KAM Theory and Applications in Celestial Mechanics – Fourth Lecture: Lower Dimensional Invariant Tori and KAM Global Point of View on the Dynamics

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The Hamiltonian of a planar “SJSU-like” system

The Hamiltonian of a planetary 4–body system writes as

$$F(\mathbf{r},\tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r})\,,$$

where $\mathbf{r}$ are the heliocentric coordinates, $\tilde{\mathbf{r}}$ the conjugated momenta and

$$\begin{aligned}
T^{(0)}(\tilde{\mathbf{r}}) &= \frac{1}{2}\sum_{j=1}^{3}\|\tilde{\mathbf{r}}_j\|^2\left(\frac{1}{m_0}+\frac{1}{m_j}\right)\,,\\
U^{(0)}(\mathbf{r}) &= -\mathcal{G}\sum_{j=1}^{3}\frac{m_0\,m_j}{\|\mathbf{r}_j\|}\,,\\
T^{(1)}(\tilde{\mathbf{r}}) &= \frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_2}{m_0}+\frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_3}{m_0}+\frac{\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_3}{m_0}\,,\\
U^{(1)}(\mathbf{r}) &= -\mathcal{G}\left(\frac{m_1\,m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|}+\frac{m_1\,m_3}{\|\mathbf{r}_1-\mathbf{r}_3\|}+\frac{m_2\,m_3}{\|\mathbf{r}_2-\mathbf{r}_3\|}\right)\,.
\end{aligned}$$

We fixed the masses and the average semi-major axes so to be equal to those of the real Sun–Jupiter–Saturn–Uranus system.

The Poincaré variables in the plane

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)$$

where a_j, e_j, M_j and ω_j are the orbital elements of the j–th planet.

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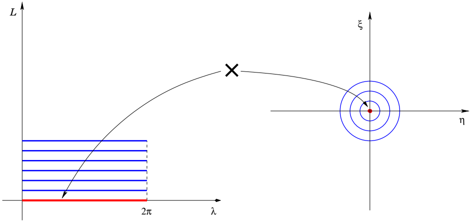
where a_j, e_j, M_j and ω_j are the orbital elements of the j–th planet.

Let us introduce new actions $\mathbf{L} = \mathbf{\Lambda} - \mathbf{\Lambda}^*$, where Λ_j^* is calculated with the average value a_j^* of the semi-major axis of the j–th planet.

Elliptic tori as "extensions" of equilibrium points

- **Question**: what is *the equilibrium point of the secular part* (that is a 3 d.o.f. system) *with respect to the flow of the averaged Hamiltonian* $\langle F\rangle_{\boldsymbol{\lambda}}$ (6 d.o.f. system)?
- **Answer**: **an elliptic torus!**



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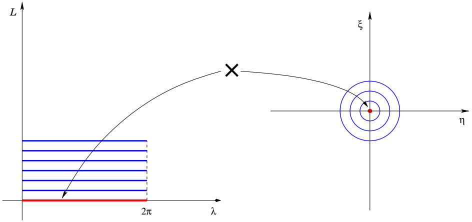

- **Problem**: **can we locate elliptic tori in the complete** *(i.e. non-averaged)* **Hamiltonian** F?

Elliptic tori in (already seen) “resonant regions”

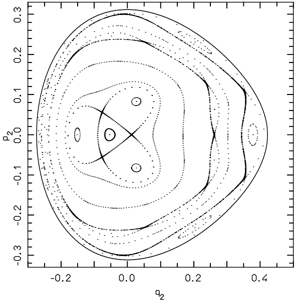

Figure: Poincaré sections for the Hénon–Heiles model $H(\mathbf{p},\mathbf{q}) = \omega_1\left(p_1^2+q_1^2\right)/2 + \omega_2\left(p_2^2+q_2^2\right)/2 + q_1^2 q_2 - q_2^3/3$ with $\omega_1 = 1$ and $\omega_2 = (\sqrt{5}-1)/2$. The energy level is fixed so that $E = 0.030$. In this case the escape energy value is $E_e = 0.03934466$.

