g)xU(g)^*\Omega = U(g)x\Omega = x\Omega$$

for all $g \in \mathbf{G}$. Take g so that $gI \cup I$ is not dense in S^1; then $U(g)xU(g)^* = x$ by the Reeh-Schlieder separating property of Ω. By the group property $U(g)xU(g)^* = x$ for all $g \in \mathbf{G}$. Thus x belongs to $\mathcal{A}(gI)$ for all $g \in \mathbf{G}$, namely to all local algebras, so x is a scalar as desired. □

Corollary 6.2.10. *Assume all properies* $1-5$ *for* $\mathcal{A}$ *except for the uniqueness of the vacuum. Then the center* $\mathcal{Z}$ *of* $\mathcal{A}(I)$ *is independent of* I *and there is a unique direct integral disintegration*

$$\mathcal{A} = \int^{\oplus} \mathcal{A}_\lambda \mathrm{d}\mu(\lambda)$$

with $\mathcal{A}_\lambda$ *an irreducible, local Möbius covariant net for almost all* λ*.*

Proof. The argument in the proof $(v) \Leftrightarrow (i)$ of the above corollary shows that if $x \in \mathcal{A}(I)$ is fixed by the modular group of $(\mathcal{A}(I), \Omega)$ then x belongs to all local algebras, showing that $\mathcal{Z}$ is independent of the interval I. Then we may identify $\mathcal{Z}$ with some $L^\infty(X, \mu)$ and consider the common factor disintegration of the $\mathcal{A}(I)$'s. Then each fiber is almost everywhere an irreducible net. By Prop. 6.6.3 the unitary $U(g)$ of the unitary representations of $\mathbf{G}$ belong to the von Neumann algebra generated by all the $\mathcal{A}(I)$'s so the commutes with $\mathcal{Z}$ (alternatively we can use the fact that the modular group fixed the center and apply the vanishing of the matrix coefficients theorem) and decompose to give the covariance unitary representation of $\mathbf{G}$ for $\mathcal{A}_\lambda$. □

6.3 Borchers theorem and half-sided modular inclusions of von Neumann algebras

Having already described the one-particle versions of Borchers and Wiesbrock theorems, the von Neumann algebra case is now almost immediate.

Theorem 6.3.1 (Borchers)**.** *Let M be a von Neumann algebra a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for M. Let U be a one-parameter group on $\mathcal{H}$ fixing Ω, with generator P, satisfying*

$$U(s)MU(-s) \subset M\,, \quad s \geq 0.$$

If $\pm P > 0$, the following commutation relations hold:

$$\begin{cases} \Delta^{it}U(s)\Delta^{-it} = U(e^{\mp 2\pi t}s), \\ JU(s)J = U(-s), \qquad t, s \in \mathbb{R}. \end{cases}$$

Proof. The theorem is immediate by its one-particle version Thm. 2.2.1 by setting $H \equiv \overline{M_{\mathrm{sa}}\Omega}$. □

The converse of Borchers theorem does not hold in the von Neumann algebra case as in the standard subspace case. However the following weaker result holds true.

Proposition 6.3.2. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for M. Let U be a one-parameter group on $\mathcal{H}$, with generator $P \geq 0$, fixing Ω, satisfying*

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s)\,, \quad t, s \in \mathbb{R}\,,$$

Suppose there exists $s_0 > 0$ and such that Ω is cyclic for $M \cap U(s_0)MU(-s_0)$. Then

$$U(s)MU(-s) \subset M\,, \quad s \geq 0.$$

Proof. First we show that $U(s_0)MU(-s_0) \subset M$. This is equivalent to say that $U(s_0)M'U(-s_0) \supset M'$. Set $H \equiv \overline{M_{\mathrm{sa}}\Omega}$. From Thm. 2.2.4 we have $U(s_0)H \subset H$, so $U(s_0)H' \supset H'$.

So both $U(s_0)M'U(-s_0)$ and M' are von Neumann subalgebras of M'_{s_0} and $\overline{(U(s_0)M'U(-s_0))_{\mathrm{sa}}\Omega} \supset \overline{M'_{\mathrm{sa}}\Omega}$. By Prop. 2.5.1 we conclude that $U(s_0)M'U(-s_0) \supset M'$, namely $U(s_0)MU(-s_0) \subset M$.

The proposition is then proved because with $s = e^{-2\pi t}s_0$ we have

$$\begin{aligned} \mathrm{Ad}U(s)M = \mathrm{Ad}U(e^{-2\pi t}s_0)M = \mathrm{Ad}\Delta^{it}U(s_0)\Delta^{-it}M \\ = \mathrm{Ad}\Delta^{it}U(s_0)M \subset \mathrm{Ad}\Delta^{it}M = M. \end{aligned}$$

□

Let $N \subset M$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for both M and N. We shall say that $(N \subset M, \Omega)$ (or simply $N \subset M$) is a *half-sided modular inclusion of von Neumann algebras* if

$$\Delta_M^{-it} N \Delta_M^{it} \subset N\,, \quad t \geq 0\,. \tag{6.3.1}$$

We shall also abbreviate half-sided modular inclusion by hsm of +hsm. If the above equation (6.3.1) holds for $t \leq 0$ instead we shall say that $(N \subset M, \Omega)$ is a -hsm inclusion of von Neumann algebras.

We now pass to the main theorem on hsm inclusions. This theorem was originally proved by Wiesbrock [175] but its proof contained a gap. Zsido then announced a different complete proof. A complete proof later appeared in papers by Borchers [18] and by Araki and Zsido [6]. This last paper contained a generalization to the weight case.

Having proved the real Hilbert subspace analog of this theorem, its von Neumann algebra version is now almost immediate.

Theorem 6.3.3 (Wiesbrock, Borchers, Araki-Zsido)**.** *Let $(N \subset M, \Omega)$ be a hsm inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for both M and N.*

Then there exists a positive energy unitary representation of **P** *on $\mathcal{H}$ determined by*

$$V(\delta(2\pi s)) = \Delta_M^{-is}, \qquad V(\delta_{(1,\infty)}(2\pi s)) = \Delta_N^{-is}\,.$$

The generator P of the translation one-parameter group $U \equiv V(\tau(\cdot))$ is the closure of $\frac{1}{2\pi}(\log \Delta_N - \log \Delta_M)$ and we have

$$U(t)MU(-t) \subset M\,,\ t \geq 0, \quad \text{and} \quad U(1)MU(-1) = N\,.$$

In other words setting $\mathcal{A}(t,\infty) \equiv U(1)MU(-1)$, $\mathcal{A}(\infty,t) = \mathcal{A}(t,\infty)'$ we get a local net of von Neumann algebras on the half-lines of $\mathbb{R}$ which is **P**-covariant with positive energy. We shall later discuss when we get a net on S^1.

Proof. Setting $H \equiv \overline{M_{\mathrm{sa}}\Omega}$, $K \equiv \overline{N_{\mathrm{sa}}\Omega}$, we may apply the standard subspace version of Wiesbrock theorem 2.4.1 and define the representation V of **P**. The equality $U(e^{2\pi t} - 1) = \Delta_M^{-it}\Delta_N^{it}$ and the assumed half-side modular invariance imply $U(t)MU(-t) \subset M$ for $t \geq -1$. Moreover $U(1)NU(-1) = N$ as the corresponding standard subspace coincide (see Prop. 2.5.1). So $U(t)MU(-t) \subset M$ for $t \geq 0$. □

Corollary 6.3.4. *Let $N \subset M$ be an inclusion of von Neumann algebras. Then $(N \subset M, \Omega)$ is a hsm inclusion iff $(M' \subset N', \Omega)$ is a -hsm inclusion.*

6.4 Extending nets from $\mathbb{R}$ to S^1

We now pass to the characterization of Möbius covariant local nets in terms of a factorization.

A *hsm factorization* (N_0, N_1, N_2, Ω) of von Neumann algebras on a Hilbert space $\mathcal{H}$ is a quadruple where N_k, $k \in \mathbb{Z}_3$, are mutually commuting von Neumann algebras on $\mathcal{H}$, Ω is a cyclic (thus separating) vector for each N_k such that $(N_k \subset N'_{k+1}, \Omega)$, $k \in \mathbb{Z}_3$, is a hsm modular inclusion of von Neumann algebras.

Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras. Choose three intervals $I_0, I_1, I_2 \in \mathcal{I}$ forming a partition of S^1 in the counter-clockwise order (up to the boundary points). Then $(\mathcal{A}(I_0), \mathcal{A}(I_1), \mathcal{A}(I_2), \Omega)$ is a hsm factorization of von Neumann algebras. Indeed the $\mathcal{A}(I_k)$'s mutually commute by locality and $\mathcal{A}(I_k) \subset \mathcal{A}(I_{k+1})' = \mathcal{A}(I'_{k+1})$ is a hsm inclusion w.r.t. Ω by the Bisognano-Wichmann property because $\delta_{I'_{k+1}}(t) I_k \subset I_k$, $t \geq 0$. As $\mathbf{G}$ acts transitively on the set of three different points of S^1 the isomorphism class of the factorization $(\mathcal{A}(I_0), \mathcal{A}(I_1), \mathcal{A}(I_2), \Omega)$ does not depend on the choice of I_0, I_1, I_2 as above.

We shall now show that every hsm factorization of von Neumann algebras arises from a local Möbius covariant net as above.

Theorem 6.4.1. *Let (N_0, N_1, N_2, Ω) be a hsm factorization of von Neumann algebras and let I_0, I_1, I_2 be intervals forming a partition of S^1, up to boundary points, in counter-clockwise order. There exists a unique local Möbius covariant net $\mathcal{A}$ on S^1 such that $\mathcal{A}(I_k) = N_k$, $k \in \mathbb{Z}_3$, with Ω the vacuum vector. The (unique) positive energy unitary representation $U$of $\mathbf{G}$ is determined by $U(\delta_{I_k}(2\pi t)) = \Delta_k^{-it}$, where Δ_k is the modular operator of (N_k, Ω).*

Proof. Setting $H_i \equiv \overline{N_{k\,\mathrm{sa}}}\Omega$, we obtain a factorization of standard subspaspaces (H_0, H_1, H_2). Let U be the associated representation of $\mathbf{G}$ given by Thm. 3.6.2, in particular $U(\delta_{I_k}(2\pi t)) = \Delta_k^{-it}$, and set

$$\mathcal{A}(I) \equiv U(g)\mathcal{A}(I_0)U(g)^*$$

if $gI_0 = I$. Then $\mathcal{A}(I)$ is well defined because if $h \in \mathbf{G}$ also satisfies $hI_0 = I$ then $h = g\delta_{I_0}(2\pi t)$ for some t and $\mathrm{Ad}U(\delta_{I_0}(2\pi t))\mathcal{A}(I_0) = \mathrm{Ad}\Delta_k^{-it}\mathcal{A}(I_0) = \mathcal{A}(I_0)$. The rest follows as in the proof of Thm. 3.6.2 for the standard subspace case. □

By *a net of von Neumann algebras on* $\mathbb{R}$ (or on the intervals of $\mathbb{R}$) we shall mean an isotonic map

$$I \in \mathcal{I}_0 \mapsto \mathcal{A}(I)$$

from $\mathcal{I}_0$, the set of intervals of $\mathbb{R}$, i.e. bounded open connected subsets of $\mathbb{R}$, to the set of von Neumann algebras on a fixed Hilbert space $\mathcal{H}$.

$\mathcal{A}$ is local means as usual that $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute if I_1 and I_2 are disjoint intervals.

If $\mathcal{A}$ is a a net of von Neumann algebras on the intervals of $\mathbb{R}$ we define by additivity the von Neumann algebra $\mathcal{A}(E)$ associated with any set $E \subset \mathbb{R}$ as the von von Neumann algebra generated by all the $\mathcal{A}(I)$'s with $I \subset E$, $I \in \mathcal{I}_0$. In particular $\mathcal{A}$ is local iff $\mathcal{A}(-\infty, a)$ and $\mathcal{A}(a, \infty)$ commute for all $a \in \mathbb{R}$.

We have the following version of Thm. 3.6.4.