- **Remark**: **in Poincaré sections, elliptic tori**, that are invariant with respect to the flow of H , **are seen as single points contoured by closed 1D–curves.** *In figure above, they are visible in the so called “chains of ordered islands”.*

The wanted normal form

Let us imagine to have already performed an infinite sequence of can. transf. so to bring the Hamiltonian in the wanted normal form

$$H^{(\infty)}(\mathbf{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) = \boldsymbol{\omega}^{(\infty)}\cdot\mathbf{L} + \sum_{j=1}^{n_2}\frac{\Omega_j^{(\infty)}}{2}(\xi_j^2+\eta_j^2) + \mathcal{R}^{(\infty)}(\mathbf{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) \ ,$$

where the frequency vectors $\boldsymbol{\omega}^{(\infty)}\in\mathbb{R}^{n_1}$ and $\boldsymbol{\Omega}^{(\infty)}\in\mathbb{R}^{n_2}$, with $n_1+n_2=n$, being n the number of d.o.f. (in our model $n_1=n_2=3$). Moreover, the remainder term is such that

$$\mathcal{R}^{(\infty)}(\mathbf{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) = \mathcal{O}\big(\|\mathbf{L}\|^2\big) + \mathcal{O}\big(\|\mathbf{L}\|\,\|(\boldsymbol{\xi},\boldsymbol{\eta})\|\big) + \mathcal{O}\big(\|(\boldsymbol{\xi},\boldsymbol{\eta})\|^3\big) \ .$$

$\forall\ (\mathbf{0},\boldsymbol{\lambda},\mathbf{0},\mathbf{0})\in(\mathbf{0},\mathbb{T}^{n_1},\mathbf{0},\mathbf{0})$, the Hamilton eq.s can be easily solved

$$\dot{\mathbf{L}}=\mathbf{0}\ ,\quad \dot{\boldsymbol{\lambda}}=\boldsymbol{\omega}^{(\infty)}\ ,\quad \dot{\boldsymbol{\xi}}=\mathbf{0}\ ,\quad \dot{\boldsymbol{\eta}}=\mathbf{0}\ .$$

Thus, **the flow induced by $H^{(\infty)}$ on the invariant lower-dimensional torus $(\mathbf{0},\mathbb{T}^{n_1},\mathbf{0},\mathbf{0})$ is quasi-periodic with frequency vector $\boldsymbol{\omega}^{(\infty)}$**, while $\boldsymbol{\Omega}^{(\infty)}$ *is the limit frequency vector of the small oscillations tranversal to the elliptic torus.*

The normal form Hamiltonian up to order $r-1$

Where it is convenient, we refer to the secular variables with action–angle coordinates $(\mathbf{I},\varphi)$ such that $\xi_j=\sqrt{2I_j}\cos\varphi_j$, $\eta_j=\sqrt{2I_j}\sin\varphi_j$, $\forall\ j=1,\ \ldots\ ,\ n_2$. The Hamiltonian F representing our planar "SJSU-like" model can be written in the following general form (with $r=1$):

$$
\begin{aligned}
H^{(r-1)} = &\ \omega^{(r-1)}\cdot\mathbf{L}+\mathbf{\Omega}^{(r-1)}\cdot\mathbf{I}+\sum_{j_1\ge 2}h_{j_1,0}(\mathbf{L})+\sum_{s\ge r}f_{0,0}^{(r-1,s)}(\boldsymbol{\lambda})+\\
&\sum_{s\ge r}f_{0,1}^{(r-1,s)}(\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})+\sum_{s\ge r}f_{1,0}^{(r-1,s)}(\mathbf{L},\boldsymbol{\lambda})+\sum_{s\ge r}f_{0,2}^{(r-1,s)}(\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})\\
&+\sum_{2j_1+j_2\ge 3}\ \sum_{s>0}f_{j_1,j_2}^{(r-1,s)}(\mathbf{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})\ ,
\end{aligned}
$$

where $r-1$ means the normalization step, $h_{j_1,0}$ are homogeneous polynomials of degree j_1 in $\mathbf{L}$, $f_{j_1,j_2}^{(r-1,s)}$ are hom. pol. of degree j_1 and j_2 in $\mathbf{L}$ and in $(\boldsymbol{\xi},\boldsymbol{\eta})$, resp., while they are also trig. pol. of degree at most $2s$ in the angles $\boldsymbol{\lambda}$. Thus, *in the previous formula,* ***each term has a finite Taylor–Fourier expansion.*** *The normalization procedure has to eliminate the (red color) perturbing terms.*

Normalization procedure: the homological equations

$$\left\{\chi_0^{(r)}, \boldsymbol{\omega}^{(r-1)} \cdot \mathbf{L}\right\} + \sum_{s=1}^{r} f_{0,0}^{(r-1,s)}(\boldsymbol{\lambda}) = 0 \ ,$$

$$\left\{\chi_1^{(r)}, \boldsymbol{\omega}^{(r-1)} \cdot \mathbf{L} + \boldsymbol{\Omega}^{(\mathbf{r-1})} \cdot \mathbf{I}\right\} + \sum_{s=0}^{r} f_{0,1}^{(\mathrm{I};r,s)}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \ ,$$

$$\left\{X_2^{(r)}, \boldsymbol{\omega}^{(r-1)} \cdot \mathbf{L}\right\} + \sum_{s=1}^{r} f_{1,0}^{(\mathrm{II};r,s)}(\mathbf{L}, \boldsymbol{\lambda}) = 0 \ ,$$

$$\left\{Y_2^{(r)}, \boldsymbol{\omega}^{(r-1)} \cdot \mathbf{L} + \boldsymbol{\Omega}^{(\mathbf{r-1})} \cdot \mathbf{I}\right\} + \sum_{s=1}^{r} f_{0,2}^{(\mathrm{II};r,s)}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \ ,$$

$$\left\{\mathcal{D}_2^{(r)}, \boldsymbol{\Omega}^{(\mathbf{r-1})} \cdot \mathbf{I}\right\} + f_{0\,,\,2}^{(\mathrm{II};r,0)}(\mathbf{I}, \boldsymbol{\varphi}) - \left\langle f_{0\,,\,2}^{(\mathrm{II};r,0)}\right\rangle_{\boldsymbol{\varphi}} = 0 \ .$$

Normalization procedure for elliptic tori: main ideas

- Each r–th normalization step is performed by *composing of three canonical transformations* $\exp\mathcal{L}_{\chi_0^{(r)}}$, $\exp\mathcal{L}_{\chi_1^{(r)}}$ and $\exp\mathcal{L}_{\chi_2^{(r)}}$, where *the generating functions* $\chi_0^{(r)}$, $\chi_1^{(r)}$ and $\chi_2^{(r)}$ **are determined so to eliminate the perturbing terms**
  - *independent from both* $\mathbf{L}$ *and* $(\boldsymbol{\xi},\boldsymbol{\eta})$,
  - *independent from* $\mathbf{L}$ *and of degree* 1 *in* $(\boldsymbol{\xi},\boldsymbol{\eta})$,
  - **either** *of degree* 1 *in* $\mathbf{L}$ *and independent from* $(\boldsymbol{\xi},\boldsymbol{\eta})$, **or** *independent from* $\mathbf{L}$ *and of degree* 2 *in* $(\boldsymbol{\xi},\boldsymbol{\eta})$,