Theorem 6.4.2. *Let $\mathcal{A}$ be a local net of von Neumann algebras on the intervals of $\mathbb{R}$, Ω a cyclic and separating vector for the von Neumann algebra $\mathcal{A}(I)$ associated with each interval $I \subset \mathbb{R}$ and U a Ω-fixing unitary representation of* $\mathbf{P}$ *acting covariantly on $\mathcal{A}$. The following are equivalent:*

(*i*) *$\mathcal{A}$ extends to a Möbius covariant net on S^1.*

(*ii*) *The Bisognano-Wichmann property holds for $\mathcal{A}$, namely*

$$\Delta^{it}_{\mathbb{R}^+} = U(\delta_{\mathbb{R}^+}(-2\pi t)), \tag{6.4.1}$$

where $\Delta_{\mathbb{R}^+}$ is the modular operator of $(\mathcal{A}(\mathbb{R}^+), \Omega)$.

Proof. (*i*) $\Rightarrow$ (*ii*): Immediate.

(*ii*) $\Rightarrow$ (*i*): The proof follows the same reasoning of that of Thm. 3.6.4. Note first that, by translation covariance, $\Delta^{it}_{(a,\infty)} = U(\delta_{(a,\infty)}(-2\pi t))$ for all $a \in \mathbb{R}$. Hence $\mathcal{A}(-\infty, a)$ is a von Neumann subalgebra of $\mathcal{A}(a, \infty)'$ that is cyclic on Ω and globally invariant under the modular group of $\mathcal{A}(a, \infty)'$ with respect to Ω, hence, by the Tomita-Takesaki theory, duality for half-lines holds

$$\mathcal{A}(a, \infty)' = \mathcal{A}(-\infty, a). \tag{6.4.2}$$

Then it is immediate to check $(\mathcal{A}(-\infty, -1), \mathcal{A}(-1, 1), \mathcal{A}(1, \infty), \Omega)$ to be a +hsm factorization of von Neumann algebras, so we get a Möbius covariant net from Theorem 6.4.1. Due to Bisognano-Wichmann property this is indeed an extension to S^1 of the original net. □

Note that positivity of the energy is not an a priori requirement in (*ii*) of the above theorem.

Although a Möbius covariant net satisfies Haag duality on S^1, duality on $\mathbb{R}$ does not necessarily hold.

Lemma 6.4.3. *Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras on S^1. The following are equivalent:*

(*i*) *The restriction of $\mathcal{A}$ to $\mathbb{R}$ satisfies Haag duality:*

$$\mathcal{A}(I) = \mathcal{A}(\mathbb{R}\backslash I)'$$

(*ii*) *$\mathcal{A}$ is* strongly additive*: If I_1, I_2 are the connected components of the interval I with one internal point removed, then*

$$\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$$

(*iii*) *if I, I_1, I_2 are intervals as above*

$$\mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2)$$

Proof. As in the standard subspace case. □

Let $\mathcal{A}$ be a net on $\mathbb{R}$; we define the *dual net* $\mathcal{A}^d$ by

$$\mathcal{A}^d(I) \equiv \mathcal{A}(-\infty, b) \cap \mathcal{A}(a, \infty)$$

for every interval $I \equiv (a, b)$.

Clearly $\mathcal{A}^d$ is a net on the intervals, namely it is isotonic, and $\mathcal{A}^d$ is larger than $\mathcal{A}$:

$$\mathcal{A}(I) \subset \mathcal{A}^d(I), \quad I \in \mathcal{I}_0\ ,$$

however the two nets coincide on half-lines:

$$\mathcal{A}(I_1) = \mathcal{A}^d(I_1), \quad \text{if } I_1 \text{ is a half line.}$$

Indeed if I_1 is a half-line, $I \in \mathcal{I}_0$ and $I \subset I_1$, then directly by the definition of $\mathcal{A}^d$ we have $\mathcal{A}^d(I) \subset \mathcal{A}(I_1)$ and so $\mathcal{A}^d(I_1) \subset \mathcal{A}(I_1)$. It thus follows that $\mathcal{A}^d$ is local if $\mathcal{A}$ is local.

Duality for half-lines (6.4.2) is also called *essential duality* because of the following lemma.

Lemma 6.4.4. *Let $\mathcal{A}$ be a net on the intervals of $\mathbb{R}$ satisfying essential duality. The dual net $\mathcal{A}^d$ is satisfies* duality on $\mathbb{R}$*:*

$$\mathcal{A}^d(I) \equiv \mathcal{A}(\mathbb{R}\backslash I)', \qquad I \in \mathcal{I}_0.$$

Proof. With $I \equiv (a, b) \in \mathcal{I}_0$ we have

$$\mathcal{A}^d(a,b) = \mathcal{A}(-\infty, b) \cap \mathcal{A}(a, \infty) = \mathcal{A}(b, \infty)' \cap \mathcal{A}(-\infty, a)' \\ = (\mathcal{A}(b,\infty) \cup \mathcal{A}(-\infty, a))' = (\mathcal{A}^d(b,\infty) \cup \mathcal{A}^d(-\infty, a))' = \mathcal{A}^d(\mathbb{R} \setminus I)' \ .$$

□

Note that, if essential duality holds for $\mathcal{A}$, the dual net is also given as in the above proof by

$$\mathcal{A}^d(I) = \mathcal{A}(\mathbb{R} \setminus I)', \quad I \in \mathcal{I}_0 \ .$$

because $\mathcal{A}(\mathbb{R} \setminus I) = \mathcal{A}^d(\mathbb{R} \setminus I)$ for any interval I.

Let now $\mathcal{A}$ be a local Möbius covariant net on S^1. Haag duality on S^1 entails essential duality for the restriction $\mathcal{A}_0$ to $\mathbb{R}$. Hence $\mathcal{A}_0^d$ obeys Haag duality on $\mathbb{R}$. Now $\mathcal{A}_0^d$ is a net on $\mathbb{R}$ that does not in general transform covariantly under the covariance unitary representation U of $\mathbf{G}$ of $\mathcal{A}$, however $\mathcal{A}_0^d$ is clearly $\mathbf{P}$-covariant with respect to the restriction of U to $\mathbf{P}$. By Thm. 6.4.2 $\mathcal{A}_0^d$ extends to a local, Möbius covariant net on S^1 that we still call the dual net of $\mathcal{A}$ and denote by $\mathcal{A}^d$.

Corollary 6.4.5. *The dual net $\mathcal{A}^d$ of a local, Möbius covariant net $\mathcal{A}$ on S^1 is a strongly additive local Möbius covariant net on S^1.*

Proof. By construction, the $\mathcal{A}^d$ satisfies Haag duality on $\mathbb{R}$, hence strong additivity by Lemma 6.4.3. □

Clearly the unitary representation of $\mathbf{G}$ associated with $\mathcal{A}^d$ differs from the one associated with $\mathcal{A}$ although both of them have the same restriction to $\mathbf{P}$. Moreover $\mathcal{A}(I)$ is in general not contained in $\mathcal{A}^d(I)$ if I is not contained in $\mathbb{R} \simeq S^1 \setminus \{-1\}$; indeed $\mathcal{A}^d(I) \subset \mathcal{A}(I)$ if the point at infinity $-1 \in I$.

We shall now discuss some consequence of the above results.

An inclusion $N \subset M$ of von Neumann algebras is said to be *normal* if $N = N^{cc}$, where $R^c = R' \cap M$ denotes the relative commutant of R in M, and conormal if M is generated by N and its relative commutant w.r.t. M, i.e., $M = N \vee N^c$ (i.e. $M' \subset N'$ is normal).

We shall then say that a local Möbius covariant net $\mathcal{A}$ is (co-)normal if the inclusion $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is (co-)normal for any pair $I_1 \subset I_2$ of proper intervals of S^1. By Haag duality, normality and conormality are equivalent properties of conformal nets.

Theorem 6.4.6. *Let $\mathcal{A}$ be a local Möbius covariant net on S^1. For any pair $I_1 \subset I_2$ of intervals of S^1 the inclusion of von Neumann algebras $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is normal and conormal. In particular the relative commutant $\mathcal{A}(I_1)' \cap \mathcal{A}(I_2)$ is a factor.*

Proof. Let us consider first an inclusion of two intervals $I_1 \subset I_2$ with a common boundary point.

Assume first that $\mathcal{A}$ is strongly additive, then the inclusion of von Neumann algebras $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is conormal as in this case $\mathcal{A}(I_1)' \cap \mathcal{A}(I_2) = \mathcal{A}(I_2 \backslash I_1)$. In the general case, by conformal invariance we may assume that I_1 and I_2 are respectively the intervals of the real line $(1, +\infty)$ and $(0, +\infty)$. By definition then $\mathcal{A}(I_1) = \mathcal{A}^d(I_1)$, $\mathcal{A}(I_2) = \mathcal{A}^d(I_2)$, with $\mathcal{A}^d$ the dual net, hence the inclusion $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is conormal by Corollary 3.6.5 and the above argument. As $\mathcal{A}(I_2)' \subset \mathcal{A}(I_1)'$ is conormal, $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ is also normal.

It remains to show the normality of $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ when $I_1 \subset I_2$ are intervals with no common boundary point, e.g. $I_1 = (b, c)$ and $I_2 = (a, d)$, with $a < b < c < d$. Then we set $I_3 = (a, c)$ and $I_4 = (b, d)$, therefore $I_1 = I_3 \cap I_4$ and both I_3 and I_4 are subintervals of I_2 with a common boundary point. Then the double relative commutant of $\mathcal{A}(I_1)$ in $\mathcal{A}(I_2)$ is given by

$$\mathcal{A}(I_1)^{cc} \subset \mathcal{A}(I_3)^{cc} \cap \mathcal{A}(I_4)^{cc} = \mathcal{A}(I_3) \cap \mathcal{A}(I_4) = \mathcal{A}(I_1) \qquad (6.4.3)$$

where the last equality is a consequence of duality and additivity and implies the first inclusion; the opposite inclusion is elementary.

The factoriality of $\mathcal{A}(I_1)^c$ follows because the center of $\mathcal{A}(I_1)^c$ is contained in the center of $\mathcal{A}(I_1)^{cc} = \mathcal{A}(I_1)$ and $\mathcal{A}(I_1)$ is a factor by Prop. 6.2.9. □

Corollary 6.4.7. *Let $(N \subset M, \Omega)$ be a hsm standard inclusion of von Neumann algebras. Then:*

- *The inclusion $N \subset M$ is normal and conormal.*
- *There exists a unique strongly additive local Möbius covariant net $\mathcal{A}$ of von Neumann algebras on S^1 with $M = \mathcal{A}(0, +\infty)$, $N = \mathcal{A}(1, +\infty)$, and Ω the vacuum vector.*
- *There exists a bijection between local Möbius covariant nets $\mathcal{A}$ of von Neumann algebras on S^1 with $M = \mathcal{A}(0, +\infty)$, $N = \mathcal{A}(1, +\infty)$, Ω the vacuum vector, and von Neumann subalgebras N_0 of $N' \cap M$ cyclic on Ω such that $(N_0 \subset M, \Omega)$ is a $-$hsm inclusion and $(N_0 \subset N', \Omega)$ is a $+$hsm inclusion.*

Proof. Starting with the last point, notice that (M', N_0, N, Ω) is a hsm factorization of von Neumann algebras, and clearly any hsm factorization arises in this way, therefore the thesis is a consequence of Theorem 6.4.1.

In the special case $N_0 = N' \cap M$ we then obtain the second statement by Lemma 6.4.3 (ii)⇔ (iii).

The first point is then a consequence of Theorem 6.4.6. □

We end this section by introducing the n-regularity notion.

Let $\mathcal{A}$ be a local, Möbius covariant net of von Neumann algebras on S^1. We shall say that $\mathcal{A}$ is *n-regular* if $\mathcal{A}(S^1 \setminus \{p_1, \dots p_n\} = B(\mathcal{H})$ for any choice of n distinct points $\{p_1, \dots p_n\} \in S^1$.

Clearly n-regularity implies $n-1$-regularity and 2-regularity always holds due to Prop. 6.2.9 (*iii*). We shall see in Sect. 6.5 examples with $\mathcal{A}$ n-regular but not $n+1$-regular, $n \geq 3$.