  respectively; moreover, *each generating function "kills" perturbing terms up to trigonometric degree* $2r$ *in* $\boldsymbol{\lambda}$.
- at the end of each normalization step, there are some terms $\mathcal{O}\big(\|\mathbf{L}\|\big)$ and $\mathcal{O}\big(\|(\boldsymbol{\xi},\boldsymbol{\eta})\|^2\big)$ that cannot be eliminated (because they do not depend on the angles); thus, they are included in the normal form terms and they induce small corrections of the frequency vectors, that are $\boldsymbol{\omega}^{(r-1)} \to \boldsymbol{\omega}^{(r)}$ and $\boldsymbol{\Omega}^{(r-1)} \to \boldsymbol{\Omega}^{(r)}$.
- *If the sequence* $\left\{\left(\boldsymbol{\omega}^{(r)},\boldsymbol{\Omega}^{(r)}\right)\right\}_{r\geq 0}$ *stays away enough from the resonances*, **the normalization procedure works!** (see *Sansottera, Locatelli & Giorgilli 2011*).

Testing the construction of the normal form for elliptic tori

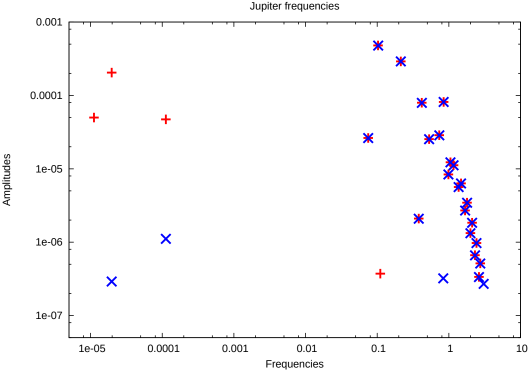

Figure: Fourier spectra with the first 30 main components of the signal $t \to \xi_1(t) + \mathrm{i}\eta_1(t)$ related to the *secular motion of* **Jupiter**, i.e. $\xi_1(t) + \mathrm{i}\eta_1(t) \simeq \sum_{s=1}^{30} c_{1,s} \exp\left[\mathrm{i}\left(\nu_{1,s} t\right)\right]$. Frequencies $\nu_{1,s}$ and amplitudes $|c_{1,s}|$ are reported on the abscissas and the ordinates, resp. $+$ symbols refer to a motion starting from "real initial conditions" of the "planar SJSU", while **x** symbols are for an approximated elliptic torus after 9 normalization steps.

Frequency analysis shows that **we are approaching an elliptic torus!**

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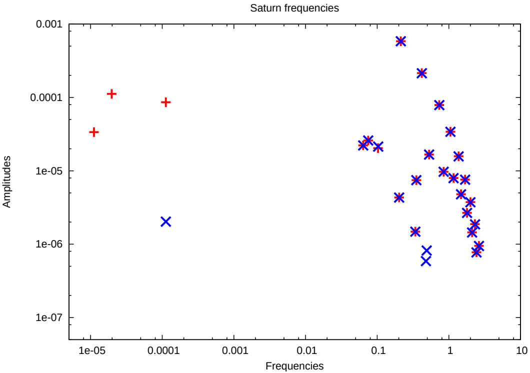

Figure: Fourier spectra with the first 30 main components of the signal $t \to \xi_2(t) + \mathrm{i}\eta_2(t)$ related to the *secular motion of* **Saturn**, i.e. $\xi_2(t) + \mathrm{i}\eta_2(t) \simeq \sum_{s=1}^{30} c_{2,s} \exp\left[\mathrm{i}\left(\nu_{2,s} t\right)\right]$. Frequencies $\nu_{2,s}$ and amplitudes $|c_{2,s}|$ are reported on the abscissas and the ordinates, resp. $+$ symbols refer to a motion starting from "real initial conditions" of the "planar SJSU", while **x** symbols are for an approximated elliptic torus after 9 normalization steps.