6.5 Second quantization nets

Let $\mathcal{H}$ be a complex Hilbert space and let $\Gamma(\mathcal{H})$ be the *exponential* of $\mathcal{H}$, i.e. the Bosonic Fock space over $\mathcal{H}$ (also denoted by $e^{\mathcal{H}}$). Thus

$$\Gamma(\mathcal{H}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_s^{\otimes^n} ,$$

$\mathcal{H}_0 \equiv \mathbb{C}\Omega$ is the one-dimensional Hilbert space a unit vector Ω calle the *vacuum*, and $\mathcal{H}_s^{\otimes^n}$ is the symmetric Hilbert n-fold tensor product, namely $\mathcal{H}_s^{\otimes^n} = \mathrm{Sym}_n(\mathcal{H} \otimes \cdots \otimes \mathcal{H})$, where Sym_n is the orthogonal projection onto the fixed point vectors for the natural unitary representation on $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ of the n-element permutation group $\mathbb{P}_n$, so $\mathrm{Sym}_n \xi_1 \otimes \cdots \otimes \xi_n = \frac{1}{n!} \sum_{\pi \in \mathbb{P}_n} \xi_{\pi(1)} \otimes \cdots \otimes \xi_{\pi(n)}$.

If $\xi \in \mathcal{H}$, we denote by e^{ξ} the coherent vector of $e^{\mathcal{H}}$:

$$e^{\xi} \equiv \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi \otimes \cdots \otimes \xi \quad n\text{-times}$$

It is immediate to see that

$$(e^{\xi}, e^{\eta}) = e^{(\xi,\eta)} ,$$

so $||e^{\xi} - e^{\eta}||^2 = e^{||\xi||^2} + e^{||\eta||^2} - 2\Re e^{(\xi,\eta)}$ and the map $\xi \mapsto e^{\xi}$ is norm continuous.

Lemma 6.5.1. $\{e^{\xi},\ \xi \in \mathcal{H}\}$ *is a total family of independent vectors of* $\Gamma(\mathcal{H})$.

Proof. Let $\mathcal{K}$ be the closed linear span of $\{e^{\xi},\ \xi \in \mathcal{H}\}$, so the family to be total means that $\mathcal{K} = \mathcal{H}$. Now $\mathcal{K}$ contains

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{t\xi}|_{t=0} = \sqrt{n!}\, \xi \otimes \cdots \otimes \xi \in \mathcal{H}_s^{\otimes^n} \tag{6.5.1}$$

so, in order to show that $\mathcal{K} = \mathcal{H}$, is sufficient to show that $\mathcal{H}_s^{\otimes^n}$ is the closed linear span of vectors of the form $\xi \otimes \cdots \otimes \xi$. With $\xi_i \in \mathcal{H}$, $i = 1, \ldots n$, let $\xi = \xi(\lambda_1, \cdots \lambda_n)$ be the vector $\xi \equiv \sum_i \lambda_i \xi_i$ where $\lambda_i \in \mathbb{C}$. Then

$$\frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \xi \otimes \cdots \otimes \xi|_{\lambda_1 = \lambda_2 \cdots = \lambda_n = 1} = \mathrm{Sym}_n \xi_1 \otimes \cdots \otimes \xi_n \tag{6.5.2}$$

which prove our statement.

Concerning the independence of the coherent vectors, let $\xi_i \in \mathcal{H}$, $i = 1, \ldots n$, be mutually different vectors. Assume $\sum_i c_i e^{\xi_i} = 0$ with non-zero $c_i \in \mathbb{C}$. With $\eta \in \mathcal{H}$, for all $t \in \mathbb{R}$ we then have $(\sum_i c_i e^{\xi_i}, e^{t\eta}) = \sum_i c_i e^{t(\xi_i, \eta)} = 0$. This implies that the numbers $\{(\xi_i, \eta),\ i = 1 \ldots n\}$ are not mutually different. As this is true for all η, the vectors ξ_i are all equal, which is a contradiction showing that the vectors e^{ξ_i} are indeed independent. □

If $A \in B(\mathcal{H})$ and $\|A\| \leq 1$, the norm one operator $\Gamma(A)$ (or e^A) on $B(\Gamma(\mathcal{H}))$

$$\Gamma(A) = 1 \oplus A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \cdots$$

is called the second quantization of A. Note that $e^A e^{\xi} = e^{A\xi}$.

Setting

$$W(\xi) e^{\eta} \equiv e^{-\frac{1}{2}(\xi, \xi)} e^{-(\xi, \eta)} e^{\xi + \eta}$$

we get an isometry on $\{e^{\xi},\ \xi \in \mathcal{H}\}$, that extends to a unitary operator $W(\xi)$ on $e^{\mathcal{H}}$. The $W(\xi)$'s are called *Weyl unitaries* as the map $\xi \mapsto W(\xi)$ is norm - strong operator continuous and gives a representation of the Weyl commutation relations

$$W(\xi + \eta) = e^{i\Im(\xi, \eta)} W(\xi) W(\eta)\ .$$

Note that

$$W(\xi)\Omega = W(\xi) e^0 = e^{-\frac{1}{2}(\xi, \xi)} e^{\xi} \tag{6.5.3}$$

therefore

$$\omega(W(\xi)) = e^{-\frac{1}{2}\|\xi\|^2}$$

where $\omega \equiv (\Omega, \cdot\Omega)$ is the vacuum state.

By the uniqueness of the GNS representation, the above Fock representation is (up to unitary equivalence) the unique representation of the Weyl commutation relations (on a Hilbert space $\mathcal{H}$) with a cyclic vector Ω such that $(\Omega, W(\xi)\Omega) = e^{-\frac{1}{2}\|\xi\|^2)}$.

Let $H \subset \mathcal{H}$ be a real subspace. We put

$$R(H) \equiv \{W(\xi) : \ \xi \in H\}'' ,$$

namely $R(H)$ is the von Neumann algebra on $e^{\mathcal{H}}$ given by the weak closure of the linear span of the $W(\xi)$'s as ξ varies in $\mathcal{H}$.

Note that if U is a unitary operator on $\mathcal{H}$ then $\Gamma(U)W(\xi)\Gamma(U)^* = W(U\xi)$. In particular, if $UH = H$, then $\Gamma(U)$ implements an automorphism of $R(H)$,

Proposition 6.5.2. (a) : *If K, H are real linear subspaces then* $R(K) = R(H)$ *iff K and H have the same closure:* $\bar{K} = \bar{H}$.

(b) : *Let H be closed. H is a cyclic (resp. separating) real subspace of* $\mathcal{H}$ *iff* Ω *is a cyclic (resp. separating) vector for* $R(H)$.

(c) : *Let H be standard. Then the modular unitaries and conjugation associated with* $(R(H), \Omega)$ *are given by*

$$\Delta^{it}_{R(H)} = \Gamma(\Delta^{it}_H), \quad J_{R(H)} = \Gamma(J_H) .$$

(c) : $R(H') = R(H)'$.

Proof. (a): $R(H) = R(\bar{H})$ follows immediately by the continuity of the Weyl representation $\xi \mapsto W(\xi)$. To prove the second assertion we may assume that $K$and H are closed and $K \subset H$, otherwise replacing K with $\bar{K}$ and H with $\overline{K + H}$. We shall show this at the end of this proof.

(b): By the Weyl commtation relation $R(H)$ and $R(H')$ commute, so it is sufficient to show that Ω is cyclic for $R(H)$ if H is cyclic. As $W(\xi)\Omega = e^{-\frac{1}{2}(\xi,\xi)}e^{\xi}$, it follows that $R(H)\Omega$ contains all coherent vector associated with $\xi \in H$. By (6.5.1) $\overline{R(H)\Omega}$ contains all vectors $\xi \otimes \cdots \otimes \xi \in \mathcal{H}^{\otimes^n}_s$ with $\xi \in H$, thus all vectors $\overline{\mathrm{Sym}_n \xi_1 \otimes \cdots \otimes \xi_n} \in \mathcal{H}^{\otimes^n}_s$ with $\xi_k \in H$ by (6.5.2). As H is cyclic, we then have $\overline{R(H)\Omega} = \mathcal{H}$.

(c): As $\Gamma(\Delta^{it}_H)W(\xi)\Gamma(\Delta^{it}_H)^* = W(\Delta^{it}_H\xi)$, we see that $\Gamma(\Delta^{it}_H)$ implements automorphisms of $R(H)$ and we check the KMS condition. With $\sigma_t \equiv \mathrm{Ad}\Gamma(\Delta^{it}_H)$ we have by (6.5.3) and the one-particle KMS condtion (2.1.6)

$$
\begin{aligned}
(\sigma_t(W(\xi))W(\eta)\Omega,\Omega)|_{t=-i} &= e^{-\frac{1}{2}(\xi,\xi)}e^{-\frac{1}{2}(\eta,\eta)}(e^{\eta}, e^{-\Delta_H^{it}\xi})|_{t=-i} \\
&= e^{-\frac{1}{2}(\xi,\xi)}e^{-\frac{1}{2}(\eta,\eta)}e^{-(\eta,\Delta_H^{it}\xi)}|_{t=-i} = e^{-\frac{1}{2}(\xi,\xi)}e^{-\frac{1}{2}(\eta,\eta)}e^{-(\eta,\Delta\xi)} \\
&= e^{-\frac{1}{2}(\xi,\xi)}e^{-\frac{1}{2}(\eta,\eta)}e^{-(\xi,\eta)} = (W(\eta)W(\xi)\Omega,\Omega)
\end{aligned}
$$

that is a form of the KMS condition.

(d): This follows at once because $R(H') = R(J_H H) = J_{R(H)}R(H)J_{R(H)} = R(H)'$.

It remain to complete the proof of (a). Let $K \subset H$ be closed real linear subspaces and suppose that $R(K) = R(H)$. We first assume that H (resp. K) is standard; then also K (resp. H) is standard and by (b) and by (c) we have

$$
\Gamma(\Delta_K^{it}) = \Delta_{R(K)}^{it} = \Delta_{R(H)}^{it} = \Gamma(\Delta_H^{it})
$$

so $\Delta_K^{it} = \Delta_H^{it}$ and $K = H$ by Prop. 2.1.10. Now we only assume that K is separating, then also H is separating by (b); replacing $\mathcal{H}$ with $\overline{H + iH}$ we may again in the case H standard. Now we only assume that H is cyclic; by considering $H' \subset K'$ we are back in the previous case. Finally, in the general case, replacing $\mathcal{H}$ with $\overline{H + iH}$ we may assume that H is cyclic, so our proof is complete. □

Proposition 6.5.3. *let $\mathcal{H}_k$ be a family of Hilbert spaces. Then $\Gamma(\bigoplus_k \mathcal{H}_k)$ can be identified wirh $\bigotimes_k^{\Omega_k} \Gamma(\mathcal{H}_k)$, where Ω_k is the vacuum vector of $\Gamma(\mathcal{H}_k)$. In this identification, $W(\bigoplus \xi_k) = \bigotimes_k^{\Omega_k} W_k(\xi_k)$, where W_k and W are the Weyl operators associated with $\mathcal{H}_k$ and $\bigoplus_k \mathcal{H}_k$. Also $\Gamma(A_1 \oplus A_2 \oplus \cdots) = \Gamma(A_1) \otimes \Gamma(A_2) \otimes \cdots$ for contractions A_k on $\mathcal{H}_k$.*

Proof. For simplicity we assume $k = 1, 2$. We show that the vacuum state ω associated with $\mathcal{H}$ is the tensor product of the vacuum states ω_1, ω_2 associated with $\mathcal{H}_1, \mathcal{H}_2$. The rest follows at once.

Indeed we have $\xi_k \in \mathcal{H}_k$ we have

$$
\omega(W(\xi_1 \oplus \xi_2)) = e^{-\frac{1}{2}\|\xi_1 \oplus \xi_2\|^2} = e^{-\frac{1}{2}\|\xi_1\|^2}e^{-\frac{1}{2}\|\xi_2\|^2} = \omega(W(\xi_1))\omega(W(\xi_2)) \,.
$$

□

Let now H be a local, Möbius covariant net of standard subspaces of $\mathcal{H}$, U the corresponding unitary representation of $\mathbf{G}$. Assume that H is non-degenerate, namely U does not contain the identity representation. Then, setting

$$
\mathcal{A}(I) \equiv R(H(I)), \quad I \in \mathcal{I},
$$

we obtain a local Möbius covariant net of von Neumann algebras on $\Gamma(\mathcal{H})$ where the unitary covariance action of $\mathbf{G}$ is $\Gamma(U)$. Note that H non-degenerate ensures the uniqueness of the vacuum vector Ω.