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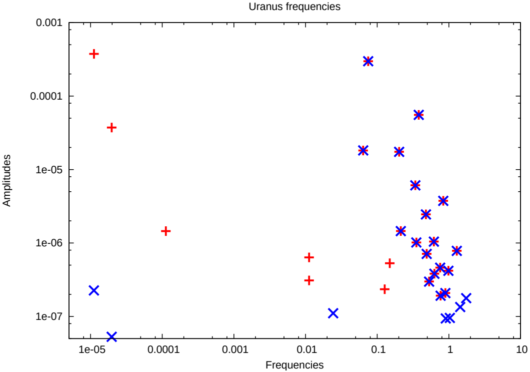


Figure: Fourier spectra with the first 30 main components of the signal $t \to \xi_3(t) + \mathrm{i}\eta_3(t)$ related to the *secular motion of* **Uranus**, i.e. $\xi_3(t) + \mathrm{i}\eta_3(t) \simeq \sum_{s=1}^{30} c_{3,s} \exp\left[\mathrm{i}\left(\nu_{3,s} t\right)\right]$. Frequencies $\nu_{3,s}$ and amplitudes $|c_{3,s}|$ are reported on the abscissas and the ordinates, resp. $+$ symbols refer to a motion starting from "real initial conditions" of the "planar SJSU", while **x** symbols are for an approximated elliptic torus after 9 normalization steps.

Frequency analysis shows that **we are approaching an elliptic torus!**

Main differences in the construction of the Kolmogorov's normal form: KAM tori vs. elliptic lower dimensional tori

- The small parameter is the mass ratio $\max_{j=1,2,3} m_j/m_0$.
- At each normalization step for elliptic tori, we need a non-resonance condition of type

$$\min_{0<|\mathbf{k}|\le 2r} \left|\mathbf{k}\cdot\boldsymbol{\omega}^{(r-1)}\right| > 0 \ ,$$

  that is similar to that needed by the usual KAM tori, but we need also the so called **Melnikov's conditions**:

$$\min_{|\mathbf{k}|\le 2r\ \ 1\le|\boldsymbol{\ell}|\le 2} \left|\mathbf{k}\cdot\boldsymbol{\omega}^{(r-1)} + \boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right| > 0 \ .$$

- In case of elliptic tori, **the frequencies are not fixed** *"a priori"*, and the algorithm let them (very slightly) change at each step. Thus, eventual resonances can show up after some normalization step.
- The system is usually parameterized with respect to n_1 independent variables (e.g., the average semi-major axes where the initial expansions are centered about). Thus, *the final result must hold true on a set having positive Lebesgue measure.* **This is really in the spirit of the original Arnold's proof scheme.**

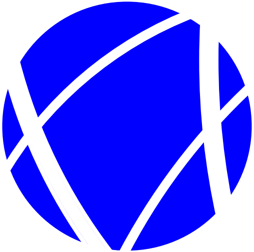
Arnold's proof scheme of KAM theorem

At the first normalization step, you cut out a first group of main resonances from the phase space and you perform a canonical transformation so to eliminate those resonant perturbing terms from the Hamiltonian, which is lead to the form:

$$H^{(1)}(\mathbf{p},\mathbf{q}) = h^{(1)}(\mathbf{p}) + \mathcal{R}^{(1)}(\mathbf{p},\mathbf{q}) \ ,$$

where $\mathcal{R}^{(1)}$ is a small remainder term.

Arnold's proof scheme of KAM theorem



At the second normalization step, you cut out a second group of main resonances from the phase space and you perform another canonical transformation so to write the Hamiltonian in the form:

$$H^{(2)}(\mathbf{p},\mathbf{q}) = h^{(2)}(\mathbf{p}) + \mathcal{R}^{(2)}(\mathbf{p},\mathbf{q}) \ ,$$

where $\|\mathcal{R}^{(2)}\| = \mathcal{O}\big(\|\mathcal{R}^{(1)}\|^2\big)$ is a remainder term much smaller than the previous one. *After having iterated infinitely many times this procedure the Hamiltonian is convergent* (if the initial perturbation is small enough) *on a Cantor set of invariant tori with positive Lebesgue measure.*

Arnold web in numerics for a 4D–symplectic map

The *Coupled Rational Shifted Standard Map* (**CRSSM**) *is defined so that*

$$
\begin{aligned}
y_1' &= y_1 + \varepsilon_1 f_1(x_1) + \gamma_+ f_3(x_1 + x_2) + \gamma_- f_3(x_1 - x_2),\\
y_2' &= y_2 + \varepsilon_2 f_2(x_2) + \gamma_+ f_3(x_1 + x_2) - \gamma_- f_3(x_1 - x_2),\\
x_1' &= x_1 + \varepsilon_1 y_1' \qquad \mathrm{mod}\ 2\pi\ ,\\
x_2' &= x_2 + \varepsilon_2 y_2' \qquad \mathrm{mod}\ 2\pi\ ,
\end{aligned}
$$

where angles $x_i \in [0, 2\pi)$, *actions* $y_i \in [0, 2\pi/\varepsilon_i)$, $\forall\ i = 1, 2$, *while the perturbing functions are such that* $f_i(x) = -\sin x/(1 - \mu_i \cos x)$, $\forall\ i = 1\,, 2\,, 3$, *being* ε_1 , ε_2 , $\gamma_\pm$, μ_1 , μ_2 *and* μ_3 *fixed small parameters.*

- *Consider a regular grid of initial values* $(y_{1;0}\ , \ y_{2;0})$ *of the actions;*
- *for each initial condition of type* $(y_{1;0}\ , \ y_{2;0}\ , \ 0\ , \ 0)$, *iterate* N *times the map, so to produce a finite sequence of points* $(y_{1;j}\ , \ y_{2;j}\ , \ x_{1;j}\ , \ x_{2;j})$ *with* $j = 0, \ldots, N$ *;*
- *make the frequency analysis of the signals* $j \to \sqrt{2y_{1;j}}\,\exp(\mathrm{i}x_{1;j})$ *and* $j \to \sqrt{2y_{2;j}}\,\exp(\mathrm{i}x_{2;j})$ *by separating two "windows":* $j \in [0,\ N/2]$ *and* $j \in [N/2,\ N]$ *. Draw the variations of the main frequencies as a function of the initial actions* $(y_{1;0}\ , \ y_{2;0})$ *.*

Arnold web in numerics for a 4D–symplectic map

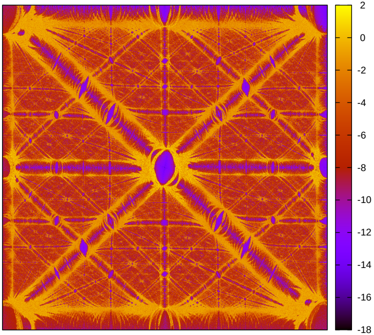


Figure: Color plot describing the Arnold web for the **CRSSM** with $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.2$, $\mu_1 = \mu_2 = \mu_3 = 0.5$, $\gamma_+ = 0.1$, $\gamma_- = 0.05$. Initial values of actions $y_{1;0}$ and $y_{2;0}$ are on abscissas and on ordinates, respectively. For each point the corresponding variation of the frequencies is reported by the color scale on the right.

Frequency analysis as a chaos indicator: *lighter colors mean chaotic motions, while darker ones are related to invariant tori.*

Diffusion along the Arnold web

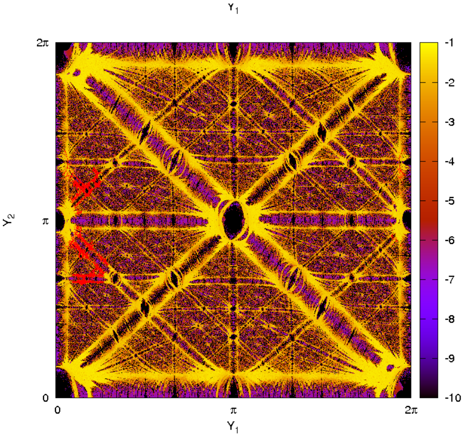


Figure: Evolution of a chaotic orbit (look at red points) along the Arnold web. *The diffusion is faster in larger resonant regions. When some resonant regions cross each other, motions can pass from one to another.*

Some more numerical experiments: the Sup–Map graph

The **standard map** is $y' = y + \varepsilon \sin x$, $x' = x + y' \pmod{2\pi}$.