Lemma 6.5.4. *The above net $\mathcal{A} = R(H(\cdot))$ is strongly additive (resp. n-regular) iff H is strongly additive (resp. n-regular) .*

Proof. Let $I, I_1, I_2 \in \mathcal{I}$ be disjoint intervals with $\bar{I} = \overline{I_1 \cup I_2}$. If H is strongly additive then $\overline{H(I_1) + H(I_2)} = H(I)$ so $\mathcal{A}(I)$ is the von Nemann algebra $\mathcal{B}$ generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ by the continuity of the Weyl representation. If H is not strongly additive then $\overline{H(I_1) + H(I_2)} \neq H(I)$ so by Prop. 6.5.2 we have $\mathcal{B} \neq \mathcal{A}(I)$ because $\mathcal{B} = R(H(I_1) + H(I_2))$.

The n-regular case is proved similarly. □

Because of the correspondence between positive energy representations of $\mathbf{G}$ and local, Möbius covariant nets of standard subspaces given by Theorem 3.6.7, we then have an arrow

$$U \longrightarrow \mathcal{A}_U$$

that associates a local Möbius covariant net of von Neumann algebras $\mathcal{A}_U$ with any unitary positive energy representation of $\mathbf{G}$, provided U has no non-zero fixed vector. These are the *second quantization nets*.

If $U^{(1)}$ is the irreducible representation of $\mathbf{G}$ with lowest weight 1, then $\mathcal{A}_U$ is called the $U(1)$ *current algebra net*.

If $U^{(n)}$ is the irreducible representation of $\mathbf{G}$ with lowest weight n, then $\mathcal{A}_{U^{(n+1)}}$ is called the net associated with the *n-derivative of* $U(1)$*-current*.

Note that

$$\mathcal{A}_{U_1 \oplus U_2} = \mathcal{A}_{U_1} \otimes \mathcal{A}_{U_2}$$

therefore any second quantization net $\mathcal{A}_U$ is given by

$$\mathcal{A}_U = \bigotimes_{n=1}^{\infty} \mathcal{A}_{U^{(n)}}^{\otimes m_n}$$

where $U = \bigoplus_n m_n U^{(n)}$ and $\mathcal{A}_{U^{(n)}}^{\otimes m_n}$ is the tensor product of m_n copies of $\mathcal{A}_{U^{(n)}}$. So every second quantization net is determined by the multiplicity coefficients m_n.

Proposition 6.5.5. *$\mathcal{A}_{U^{(n)}}$ is strongly additive if and only if $n = 1$, hence $\mathcal{A}_{U^{(1)}}$ n-regular for any n.*

$\mathcal{A}_{U^{(2)}}$ is 3-regular but not 4-regular. $\mathcal{A}_{U^{(n)}}$ is not 3-regular if $n \geq 3$.

$\mathcal{A}_{U^{(1)}}$ is the dual net of $\mathcal{A}_{U^{(n)}}$ for every n.

Proof. Immediate by Cor. 4.2.4 and Lemma 6.5.4. □

6.6 Appendix

6.6.1 Innerness of the Möbius action

We first recall without proof the following theorem by Borchers.

Theorem 6.6.1. *Let M be a von Neumann algebra and α a one-parameter group of automorphisms of M. If $\alpha_t = \mathrm{Ad}U(t)$, where the unitary one-parameter group has semi-bounded generator, shows that α is inner, namely $\alpha_t = \mathrm{Ad}v(t)$ here v is a one-parameter unitary group and $v(t) \in M$.*

The above theorem generalizes the Kadison-Sakai derivation theorem, which is equivalent to Thm. 6.6.1 in the case the generator is bounded. We treat here a variation of the above theorem.

Lemma 6.6.2. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $U(t) \equiv e^{itP}$ a one-parameter unitary group on $\mathcal{H}$ such that $\mathrm{Ad}U(t)M = M$, $t \in \mathbb{R}$. Suppose $sp(P) \subset [a, \infty)$ and let $\mathcal{H}_a$ the space of a-eigenvectors for P. If $\mathcal{H}_a$ is cyclic for M, then $U(t) \in M$.*

Proof. Replacing P with $P - a$ we may assume that $a = 0$, so $P \geq 0$. By considering the adjoint action of U on M', we may equivalently prove the dual statement, namely that if $\mathcal{H}_a$ is separating then $\mathrm{Ad}U$ is trivial. To this end we first assume that $\mathcal{H}_a$ is both cyclic and separating. Choose selfadjoint $x \in M$, $x' \in M'$ and $\xi \in \mathcal{H}_a$. The funtion $f(t) \equiv (U(t)x\xi, x'\xi)$ is real and, since $P \geq 0$, f extends to a bounded continuous function on the upper plan $\Im z \geq 0$, analytic in $\Im z > 0$. So f is constant, namely $U(t) = 1$, by the cyclic and separating assumption for $\mathcal{H}_a$. Now assume that $\mathcal{H}_a$ is only separating for $\mathcal{H}_a$ and let $E \in M'$ be the projection onto $\overline{M\mathcal{H}_a}$. Then U commutes with E, by considering the action of $U(t)E$ on M_E, we see that $U(t)E = E$ as $\mathcal{H}_a$ is cyclic and separating for M_E. Then for every $x \in M$ the operators x and $\mathrm{Ad}U(t)x$ have the same restriction to $\mathcal{H}_a$, hence $x = \mathrm{Ad}U(t)x$ because $\mathcal{H}_a$ is separating. □

Proposition 6.6.3. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\alpha : g \in \mathbf{G} \mapsto \alpha_g \in \mathrm{Aut}(M)$ an automorphism action. If U a positive energy unitary representation of $\bar{\mathbf{G}}$ on $\mathcal{H}$ such that $\alpha_g = \mathrm{Ad}U(g)$, $g \in \bar{\mathbf{G}}$. Then α is inner* *and there exists a positive energy unitary representation V of $\bar{\mathbf{G}}$ on $\mathcal{H}$ such that $\alpha_g = \mathrm{Ad}V(g)$ and $V(g) \in M$.*

Moreover, if the lowest eigenvector space $\mathcal{H}_a$ for the conformal Hamiltonian L_0 is cyclic for M, then $U(g) \in M$.

Proof. We first prove the last statement. By the above lemma $U(R(t)) \in M$ so $U(gR(t)g^{-1}) = U(g)U(R(t))U(g)^* \in M$, namely $U(h) \in M$ for every $h \in \bar{\mathbf{G}}$ conjugate to rotation, thus for all $g \in \bar{\mathbf{G}}$ because these elements generate $\bar{\mathbf{G}}$ ($\mathbf{G}$ is a simple group).

In general, by the lemma and the argument here above, α_g is inner for all $g \in \mathbf{G}$. So there exists unitaries $V(g) \in M$ such that $\alpha_g = \mathrm{Ad}V(g)$. Then $V(gh) = Z(g,h)V(g)V(h)$ with $Z(g,h)$ in the center of M. Since the cohomology of $\bar{\mathbf{G}}$ is trivial, by replacing $V(g)$ with $c(g)V(g)$ for a suitable central unitary $c(g)$, we have the group property $V(gh) = V(g)V(h)$. □

6.6.2 On the fixed point algebra under a group action

Proposition 6.6.4. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$, Ω a cyclic and separating vector for M, U a Ω-fixing unitary representation of a semigroup G on $\mathcal{H}$ such that $\alpha_g(M) \subset M$, $g \in G$, where $\alpha_g \equiv \mathrm{Ad}U(g)$. Then there exists a ω-invariant normal faithful conditional expectation ε of M onto the fixed point algebra M^α, where $\omega \equiv (\cdot\Omega, \Omega)$. Moreover $\overline{M^\alpha\Omega} = E\mathcal{H}$, where E is the projection onto the space of U-invariant vectors, and $\varepsilon(x)E = ExE$, $x \in M$.*

Proof. Suppose first that $G = \mathbb{N}$. Then by the mean ergodic theorem $\frac{1}{n}\sum_{k=1}^{n} U^k\xi$ weakly converges to $E\xi$ for every $\xi \in \mathcal{H}$. Let $x \in M$ and consider the bounded sequence $x_n \equiv \frac{1}{n}\sum_{k=1}^{n}\alpha^n(x)$. Clearly $x_n\Omega = \frac{1}{n}\sum_{k=1}^{n} U^k x\Omega$, so every x_n-weak limit point $\bar{x}$ satisfies $\bar{x}\Omega = Ex\Omega$. So $\bar{x}$ is unique, $\alpha(\bar{x}) = \bar{x}$, and we may define ε by $\varepsilon(x) \equiv \bar{x}$. Clearly $\varepsilon(x)E = ExE$ and the rest is clear. The case of a general G can be proved by replacing $\frac{1}{n}\sum_{k=1}^{n} U^k$ with a net in the convex hull of $U(G)$ weakly converging to E (consider sub-semigroups generated by single elements $g \in G$). □

Proposition 6.6.5. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If there exists a faithful normal state ω with trivial centralizer, i.e. the modular group σ^ω is ergodic. Then M is a type III_1 factor in Connes classification, or $M = \mathbb{C}$.*

Proof. As the center is contained in the centralizer, M is a factor. If M is semifinite then $\sigma^\omega = \mathrm{Ad}u_t$ for some one-parameter unitary group in M, thus $\sigma^\omega(u_s) = u_s$; as σ^ω is ergodic $u_s \in \mathbb{C}$ so σ^ω is trivial so M is equal to its centralizer $\mathbb{C}$.

If $M \neq \mathbb{C}$ then M is of type III. To infer that M is actually a type III_1 factor we shall rely on the fact that $\mathrm{sp}\Delta_\omega \setminus \{0\}$ is minimal, i.e. equal to Connes invariant $S(M)$, if the centralizer is a factor. Thus $S(M) \neq \{1\}$ namely M is not of type III_0. Moreove M cannot be of type III_λ with $\lambda \in (0,1)$ as otherwise the centralizer would be a factor of type II_1. So M has to be of type III_1. □

Corollary 6.6.6. *Let M be a von Neumann algebra a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic vector for M. Let U be a one-parameter group on $\mathcal{H}$ with generator P satisfying $U(s)MU(-s) \subset M$, $s \geq 0$. Suppose that $P > 0$ and Ω is a simple eigenvector of P.*

Then either

(*a*) *$M = B(\mathcal{H})$: this is the case iff $U(s)MU(-s) = M$, $\forall s \in \mathbb{R}$; or*

(*b*) *M is a factor of type III_1: this is the case iff $U(s)MU(-s) \neq M$ for some $s \in \mathbb{R}$.*

Case (a) holds if $sp(P) \neq [0,\infty)$. Case (b) holds if Ω is separating.

Chapter 7

The split property

This chapter deals with a fundamental property for local Möbius covariant nets of von Neumann algebras, the split property. It is an algebraic property of the net that is closely related to more analytical properties of nuclearity type, as the trace class property for the semigroup generated by the conformal Hamiltonian. The split property selects physically, and mathematically, interesting models leaving outside models with too many degrees of freedom.

We begin with a discussion of the split propery in the abstract setting of inclusions of von Neumann algebras.

7.1 Standard and split inclusions of von Neumann algebras

7.1.1 Split inclusions

Let M_1, M_2 be a commuting pair of von Neumann algebras on a Hilbert space $\mathcal{H}$. We shall say that the pair (M_1, M_2) is *split* if there exists a von Neumann algebra isomorphism

$$\Phi : M_1 \vee M_2 \to M_1 \otimes M_2$$

such that

$$\Phi(m_1 m_2) = m_1 \otimes m_2 \ , \quad m_i \in M_i \ ,$$

in other words a natural von Neumann algebra isomorphism between $M_1 \vee M_2$ and $M_1 \otimes M_2$.

Note that if one of the M_i is a factor, by Murray-von Neumann lemma there is a natural isomorphism between the algebraic tensor product $M_1 \odot M_2$ and the *-algebra generated by M_1 and M_2. By definition, the pair is split if it extends to an isomorphism of the weak closures.

Lemma 7.1.1. *Let M_1, M_2 be a commuting pair of von Neumann algebras with $M \equiv M_1 \vee M_2$ σ-finite*[1]*. The following are equivalent:*

(i) *: (M_1, M_2) is split ;*

(ii) *: For any given normal states φ_1 on M_1 and φ_2 on M_2 there exists a normal state φ on M such that*

$$\varphi(m_1 m_2) = \varphi_1(m_1)\varphi_2(m_2)\ , \qquad m_i \in M_i\ .$$

and φ is faithful if both φ_1 and φ_2 are faithful.

(iii) *: There exists a faithful normal state φ on M such that*

$$\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)\ , \qquad m_i \in M_i\ .$$

Proof. Clearly $(i) \Rightarrow (ii)$ by the tensor product identification and $(ii) \Rightarrow (iii)$ is obvious.

Assuming (iii) we have to show that (i) holds. Let $\varphi_i \equiv \varphi|_{M_i}$. With π_i, π the GNS representations of (M_i, φ_i) and (M, φ), π is normal and faithful. Now the restriction of π to the *-algebra generated by M_1 and M_2 is $\pi_1 \odot \pi_2$ (by uniqueness of the GNS representation), so π provides a natural identification of $M_1 \vee M_2$ with $M_1 \otimes M_2$. □

Let now $N \subset M$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$. We shall say that $N \subset M$ is a *split inclusion* (of von Neumann algebras) if the commuting pair (N, M') is split. We shall frequently pass from an inclusion to a commuting pair and back.

Next lemma provides a Hilbert space free equivalent definition of split inclusion. Note that the trivial inclusion with $N = M$ is split iff M is a type I factor.

[1]A von Neumann algebra is σ-finite if every family of mutually orthogonal projections of M is countable; equivalently if there exists a faithful normal state on M. This automatic if the underlying Hilbert space is separable.

Lemma 7.1.2. *Let $N \subset M$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume that either a): $N' \cap M$ has a cyclic and separating vector or b): $\mathcal{H}$ is separable and $N' \cap M$ is properly infinite. Then $N \subset M$ is split iff there exists an intermediate type I factor F, $N \subset F \subset M$.*

Proof. Suppose there exists an intermediate type I factor F. The pair (F, F') is split and so is (M, N') because $N \subset F$ and $M' \subset F'$.

Conversely assume the pair $(M_1, M_2) \equiv (N, M')$ to be split and let $\Phi : M_1 \vee M_2 \to M_1 \otimes M_2$ be the natural isomorphism. Suppose that Φ is spatial, namely $\Phi(x) = UxU^*$ for some unitary U from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$. Then $F \equiv U^*(B(\mathcal{H}) \otimes 1)U$ is an intermediate type I factor. So any condition that ensures Φ to be spatial also gives an intermediate type I factor for $N \subset M$. For the case b) note that the commutant of $M_1 \vee M_2$ is properly infinite (by assumptions) and the commutant of $M_1 \otimes M_2$ is properly infinite (because $M_i' \supset (M_1 \vee M_2)'$). Thus Φ is spatial, i.e. implementented by a unitary U from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$. The case assuming a) is similar, cf. also the following Sect. 7.1.2. □

Let M_1 and M_2 be commuting factors on a Hilbert space $\mathcal{H}$. By Murray-von Neumann lemma the *-algebra $\mathfrak{A}_0$ generated by M_1 and M_2 is naturally isomorphic to the algebraic tensor product $M_1 \odot M_2$, so the norm closure $\mathfrak{A}_0$ of $\mathfrak{A}$ is a the C^* tensor product of M_1 and M_2 w.r.t. some C^* tensor product norm. Then we may consider the linear functionals on $\mathfrak{A}$ of the form

$$\sum_{k=1}^{n} \varphi_k \otimes \psi_k \tag{7.1.1}$$

with $\varphi_k \in M_{1*}$, $\psi_k \in M_{2*}$.

Lemma 7.1.3. *In the above situation, suppose there is a normal state φ on M such that $\varphi|_{\mathfrak{A}}$ is norm limit of sums of product functionals of the form* (7.1.1) *and that φ has central cover 1 in M (the last condition is satisfied if either φ is faithful or M is a factor). Then the pair (M_1, M_2) is split.*

Proof. Let π_0 be the defining representation of $\mathfrak{A}$ on $\mathcal{H}$ and π_p the representation of $\mathfrak{A}$ on $\mathcal{H} \otimes \mathcal{H}$ determined by $\pi_p(m_1 m_2) = m_1 \otimes m_2$, $m_i \in M_i$ (this is first defined on $\mathfrak{A}_0$ and then it extends to $\mathfrak{A}$ because the spatial tensor product norm is minimal).

We have to show that π_0 and π_p are quasi-equivalent. By assumption $\varphi|_{\mathfrak{A}}$ is norm limit of functionals that are normal w.r.t. π_p. So the GNS representation of $\varphi|_{\mathfrak{A}}$ is contained in π_p, thus it is quasi-equivalent to π_p because π_p is factorial. But the GNS representation of $\varphi|_{\mathfrak{A}}$ is quasi-equivalent to π_0 if φ has central cover 1 on M. □

7.1.2 Standard inclusions

Let $N \subset M$ be an inclusion of von Neumann algebras on a Hilbert space $\mathcal{H}$. A *semi-standard vector* (resp. *standard vector*) for $N \subset M$ is a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for $N' \cap M$ (resp. $N' \cap M$, N and M).

A *(semi)-standard inclusion* of von Neumann algebra is a triple $\Lambda \equiv (N \subset M, \Omega)$, where $N \subset M$ be an inclusion of von Neumann algebras and Ω is a (semi)-standard vector for $N \subset M$. We shall say that Λ is split if $N \subset M$ is split.

Proposition 7.1.4. *Let $\Lambda = (N \subset M, \Omega)$ be a standard split inclusion on $\mathcal{H}$. Then*

(a) *$\mathcal{H}$ is separable;*

(b) *N, M and $N' \cap M$ are properly infinite (unless $M = \mathbb{C}$).*

Proof. (a): The state $(\Omega, \cdot\Omega)$ is normal faithful on M, hence on any intermediate type I factor F. But a type I factor admits a normal faithful state iff it is separable in the strong topology. Thus $\overline{F\Omega}$ is a separable Hilbert space. As Ω is cyclic for N, it is also cyclic for F, so $\overline{F\Omega} = \mathcal{H}$.

(b): Suppose that an intermediate type I factor F is finite-dimensional. Then $M = F = N$ and $M = N' \cap N$ because Ω is separating, so M is abelian. F is thus one-dimensional and so is $\mathcal{H}$. So, if Λ is non-trivial, F is infinite-dimensional. As the von Neumann algebras are in a standard form, N' ($\supset F'$) and N, $(N' \cap M)'$ ($\supset N$) and $N' \cap M$ are properly infinite too. □

Lemma 7.1.5. *Let $N_i \subset M_i$ be an inclusion of von Neumann algebra on a Hilbert space $\mathcal{H}_i$ and $\Omega_i \in \mathcal{H}_i$ a cyclic vector for N_i, thus for M_i, $i = 1, 2$. Let Φ be an isomorphism of M_1 onto M_2 such that $(\Omega_2, \Phi(\cdot)\Omega_2)|_{M_1} = (\Omega_1, \cdot\,\Omega_1)|_{M_1}$. Then the unitary from $\mathcal{H}_1$ onto $\mathcal{H}_2$ determined by*

$$U n\Omega_1 = \Phi(n)\Omega_2 , \quad n \in N_1 , \tag{7.1.2}$$

implements Φ on M_1.

In particular if Ω_i is also separating for M_i, thus for N_i, and $\Phi(N_1) = N_2$, then the unitary standard implementations of Φ and of $\Phi|_{N_1}$ w.r.t. Ω_1, Ω_2 coincide.

Proof. The first part follows because the unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ determined by

$$U m\Omega_1 = \Phi(m)\Omega_2 , \quad m \in M_1 , \tag{7.1.3}$$

implements Φ and U is determined by its restriction to $N_1\Omega_1$ as $\overline{N_1\Omega_1} = \mathcal{H}_1$.

Now, if Ω_i is separating for M_i, then the standard unitary implementation U of Φ is given by (7.1.3). Also, if $\Phi(N_1) = N_2$, the standard unitary implementation of $\Phi|_{N_1}$ is given by (7.1.2), so it is equal to U by the above argument. □

Let $\Lambda_i = (N_i \subset M_i, \Omega_i)$ be a (semi)-standard inclusion, $i = 1, 2$ on the Hilbert space $\mathcal{H}_i$. An *isomorphism* $\Phi : \Lambda_1 \to \Lambda_2$ is an isomorphism of $\Phi : M_1 \to M_2$ such that $\Phi(N_1) = N_2$ and $(\Omega_2, \Phi(\cdot)\Omega_2) = (\Omega_1, \cdot\Omega_1)$. Note that Φ is spatial, indeed $\Phi = \mathrm{Ad}V_\Phi$ where $V_\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ the unitary defined by

$$V_\Phi x\Omega_1 = \Phi(x)\Omega_2, \quad x \in M_1 .$$

As Ω_i is cyclic for $N_i' \cap M_i$, the unitary V_Φ is determined by taking $x \in N_1' \cap M_1$ or, if Ω_i is cyclic for N_i, by taking $x \in N_1$. So, if the Λ_i are standard, V_Φ is the standard unitary implementation of Φ, of $\Phi|_{N_1 \cap M_1}$, or of $\Phi|_{N_1}$ as in Lemma 7.1.5.

Let now $\Lambda = (N \subset M, \Omega)$ be a standard split inclusion of von Neumann algebras Since Λ is split, there exists an isomorphism $\Phi_\Lambda : N \vee M' \to N \otimes M'$, $\Phi_\Lambda(nm') = n \otimes m'$, $n \in N, m' \in M'$. Since Ω and $\Omega \otimes \Omega$ are cyclic and separating repectively for $N \vee M'$ and $N \otimes M'$, we may consider the standard unitary implementation of Φ_Λ with respect to Ω and $\Omega \otimes \Omega$; namely the unique unitary $U_\Lambda : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $U_\Lambda nm'U_\Lambda^* = n \otimes m'$, $n \in N, m' \in M'$, and $U_\Lambda \mathcal{P}^\natural_\Omega(N \vee M') = \mathcal{P}^\natural_{\Omega\otimes\Omega}(N \otimes M')$. Note that

$$U_\Lambda N U_\Lambda^* = N \otimes 1, \quad U_\Lambda M' U_\Lambda^* = 1 \otimes M', \quad U_\Lambda M U_\Lambda^* = B(\mathcal{H}) \otimes M,$$

where the third equality follows by the second one by taking commutants.

We call $\Phi_\Lambda = \mathrm{Ad}U_\Lambda$ the *canonical tensorial representation* of Λ. Note that the canonical tensorial representation is an isomorphism of Λ with $(N \otimes 1 \subset M \otimes B(\mathcal{H}), \xi_\Lambda)$, hence a spatial identification, where $\xi_\Lambda \equiv U_\Lambda\Omega$. In the canonical tensorial representation many things become visible and trivialize.