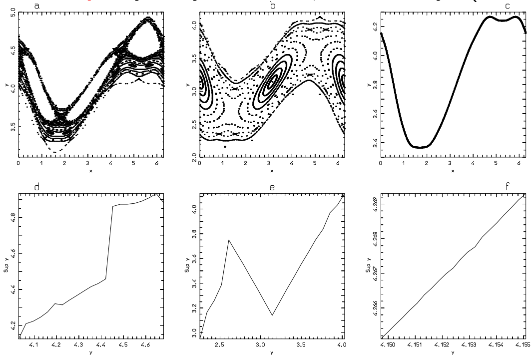


Figure: Archetypical situations occurring in a Sup–Map graph. Boxes a–c contain plots of some orbits of the standard map with $\varepsilon = 0.9$. In boxes d–f, the values of the sup of an orbit as a function of the action y_0, being $(0, y_0)$ the initial point starting from which the orbit is generated by iterations of the standard map. The pair of *boxes (a,d), (b,e) and (c,f) refer to the vertical crossing of* **the chaotic zone close to an hyperbolic point, of the resonant region close to an elliptic point and to a set of invariant tori, resp.**

Some more numerical experiments: the Sup–Map graph

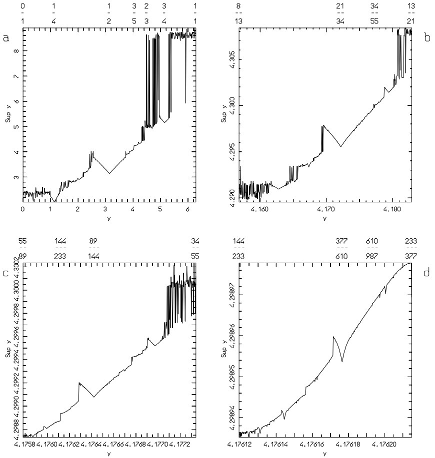


Figure: Sup–Map graph of the standard map with $\varepsilon = 0.9$. Boxes a–d are enlargements focused in a neighborhood smaller and smaller of the "golden mean" torus, that is related to the frequency $(\sqrt{5}-1)/2$. On the top of each box, the location of the main resonances of type F_k/F_{k+1} is reported (being $\{F_k\}_{k\geq 0}$ the sequence of Fibonacci's numbers).

Dynamics in the neighborhood of an invariant torus

Let us focus on a neighborhood of a KAM torus characterized by a Diophantine frequency vector $\boldsymbol{\omega}$, that is such that $|\mathbf{k}\cdot\boldsymbol{\omega}| > \gamma/|\mathbf{k}|^{\tau}$ with some fixed $\gamma > 0$ and $\tau \geq n-1$. By constructing a Birkhoff's normal form starting from the Kolmogorov's normal form and using Nekhoroshev's estimates, we can prove the following results.

- **Let $V(\varrho)$ be the measure of the complementary set with respect to the invariant tori staying at a distance smaller than ϱ from the "ω–torus", then**

$$V(\varrho) \simeq \exp\left(-\left(\frac{\varrho_*}{\varrho}\right)^{1/(\tau+1)}\right) .$$

- **Let T_ϱ be the diffusion time needed by a motion to double its initial distance ϱ from the "ω–torus", then**

$$T_\varrho \simeq \exp\left(C\exp\left(\frac{1}{2n}\left(\frac{\varrho_*}{\varrho}\right)^{1/(\tau+1)}\right)\right) ,$$

being C and ϱ_* suitable positive constants (see *Morbidelli & Giorgilli 1995*).