The canonical tensorial representation is functorial, namely if Λ_1, Λ_2 are standard split inclusions and $\Phi : \Lambda_1 \to \Lambda_2$ is an isomorphism of standard inclusion then the following diagram commute

$$\begin{tikzcd}
\Lambda_1 = (N_1 \subset M_1, \Omega_1) \arrow[r, "\Phi_{\Lambda_1} = \mathrm{Ad}U_{\Lambda_1}"] \arrow[d, "\Phi = \mathrm{Ad}V_\Phi"] & (N_1 \otimes 1 \subset M_1 \otimes B(\mathcal{H}), \Omega) \arrow[d, "\Phi\otimes\Phi = \mathrm{Ad}V_\Phi \otimes V_\Phi"] \\
\Lambda_2 = (N_2 \subset M_2, \Omega_2) \arrow[r, "\Phi_{\Lambda_2} = \mathrm{Ad}U_{\Lambda_2}"] & (N_2 \otimes 1 \subset M_2 \otimes B(\mathcal{H}), \Omega)
\end{tikzcd} \tag{7.1.4}$$

and $(V_\Phi \otimes V_\Phi) \cdot U_{\Lambda_1} = U_{\Lambda_2} \cdot V_\Phi$. This follows immediately by the uniqueness of the standard implementation and the commutativity of the diagram

$$\begin{array}{ccc} N_1 \vee M_1' & \xrightarrow{\Phi_{\Lambda_1}} & N_1 \otimes M_1' \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi\otimes\Phi} \\ N_2 \vee M_2' & \xrightarrow{\Phi_{\Lambda_2}} & N_2 \otimes M_2' \end{array}$$

7.1.3 The canonical intermediate type I factor

If an inclusion of von Neumann algebras has an intermediate type I factor F, then in general it has infinitely many intermediate type I factors, for example factors of the form uFu^* with u a unitary in $N' \cap M$. Remarkably, a standard split inclusion has a canonical intermediate type I factor.

Theorem 7.1.6. *Let $\Lambda = (N \subset M, \Omega)$ be a semi-standard split inclusion on a Hilbert space $\mathcal{H}$. There exists a canonical intermediate type I factor F_Λ between N and M. If $\Phi : \Lambda_1 \to \Lambda_2$ is an isomorphism of standard split inclusions Λ_1, Λ_2, then $\Phi(F_{\Lambda_1}) = F_{\Lambda_2}$.*

Proof. For simplicity we assume Λ to be standard. In the canonical tensorial representation $\Phi_\Lambda = \mathrm{Ad}\,U_\Lambda$ the canonical intermediate type I factor is simply $B(\mathcal{H}) \otimes 1$. Namely we set

$$F_\Lambda \equiv U_\Lambda^*(B(\mathcal{H}) \otimes 1)U_\Lambda \, .$$

Clearly F_Λ is a type I factor such that $N \subset F_\Lambda \subset M$. The last part of the statement (functoriality character of F_Λ) follows immediately by the commutativity of the diagram (7.1.4). □

We shall call F_Λ the *canonical intermediate type I factor* for Λ.

Denote $\mathrm{Aut}(\Lambda)$ the automorphism group of Λ, i.e. the group of all isomorphisms of Λ with itself.

Corollary 7.1.7. *If $\Lambda = (N \subset M, \Omega)$ is a standard split inclusion, then*

(*a*) *F_Λ is globally stable under* $\mathrm{Aut}(\Lambda)$*,*

(*b*) *$F'_\Lambda = F_{\Lambda'}$, where $\Lambda' \equiv (M' \subset N', \Omega)$.*

Proof. (a) follows by previous Thm. 7.1.6.

(b): in the canonical tensorial representation of Λ we have:

$$\Lambda \equiv (N \otimes 1 \subset B(\mathcal{H}) \otimes M, \xi_\Lambda), \quad \Lambda' \equiv (1 \otimes M' \subset N' \otimes B(\mathcal{H}), \xi_\Lambda) .$$

We apply the characterization in the following Prop. 7.1.8. Then $F_{\Lambda'}$ is generated in the above canonical tensorial representation by $1 \otimes N'$ and $1 \otimes P_\Omega$, so it is equal to $1 \otimes B(\mathcal{H})$, the commutant of $U_\Lambda F_\Lambda U_\Lambda^*$. □

We now give another description of F_Λ. Let $\Lambda = (N \subset M, \Omega)$ be a standard split inclusion, then there exists a faithful normal *product state* φ on $N \vee M'$ given by $\varphi(nm') = (\Omega, n\Omega)(\Omega, m'\Omega)$, namely $\varphi = \omega \otimes \omega \cdot \Phi_\Lambda$ where $\omega \equiv (\Omega, \cdot\Omega)$. Let η_Λ be the a unique vector representative of φ in $\mathcal{P}^\natural_\Omega(N \vee M')$, namely $\eta_\Lambda \in \mathcal{P}^\natural_\Omega(N \vee M')$ and $\varphi = (\eta_\Lambda, \cdot\eta_\Lambda)$. As φ is faithful, then η_Λ is separating, thus cyclic, for $N \vee M'$.

Proposition 7.1.8. *F_Λ is generated by N and the projection $p_\Lambda \equiv [M'\eta_\Lambda] \in M$.*

Proof. Note that

$$U_\Lambda \eta_\Lambda = \Omega \otimes \Omega$$

by the uniqueness of the vector representative of a normal state in a natural cone.

Therefore, in the canonical tensorial representation M' goes to $1 \otimes M'$ and η_Λ to $\Omega \otimes \Omega$, thus p_Λ to $P_\Omega \otimes 1$, where P_Ω is the orthogonal projection onto $\mathbb{C}\Omega$. So we conclude by the following lemma.

Lemma 7.1.9. *Let N be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and Ω a cyclic vector for N. The von Neumann algebra generated by N and the rank one projection P_Ω onto $\mathbb{C}\Omega$ is $B(\mathcal{H})$.*

Proof. Indeed if $x \in B(\mathcal{H})$ commutes with N and P_Ω then $x\Omega = \lambda\Omega$ and thus $x = \lambda$ because $x \in N'$ and Ω is separating for N'. □

7.1.4 Compactness of $\mathrm{Aut}(\Lambda)$

By Cor. 7.1.7, $\mathrm{Aut}(\Lambda)$ leaves F_Λ globally stable. We now show that $\mathrm{Aut}(\Lambda)$ is compact and metrizable in the topology of pointwise norm convergence on the predual M_* of M.

Lemma 7.1.10. *Let F be a type I factor and ω a faithful normal state of F. The group $\mathrm{Aut}_\omega(F)$ of all automorphisms of F leaving ω invariant is a compact metrizable subgroup of $\mathrm{Aut}(F)$.*

Proof. As F is a type I factor we can assume that $F = B(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space. Since ω is faithful, F is σ-finite and hence $\mathcal{H}$ is separable. Let $h_\omega \in F$ be the non-singular trace class Radon-Nikodym derivative of ω with respect to the trace on F, $\omega(\cdot) = \mathrm{Tr}(h_\omega \cdot)$. Also consider the group $\mathcal{U}_\omega$ of all unitaries of F commuting with h_ω. $\mathcal{U}_\omega$ is compact metrizable in the strong operator topology, since it is (isomorphic to) the direct product of a countable family of metrizable compact groups, namely the groups of all unitaries acting on the finite-dimensional eigenspaces of h_ω. The map $u \in \mathcal{U}_\omega \to \mathrm{Ad}u \in \mathrm{Aut}_\omega(F)$ is continuous and surjective, and therefore $\mathrm{Aut}_\omega(F)$ is compact and metrizable. □

Theorem 7.1.11. *If $\Lambda = (N \subset M, \Omega)$ is a standard split inclusion acting on the Hilbert space $\mathcal{H}$, then* $\mathrm{Aut}(\Lambda)$ *is compact and metrizable.*

Proof. Ω is cyclic and separating for F_Λ, so $\omega_\Lambda \equiv (\Omega, \cdot\Omega)|_{F_\Lambda}$ is a faithful and normal state of F_Λ. For any $\alpha \in \mathrm{Aut}_{\omega_\Lambda}(F_\Lambda)$ let U_α be the standard implementation of α with respect to Ω. The map

$$f : \alpha \in \mathrm{Aut}_{\omega_\Lambda}(F_\Lambda) \to \mathrm{Ad}\, U_\alpha \in \mathrm{Aut}(B(\mathcal{H}))$$

is continuous, thus range(f) is compact and metrizable. Let $\mathrm{Aut}(N, M, B(\mathcal{H}))$ be the closed subgroup of $\mathrm{Aut}(B(\mathcal{H}))$ of all automorphisms of $B(\mathcal{H})$ leaving N and M globally stable. Then

$$G \equiv \mathrm{range}(f) \cap \mathrm{Aut}(N, M, B(\mathcal{H}))$$

is a compact metrizable subgroup of $\mathrm{Aut}(B(\mathcal{H}))$. But $\mathrm{Aut}(\Lambda)$ is the range of the continuous restriction map $\alpha \in G \to \alpha|_M \in \mathrm{Aut}(\Lambda)$, and thus the statement of the theorem follows. □

7.1.5 Characterizations of F_Λ

The following theorem gives an important description of the canonical intermediate type I factor.

Theorem 7.1.12. *Let $\Lambda = (N \subset M, \Omega)$ be a standard split inclusion and $J = J_{N' \cap M}$ the modular conjugation of $N' \cap M$ with respect to Ω. If both N, M are factors,*

$$F_\Lambda = N \vee JNJ = M \cap JMJ.$$

So F_Λ is the unique von Neumann algebra F with $N \subset F \subset M$ such that $JFJ = F$.

Proof. In the canonical tensorial representation ξ_Λ and $\Omega \otimes \Omega$ belong to the same natural cone $\mathcal{P}^\natural_{\Omega\otimes\Omega}(N' \otimes M)$, hence they give rise to the same modular conjugation for $N' \otimes M$ (the relative commutant in the canonical tensorial representation). Since the modular conjugation of the tensor product is the tensor product of the modular conjugations, we have

$$U_\Lambda J U_\Lambda^* = J_N \otimes J_M .$$

Since $\mathrm{Ad}\,U_\Lambda$ maps N onto $N \otimes 1$ we then have

$$\begin{aligned} \mathrm{Ad}\,U_\Lambda : JNJ &\longrightarrow J_N \otimes J_M (N \otimes 1) J_N \otimes J_M = N' \otimes 1 \\ \mathrm{Ad}\,U_\Lambda : N \vee JNJ &\longrightarrow (N \vee N') \otimes 1 = B(\mathcal{H}) \otimes 1 = \mathrm{Ad}\,U_\Lambda(F_\Lambda) \end{aligned}$$

thus $F_\Lambda = N \vee JNJ$. The equality $F_\Lambda = M \cap JMJ$ is similarly obtained.

Clearly any von Neumann algebra F intermediate between N and M such that $JFJ = F$ must be also intermediate between $N \vee JNJ$ and $M \cap JMJ$, hence it has to coincide with F_Λ. □

7.1.6 Local implementation

Given a standard split inclusion $\Lambda = (N \subset M, \Omega)$ we now construct a canonical unitary representation of $\mathrm{Aut}(\Lambda)$ in M implementing the natural action of $\mathrm{Aut}(\Lambda)$ on N with a natural covariance property.

Given a standard split inclusion Λ, the (normal, unital) endomorphism ψ_Λ of $B(\mathcal{H})$ defined by

$$\psi_\Lambda : X \in B(\mathcal{H}) \mapsto U_\Lambda^*(X \otimes 1)U_\Lambda$$

is called the *universal localizing map* associated to Λ. Note that

$$\psi_\Lambda(B(\mathcal{H})) = F_\Lambda$$

and that $\psi_\Lambda(n) = n$ for every $n \in N$.

Being an endomorphism of $B(\mathcal{H})$, ψ_Λ is *inner*, i.e. implemented by a (canonical) Hilbert space of isometries.

Remark. With $\Lambda = (N \subset M, \Omega)$ a standard split inclusion, we note that

$$\psi_\Lambda|_{N'} = \gamma$$

where $\gamma : N' \to N' \cap M$ is the canonical endomorphism w.r.t. Ω, namely $\gamma = \mathrm{Ad}\Gamma$ where $\Gamma \equiv JJ_N$. Indeed if $n' \in N'$, thus $n = J_N n' J_N \in N$, we have

$$\begin{aligned}\psi_\Lambda(n') &= U_\Lambda(n' \otimes 1)U_\Lambda^* = U_\Lambda(J_N n J_N \otimes 1)U_\Lambda^* \\ &= U_\Lambda(J_N \otimes J_M)(n \otimes 1)(J_N \otimes J_M)U_\Lambda^* = JU_\Lambda(n \otimes 1)U_\Lambda^* J = JnJ = JJ_N n' J_N J\ ,\end{aligned}$$

where we have used the relation $J_N \otimes J_M \cdot U_\Lambda = U_\Lambda J$ that holds because $J_N \otimes J_M$ is the modular conjugation of $N \otimes M'$ with respect to $\Omega \otimes \Omega$ and U_Λ is the standard implementation of the tensorial representation.

Theorem 7.1.13. *Let $\Lambda = (N \subset M, \Omega)$ be a standard split inclusion on $\mathcal{H}$. There exists a continuous unitary representation $v : \alpha \in \mathrm{Aut}(\Lambda) \to v(\alpha) \in M$ such that*

(a) $v(\alpha)nv(\alpha)^* = \alpha(n), \quad \alpha \in \mathrm{Aut}(\Lambda), n \in N\ ,$

(b) $\alpha(v(\beta)) = v(\alpha\beta\alpha^{-1})\ , \quad \alpha, \beta \in \mathrm{Aut}(\Lambda)\ ,$

(c) $v(\alpha)\mathcal{P}^\natural_\Omega(N' \cap M) = \mathcal{P}^\natural_\Omega(N' \cap M), \quad \alpha \in \mathrm{Aut}(\Lambda)\ ,$

(d) $v(\alpha) \in F_\Lambda\ ,$

(e) $Jv(\alpha)J = v(\alpha)\ ,$

(f) $v(g)\eta_\Lambda = \eta_\Lambda\ .$

Furthermore if $v' : \alpha \in \mathrm{Aut}(\Lambda) \to v'(\alpha) \in M$ is a unitary representation satisfying (a) and (c), or (a) and (f), then $v' = v$.

Proof. Let $\alpha \in \mathrm{Aut}(\Lambda)$ and U_α be the standard implementation of α as an automorphism of M. Then $U_\alpha\Omega = \Omega$ and we still denote by $\alpha = \mathrm{Ad}U_\alpha$ the adjoint action on $B(\mathcal{H})$. Then U_α is also the standard implementation of $\alpha|_{N \vee M'}$, of $\alpha|_N$ and of $\alpha|_{M'}$ by Lemma 7.1.5. Then $U_\Lambda U U_\Lambda^*$ is the unitary standard implementation of $\alpha|_N \otimes \alpha|_{M'}$ with respect to $\Omega \otimes \Omega$, thus $U_\Lambda U_\alpha U_\Lambda^* = U_\alpha \otimes U_\alpha$ Define

$$v(\alpha) \equiv \psi_\Lambda(U_\alpha) = U_\Lambda^*(U_\alpha \otimes 1)U_\Lambda$$

All the properties follow easily from the definition. Concerning the uniqueness, if v satisfies (a) and (c) then $v(\alpha) \in M$ is the standard implementation of the automorphism $\mathrm{Ad}U_\Lambda^* \cdot (\alpha|_N \otimes \iota) \cdot \mathrm{Ad}U_\Lambda : nm' \to \alpha(n)m'$ of $N \vee M'$ with respect to Ω, hence it is uniquely determined. (f) It follows from the definition that $U_\Lambda v(g)U_\Lambda^*\Omega \otimes \Omega = \Omega \otimes \Omega$ and $U_\Lambda \eta_\Lambda = \Omega \otimes \Omega$.

The rest follows easily. $\square$

We include the following table that illustrates how various objects look in the original representation and in the canonical tensorial representation:

Object	*Original rep.*	*Canonical tensorial rep.*
standard split inclusion Λ	$(N \subset M, \Omega)$	$(N \otimes 1 \subset B(\mathcal{H}) \otimes M, \xi_\Lambda)$
von Neumann algebra	N	$N \otimes 1$
von Neumann algebra	M'	$1 \otimes M'$
von Neumann algebra	M	$B(\mathcal{H}) \otimes M$
von Neumann algebra	F_Λ	$B(\mathcal{H}) \otimes 1$
von Neumann algebra	$N \vee M'$	$N \otimes M'$
von Neumann algebra	$N' \cap M$	$N' \otimes M$
vector	Ω	ξ_Λ
vector	η_Λ	$\Omega \otimes \Omega$
modular conjugation	$J \equiv J_{N' \cap M}$	$J_N \otimes J_M$
natural cone	$\mathcal{P}^\natural_\Omega(N' \cap M)$	$\mathcal{P}^\natural_{\Omega\otimes\Omega}(N' \otimes M) \supset \mathcal{P}^\natural_\Omega(N) \otimes \mathcal{P}^\natural_\Omega(M)$
unitary implem. $\alpha \in \mathrm{Aut}(\Lambda)$	U_α	$U_\alpha \otimes U_\alpha$
local unitary	$v(\alpha)$	$U_\alpha \otimes 1$

7.2 Modular nuclearity and the split property

We now introduce the concept of *modular nuclearity* for inclusions of von Neumann algebras with a distinguished cyclic and separating vector.

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and cyclic and separating unit vector Ω. We set

$$L^\infty(M) = M, \qquad L^2(M) = \mathcal{H}, \qquad L^1(M) = M_* \,.$$

Then we have the linear embeddings

$$\begin{tikzcd} L^\infty(M) \arrow[rr, "x\to(x\Omega,J\cdot\Omega)", "\Phi^M_{\infty,1}"'] \arrow[dr, "\Phi^M_{\infty,2}", "x\to\Delta^{1/4}x\Omega"'] & & L^1(M) \\ & L^2(M) \arrow[ur, "\Phi^M_{2,1}", "\xi\to(\xi,J\cdot\Omega)"'] & \end{tikzcd}$$

All embeddings are bounded with norm one.

Let now N be a von Neumann subalgebras of M . We shall say that $L^{p,q}$-*nuclearity* holds for $N \subset M$, with respect to Ω, if $\Phi^M_{p,q}|_N$ is a nuclear operator

$(p, q = 1, 2, \infty,\ p \geq q)$. $L^{\infty,2}$ nuclearity is also called *modular nuclearity* (for $(N \subset M, \Omega)$).

Denote by $H \equiv \overline{M_{\mathrm{sa}}\Omega}$, $K \equiv \overline{N_{\mathrm{sa}}\Omega}$ the associated closed real linear subspaces of $\mathcal{H}$. Recall that in Sect. 5.3 we have introduced the notion of modular nuclearity for a real linear subspace of a standard subspace and we now provide a link to this notion.

Proposition 7.2.1. *If modular nuclearity holds for $K \subset H$, then modular nuclearity holds for $(N \subset M, \Omega)$,*

Proof. $\Phi^M_{\infty,2}|_N$ is nuclear iff its restriction $\Phi^M_{\infty,2}|_{N_{\mathrm{sa}}}$ is nuclear (as real linear map). As $\Phi^M_{\infty,2}|_{N_{\mathrm{sa}}}$ is equal to $\Delta_H^{1/4}E_K$ multiplied on the right with the norm one map $n \in N_{\mathrm{sa}} \mapsto n\Omega \in \mathcal{H}$, the statement follows. □

Proposition 7.2.2. *Modular nuclearity implies $L^{\infty,1}$ nuclearity and $\|\Phi^M_{\infty,1}|_N\|_1 \leq \|\Phi^M_{\infty,2}|_N\|_1$.*

Proof. The statement is immediate by the above diagram: indeed $\Phi^M_{\infty,1}|_N = \Phi^N_{2,1} \cdot \Phi^M_{\infty,2}|_N$ and $\|\Phi^N_{2,1}\| \leq 1$. □

Proposition 7.2.3. *If N or M is a factor and $L^{\infty,1}$ holds (in particular if modular nuclearity holds) then $N \subset M$ is a split inclusion.*

Proof. By assumption $\Phi^M_{\infty,1}|_N$ is nuclear. It follows that the map $\Phi : M_1 \to M_2^*$ given by

$$\Phi : m_1 \in M_1 \mapsto \omega(m_1\ \cdot)|_{M_2} \in M_{2*} \tag{7.2.1}$$

is nuclear, where $\omega \equiv (\cdot\,\Omega, \Omega)$. This means that there exist sequences of elements $\varphi_k \in M_1^*$ and $\psi_k \in M_{2*}$ such that

$$\sum_k \|\varphi_k\|\,\|\psi_k\| < \infty \tag{7.2.2}$$

and

$$\omega(m_1 m_2) = \sum_k \varphi_k(m_1)\psi_k(m_2)\ , \quad m_1 \in M,\ m_2 \in M_2\ . \tag{7.2.3}$$

We first show that we may choose the φ_k to be normal. Let $\varphi_k^{(n)}$ and $\varphi_k^{(s)}$ the normal and the singular part of $\varphi_k \in M_1^*$, thus $\varphi_k = \varphi_k^{(n)} + \varphi_k^{(s)}$ and $\|\varphi_k^{(n)}\|, \|\varphi_k^{(s)}\| \leq \|\varphi_k\|$.

Consider the maps from M_1 to M_2^*

$$\Phi^{(n)} \equiv \sum_{k=1}^{\infty} \varphi_k^{(n)} \otimes \psi_k \,, \qquad \Phi^{(s)} \equiv \sum_{k=1}^{\infty} \varphi_k^{(s)} \otimes \psi_k \,, \tag{7.2.4}$$

in the sense that foe every $m_1 \in M_1$ we have

$$\Phi^{(n)}(m_1) = \sum_{k=1}^{\infty} \varphi_k^{(n)}(m_1)\psi_k \,, \qquad \Phi^{(s)}(m_1) = \sum_{k=1}^{\infty} \varphi_k^{(s)}(m_1)\psi_k \,. \tag{7.2.5}$$

Clearly

$$\Phi = \Phi^{(n)} + \Phi^{(s)}$$

so for any fixed $m_1 \in M_1$ we have the equality among elements of M_1^*

$$\Phi(m_1) = \Phi^{(n)}(m_1) + \Phi^{(s)}(m_1) \,.$$

Clearly $\Phi(m_1) \in M_2^*$ as is given by $m_2 \mapsto \omega(m_1 m_2)$.

Now $\Phi^{(n)}(m_1)$ and $\Phi^{(s)}(m_1)$ are respectively normal and singular elements of M_2^* because the truncated sums

$$\Phi^{(n)}(m_1) = \sum_{k=1}^{N} \varphi_k^{(n)}(m_1)\psi_k \,, \qquad \Phi^{(s)}(m_1) = \sum_{k=1}^{N} \varphi_k^{(s)}(m_1)\psi_k \tag{7.2.6}$$

are respectively normal and singular functionals and the series (7.2.5) are absolutely convergent as the series (7.2.4) are absolutely convergent. Therefore $\Phi(m_1) = \Phi^{(n)}(m_1)$ for all m_1 showing that the φ_k can be taken normal.

The proposition now follows by Lemma 7.1.3 □

Consider now the commutative diagram

$$\begin{array}{ccc}
L^\infty(N) & \xrightarrow{\Phi^M_{\infty,1}|_N} & L^1(M) \\
\Phi^N_{\infty,2}\big\downarrow & & \big\uparrow\Phi^M_{2,1} \\
L^2(N) & \xrightarrow{T_{M,N}\equiv\Delta_M^{1/4}\Delta_N^{-1/4}} & L^2(M)
\end{array}$$

Recall that the operator $\Delta_M^{1/4}\Delta_N^{-1/4}$ is densely defined with norm one; its closure $T_{M,N}$ here above is the canonical embedding of $L^2(N)$ into $L^2(M)$.

Note that the map $T_{M,N}$ is associated only with the standard subspaces $H \equiv \overline{M_{\rm sa}}$, $K \equiv \overline{N_{\rm sa}}$.

We shall now consider the condition that $T_{M,N}$ be a nuclear operator that we call the *L^2-nuclearity condition*. Obvoiusly, this is nothing else than the L^2-nuclearity condition for $K \subset H$.

Proposition 7.2.4. *L^2-nuclearity implies modular nuclearity and* $\|\Phi^M_{\infty,2}|_N\|_1 \leq \|T_{M,N}\|_1$.

Proof. Immediate by Prop. 5.3.1, 7.2.2 and 7.2.3. □

Combining Propositions 7.2.2 and 7.2.3 we then have:

$$\text{Modular nuclearity} \Rightarrow L^{\infty,2}\text{-nuclearity} \Rightarrow L^{\infty,1}\text{-nuclearity} \Rightarrow \text{Split}\,. \tag{7.2.7}$$

We make now a few comments about generalizing the above notions with more general exponents. Consider the map $\Xi^M_\lambda : M \to \mathcal{H}$

$$\Xi^M_\lambda : x \in M \mapsto \Delta^\lambda_M x\Omega \in \mathcal{H}$$

thus $\Xi^M_{1/4} = \Phi^M_{\infty,2}$. We have $\|\Xi^M_\lambda\| \leq 1$ if $0 \leq \lambda \leq 1/2$.

With $S = S_M$, $\Delta_M = \Delta$ and $J = J_M$ the Tomita operators, we shall say that *L^2-modular nuclearity* holds for $(N \subset M, \Omega)$, with exponent $\lambda \in (0, 1/2)$, if $\Xi^M_\lambda|_N$ is nuclear.

As

$$J\Delta^\lambda n\Omega = J\Delta^\lambda S n^*\Omega = J\Delta^\lambda J\Delta^{1/2} n^*\Omega = \Delta^{-\lambda}\Delta^{1/2} n^*\Omega = \Delta^{1/2-\lambda} n^*\Omega\,,$$

L^2-modular nuclearity for $(N \subset M, \Omega)$ holds with exponent λ iff it holds with exponent $1/2 - \lambda$. As $\Delta^{1/4} = B(\Delta^\lambda + \Delta^{1/2-\lambda})$ with B a bounded operator, if L^2-modular nuclearity holds with exponent λ then it holds with exponent $1/4$. If no exponent is specified for L^2-modular nuclearity, we shall implicitly assume it to be $1/4$.

Consider now the condition

$$\|T_{M,N}(i\lambda)\|_1 < \infty$$

with $T_{M,N}(\lambda) \equiv \Delta^\lambda_M \Delta^{-\lambda}_N = \Delta^\lambda_H \Delta^{-\lambda}_K$ for general exponents $0 < \lambda < 1/2$. Note that $\|T_{M,N}(\lambda)\| \leq 1$ by (7) of Th. 2.3.1.

Since

$$\Xi^M_\lambda|_N = T_{M,N}(i\lambda) \cdot \Xi^N_\lambda$$

we have

$$\|\Xi^M_\lambda|_N\|_1 \leq \|T_{M,N}(i\lambda)\|_1\,.$$

7.3 Split property and trace class condition for nets of factors

A net $\mathcal{A}$ of von Neumann algebras on S^1 is said to satisfy the *split property* if there exists an intermediate type I factor $\mathcal{A}(I_1) \subset F \subset \mathcal{A}(I_2)$ for any inclusion of intervals $I_1 \Subset I_2$.

Proposition 7.3.1. *If the split property holds, then the local algebras $\mathcal{A}(I)$ are approximately finite-dimensional.*

Proof. By continuity we may suppose that I is open. Let $I_1 \subset I_2 \subset \cdots I$ be an increasing sequence of open intervals with $\bar{I}_k \subset I_{k+1}$ and $\cup I_k = I$ and choose type I factors $\mathcal{A}(I_k) \subset F_k \subset \mathcal{A}(I_{k+1})$. Then $\mathcal{A}(I)$ is generated by the increasing sequence of type I factors F_k, hence it is approximately finite-dimensional. □

Corollary 7.3.2. *If $\mathcal{A}$ is non-trivial and the split property holds, $\mathcal{A}(I)$ is isomorphic to the unique Connes-Haagerup injective III_1-factor.*

Proof. By Prop. 6.2.9 $\mathcal{A}(I)$ is a type III_1-factor. The result is then immediate by the above proposition and the uniqueness of the injective type III_1-factor. □

Let $\mathcal{A}$ be a Möbius covariant net and L be its conformal Hamiltonian. We shall say that $\mathcal{A}$ satisfies the *trace class condition*, at inverse temperature $\beta > 0$, if

$$\mathrm{Tr}(e^{-\beta L}) < \infty \,.$$

We now define the *inner distance* $\ell(\tilde{I}, I)$ for an inclusion of intervals $I \Subset \tilde{I}$. First suppose that in the real line picture $\tilde{I} = (-1, 1)$ and $I = (-e^{-s}, e^{-s})$, then we set $\ell(\tilde{I}, I) \equiv s$. Now an arbitrary inclusion $I \Subset \tilde{I}$ of intervals of S^1 is conjugate by a Möbius tranformation to a an inclusion $(-e^{-s}, e^{-s}) \subset (-1, 1)$ as above for a unique $s > 0$ and we thus set $\ell(\tilde{I}, I) = s$.

Theorem 7.3.3. *If $\mathcal{A}$ satisfies the trace class condition at inverse temperature β, then $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is a split inclusion if $\ell(\tilde{I}, I) > \beta$. Therefore*

$$\mathrm{Tr}(e^{-\beta L}) < \infty \ \forall \beta > 0 \implies \textit{split property}$$

Proof. With $H(I) \equiv \overline{\mathcal{A}(I)_{\mathrm{sa}}\Omega}$ the associated net of standard subspaces, we know from Chapter 5 that

$$\mathrm{Tr}(e^{-sL}) < \infty \implies \|T_{\tilde{I},I}\| < \infty \,, \quad \tilde{I} \supset I, \ell(\tilde{I}, I) > s \,,$$

namely L^2 nuclearity holds for $H(I) \subset H(\tilde{I})$. By the implications (7.2.7) we then have the split property for $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$. □

We note that if $I \subset \tilde{I}$ are intervals with one common boundary point, then the split property for $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ does *not* hold. Otherwise there would exists a product state φ on $\mathcal{A}(I) \vee \mathcal{A}(\tilde{I}')$, namely $\varphi(xx') = \omega(x)\omega(x')$, $x \in \mathcal{A}(I)$, $x' \in \mathcal{A}(\tilde{I}')$. Now $\delta_{\tilde{I}}(s)$ maps $\mathcal{A}(I) \vee \mathcal{A}(\tilde{I}')$ into itself, $s \geq 0$, and for $x \in \mathcal{A}(I)$, $x' \in \mathcal{A}(\tilde{I}')$, we have

$$\varphi(\delta_{\tilde{I}}(s)(xx')) = \varphi(\delta_{\tilde{I}}(s)(x)\delta_{\tilde{I}}(s)(x')) = \omega(\delta_{\tilde{I}}(s)(x))\omega(\delta_{\tilde{I}}(s)(x')) = \omega(x)\omega(x') = \varphi(xx').$$

As $s \to +\infty$, $\delta_{\tilde{I}}(s)(X) \to \omega(X)$ weakly for all $X \in \mathcal{A}(I) \vee \mathcal{A}(\tilde{I}')$, so the above equation shows that $\varphi = \omega$ on $\mathcal{A}(I) \vee \mathcal{A}(\tilde{I}')$ which is a contradiction by the cyclicity of Ω. □

7.4 Split and nuclarity for second quantization nets

7.4.1 Trace and determinants

With a $\mathcal{H}$ be a Hilbert space and a positive contraction $A \in B(\mathcal{H})$, we now give a formula for calculating the trace of the exponential $\Gamma(A)$ of A in terms of A. If B is an operaror with discrete spectrum with eigenvalue list s_i (with multiplicity), the determinant of B id defined by $\det(B) = \prod_i s_i$ provided the product is absolutely convergent.

Lemma 7.4.1. *If* $A \in B(H)$ *is selfadjoint,* $0 \leq A < 1$*, then*

$$\operatorname{Tr}\Gamma(A) = \det(1 - A)^{-1}, \tag{7.4.1}$$

$$\log \operatorname{Tr}\Gamma(A) = \operatorname{Tr}\log(1 - A) \tag{7.4.2}$$

Proof. Assume first that $\mathcal{H}$ is one-dimensional, thus $A = \lambda$ is a scalar $0 \leq \lambda < 1$. Then $\mathcal{H}^{\otimes^n}$ is also one-dimensional for all n, thus we have $\Gamma(A) = \oplus_{n=0}^{\infty}\lambda^n$, so $\operatorname{Tr}\Gamma(A) = \sum_{n=0}^{\infty}\lambda^n = (1 - \lambda)^{-1}$.

For a general A (with discrete spectrum) we may decompose $\mathcal{H} = \oplus_i\mathcal{H}_i$ so that $\dim\mathcal{H}_i = 1$ and $A = \oplus_i\lambda_i$. Then $\Gamma(\mathcal{H}) = \otimes_i^{\{\Omega_i\}}\Gamma(\mathcal{H}_i)$, where Ω_i is the vacuum vector of $\Gamma(\mathcal{H}_i)$, and $A = \otimes_i A_i$. It follows that

$$\operatorname{Tr}\Gamma(A) = \prod_i \operatorname{Tr}\Gamma(A_i) = \prod_i (1 - \lambda_i)^{-1} = \det(1 - A)^{-1}.$$

Concerning the second formula, notice that

$$\det B = e^{\operatorname{Tr}\log B},$$

hence

$$\log \operatorname{Tr} \Gamma(A) = -\log \det(1 - A) = -\operatorname{Tr} \log(1 - A).$$

□

Corollary 7.4.2. *With A as above we have* $\operatorname{Tr} \Gamma(A) < \infty$ *iff* $\operatorname{Tr} A < \infty$.

Proof. We have $\operatorname{Tr} A = \sum_i \lambda_i$ and $\operatorname{Tr} \Gamma(A) = -\sum_i \log(1 - \lambda_i)$. As $-\log(1 - t) = t$ as $t \to 0^+$, the two series have the same character and the corollary follows. □

7.4.2 The trace class condition for $\mathcal{A}_U$

We now show the trace class property, hence the split property, for second quantization nets where the multiplicity of the irreducible components in the one-particle space increases sub-exponentially, in particular for the nets associated with the $U(1)$ current or its n-derivative. We shall denote by $U^{(n)}$ be the irreducible unitary representation of $\mathbf{G}$ with lowest weight n and by $L_0^{(n)}$ the associated conformal Hamiltonian.

Proposition 7.4.3. *Let $U = \bigoplus_n m_n U^{(n)}$ be a positive energy unitary representation of $\mathbf{G}$ and denote by L_0 the conformal Hamiltonian of U. Then*

$$\operatorname{Tr}(e^{-\beta L_0}) < \infty\,, \qquad \forall \beta > \beta_0\,, \tag{7.4.3}$$

$$\operatorname{Tr}(e^{-\beta L_0}) = \infty\,, \qquad \forall \beta < \beta_0\,, \tag{7.4.4}$$

where $\beta_0 \equiv \log \limsup_n \sqrt[n]{m_n}$.

Proof. The eigenvalues of $L_0^{(n)}$ are $\{n, n+1, n+2, \dots\}$, with multiplicity one, so $\operatorname{Tr}(e^{-\beta L_0^{(n)}}) = \sum_{k=n}^{\infty} e^{-\beta k} = \frac{e^{-\beta n}}{1 - e^{-\beta n}} < \infty$.

We then have

$$\operatorname{Tr}(e^{-\beta L_0}) = \sum_n m_n \operatorname{Tr}(e^{-\beta L_0^{(n)}}) = (1 - e^{-\beta})^{-1} \sum_n m_n e^{-\beta n}$$

that converges if $\limsup_n \sqrt[n]{m_n} e^{-\beta} < 1$ and diverges if $\limsup_n \sqrt[n]{m_n} e^{-\beta} > 1$. □

Corollary 7.4.4. *The net generated by $U(1)$ current, or by any derivative of the $U(1)$ current, satisfies the trace class condition, hence the split property.*

More generally, let $U = \bigoplus_n m_n U^{(n)}$ be a positive energy unitary representation of $\mathbf{G}$. The second quantization net $\mathcal{A}_U$ satisfies the trace class condition if $\beta > \beta_0$ where β_0 is given in Prop. 7.4.3. So the split property holds if $\beta_0 = 0$.

Proof. The semigroup generated by the conformal Hamiltonian of $\mathcal{A}_{U^{(n)}}$ is $\Gamma(e^{-\beta L_0^{(n)}})$, so by Lemma 7.4.1 we have $\mathrm{Tr}(\Gamma(e^{-\beta L_0^{(n)}})) < \infty$ if and only is $\mathrm{Tr}(e^{-\beta L_0^{(n)}}) < \infty$, which holds for all $\beta > 0$. The split property now follows by Thm. 7.3.3. The case of a general second quantization net $\mathcal{A}_U$ is treated similarly. □